18.118 DECOUPLING
LECTURE 13

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1. Loose Ends

Define a quantity

$$\mathfrak{M}_{p,q}(r,\sigma) = \operatorname*{Avg}_{B_r \subset B_R} \prod_{j=1}^{n} \left(\sum_{\theta:\sigma^{-1}-\text{caps}} \|f_{j,\theta}\|^2_{L^q_{\text{Avg}}(B_r)} \right)^{\frac{1}{2np}}.$$

Last lecture, we stated two inequalities coming from multilinear Kakeya: the first is

$$\mathfrak{M}_{\frac{2n}{n-1},2}(r,r) \lesssim r^{\varepsilon} \mathfrak{M}_{\frac{2n}{n-1},2}(r^2,r). \qquad \text{MK1}$$

If $q = \frac{n-1}{n} \cdot p \geq 2$, the second more general inequality is

$$\mathfrak{M}_{p,q}(r,r) \lesssim r^{\varepsilon} \mathfrak{M}_{p,q}(r^2,r). \qquad \text{MK2}$$

We proved MK1 last lecture, and our first loose end is to prove MK2. Our starting point is the following form of the multi-linear Kakeya inequality: for $g_j = \sum_a \omega_{j,a} T_{j,a}$, where $\omega_{j,a} \geq 0$ and $T_{j,a}$ are characteristic functions of tubes almost parallel to the j'th axis, then

$$⨍_{Q_S} \prod_{j=1}^{n} g_j^{\frac{1}{n-1}} \lesssim S^{\varepsilon} \prod_{j=1}^{n} \left(⨍ g_j \right)^{\frac{1}{n-1}} \qquad \text{MK.}$$

2. Proving MK2

The left-hand-side of MK2 equals

$$\operatorname*{Avg}_{B_{r^2} \subset B_R} \left[\operatorname*{Avg}_{B_r \subset B_{r^2}} \prod_{j=1}^{n} \left(\sum_{\theta:r^{-1}-\text{caps}} \|f_{j,\theta}\|^2_{L^q_{\text{Avg}}(B_r)} \right)^{\frac{1}{2np}} \right].$$

Notice that $|f_{j,\theta}|$ is approximately a constant on any tube almost perpendicular to the direction of θ, of radius r and length r^2. With this observation, define function $F_{j,\theta}(x) := \|f_{j,\theta}\|_{L^q_{\text{Avg}}(B_r)} \chi_{B_r}(x)$. Then the quantity within the averaging sign on the left-hand-side of MK2 is estimated as follows: with $N_j = \#\theta \subset \Omega_j$,

$$\begin{aligned}\fint_{B_{r^2}} \prod_{j=1}^{n} \left(\sum_\theta F_{j,\theta}^2\right)^{\frac{1}{2np}} &\overset{\text{Hölder}}{\leq} \fint_{B_{r^2}} \prod_{j=1}^{n} \left(\sum_\theta F_{j,\theta}^q\right)^{\frac{1}{qnp}} \prod_{j=1}^{n} N_j^{\left(\frac{1}{2}-\frac{1}{q}\right)\frac{p}{n}} \\ &\overset{\text{MK}}{\lesssim} r^\varepsilon \prod_{j=1}^{n} \left(\fint_{B_{r^2}} \sum_\theta F_{j,\theta}^q\right)^{\frac{1}{qnp}} \prod_{j=1}^{n} N_j^{\left(\frac{1}{2}-\frac{1}{q}\right)\frac{p}{n}} \\ &\leq r^\varepsilon \prod_{j=1}^{n} \left(\sum_\theta \|f_{j,\theta}\|^q_{L^q_{\text{Avg}}(B_{r^2})}\right)^{\frac{1}{qnp}} \prod_{j=1}^{n} N_j^{\left(\frac{1}{2}-\frac{1}{q}\right)\frac{p}{n}}.\end{aligned}$$

The desired inequality is

$$\left(\sum_\theta \|f_{j,\theta}\|^q_{L^q_{\text{Avg}}(B_{r^2})}\right)^{\frac{1}{q}} N_j^{\frac{1}{2}-\frac{1}{q}} \lesssim \left(\sum_\theta \|f_{j,\theta}\|^2_{L^q_{\text{Avg}}(B_{r^2})}\right)^{\frac{1}{2}}.$$

However, the Hölder inequality gives the reverse, and the desired inequality holds only if, for each j, the quantity $\|f_{j,\theta}\|_{L^q_{\text{Avg}}(B_{r^2})}$ is approximately constant in θ.

In order to overcome this obstacle, let us introduce a classification of the caps θ: for λ dyadic, define a family

$$\Theta_{i,\lambda} := \left\{\theta : \lambda/2 \leq \|f_{j,\theta}\|_{L^q_{\text{Avg}}(B_{r^2})} \leq 2\lambda\right\}.$$

Write $N_{j,\lambda} = |\Theta_{j,\lambda}|$. Notice that for $\lambda_j^+ := \max\{\lambda : N_{j,\lambda} \geq 1\}$,

$$\sum_{\substack{\lambda \leq r^{-100n}\lambda_j^+ \\ \theta \in \Theta_{j,\lambda}}} \|f_{j,\theta}\|_{L^q_{\text{Avg}}(B_{r^2})} \ll \sum_{\theta \in \Theta_{j,\lambda_j^+}} \|f_{j,\theta}\|_{L^q_{\text{Avg}}(B_{r^2})}.$$

Consequently, only those λ's such that $r^{-100n}\lambda_j^+ \leq \lambda \leq \lambda_j^+$, i.e., those λ's which are comparable to $\log r$, make the major contribution to the sum

$$\sum_\theta \|f_{j,\theta}\|^q_{L^q_{\text{Avg}}(B_{r^2})} = \sum_\lambda \sum_{\theta \in \Theta_{j,\lambda}} \|f_{j,\theta}\|^q_{L^q_{\text{Avg}}(B_{r^2})}.$$

So the quantity inside the averaging sign one left-hand-side of MK2 is controlled as follows: by the reverse Hölder inequality (for exponents which are ≤ 1),

$$\ldots \leq (\log r)^{O(1)} \max_{\lambda_1,\cdots,\lambda_n} \left\{ \operatorname*{Avg}_{B_r \subset B_{r^2}} \prod_{j=1}^{n} \left(\sum_{\theta \in \Theta_{j,\lambda_j}} \|f_{j,\theta}\|^q_{L^q_{\text{Avg}}(B_r)} \right)^{\frac{1}{2np}} \right\}$$

$$\leq (\log r)^{O(1)} r^{\varepsilon} \max_{\lambda_1,\cdots,\lambda_n} \prod_{j=1}^{n} \left(\sum_{\theta \in \Theta_{j,\lambda_j}} \|f_{j,\theta}\|^q_{L^q_{\text{Avg}}(B_r)} \right)^{\frac{1}{2np}}.$$

And this is comparable to the desired right-hand-side of MK2.

3. How to Summarize this Section...?

Theorem 3.1. *For $p = \frac{2(n+1)}{n-1}$, given any $\varepsilon > 0$, there is an $R_0 = R_0(\varepsilon)$ such that if $R \geq R_0$, then*

$$D_{p,n}(R) \leq R^{\varepsilon}.$$

We shall use a recurrence scheme to prove this theorem. For any natural number s, let $\delta = 2^{-s}$. An iteration of the broad-narrow argument gives

$$D_{p,n}(R) \lesssim K^{O(1)} D_p(R^{1-2\delta})^{1/2} \cdots D_p(R^{1/2})^{1/2^{s-1}} R^{O(\delta)} + D_{p,n-1}(K^2) D_{p,n}(R/K^2),$$

where the first term corresponds to the broad part, and the second the narrow part.

In order to gain some intuition on deriving theorem 3.1 from this recurrence relation, let us focus on simpler recurrences first. Let G be a non-negative non-decreasing function such that for all $R > 0$,

$$G(R) \leq C G(R^{1/2})^{3/2}$$

for some constant $C > 0$. Put $r = \log R$ and $g(r) = \log G(e^r)$. Then one derives, for $c = \log C$,

$$\begin{aligned} g(r) &\leq \frac{3}{2} g\left(\frac{r}{2}\right) + c \\ &\leq \left(\frac{3}{2}\right)^2 g\left(\frac{r}{4}\right) + 2c \leq \cdots \\ &\leq \left(\frac{3}{2}\right)^{\log_2 r} g(r') + c \log_2 r \\ &\leq c_1 \left(\frac{3}{2}\right)^{\log_2 r} + c \log_2 r, \end{aligned}$$

where the inequality is iterated for so many times that $1 \leq r' \leq 2$. Consequently,

$$g(r) \lesssim \left(\frac{3}{2}\right)^{\log_2 r},$$

or

$$G(R) \lesssim \exp\left(C(\log R)^{\frac{\log 3 - \log 2}{\log 2}}\right) \lesssim R^{\varepsilon}.$$

A more complicated example for simple iteration:

Proposition 3.2. *Similar general assumption on G as above. If it is such that*

$$G(R) \leq CR^{1/100} G(R^{1/2})^{3/2},$$

then

$$G(R) \lesssim R^{1/25+\varepsilon}.$$

Note that if one assumes $G(R) \sim R^p$, then this is possible if and only if $p \leq 1/25$.

Proof of proposition 3.2. Still take logarithm and iterate the inequality. Put $r = \log R$, $g(r) = \log G(e^r)$. Then iterating,

$$\begin{aligned} g(r) &\leq \frac{r}{100} + \frac{3}{2} g\left(\frac{r}{2}\right) + c \\ &\leq \frac{r}{100} + \frac{3}{4}\frac{r}{100} + \left(\frac{3}{2}\right)^2 g\left(\frac{r}{4}\right) + 2c \leq \cdots \\ &\leq \frac{r}{100}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right) + \left(\frac{3}{2}\right)^{\log_2 r} g(r') + c\log_2 r \\ &\leq \frac{r}{25} + \left(\frac{3}{2}\right)^{\log_2 r} g(r') + c\log_2 r, \end{aligned}$$

where $1 \leq r' \leq 2$. This is the desired result. □

Now we turn to multi-recurrences. The relation is

$$G(R) \leq CR^{\gamma} \prod_i G(R^{\alpha_i})^{\beta_i}.$$

Proposition 3.3. *Under this assumption,*

$$G(R) \lesssim_{\varepsilon} R^{\frac{\gamma}{1-\sum_i \alpha_i\beta_i}+\varepsilon}.$$

Note that if one assumes $G(R) \sim R^p$, then this is possible if and only if

$$p \leq \frac{\gamma}{1 - \sum_i \alpha_i \beta_i}.$$

Proposition 3.3 is proved in a similar way as proposition 3.2.

For the concrete case of decoupling constant $D_{p,n}(R)$, if the broad part dominates, then

$$D_{p,n}(R) \lesssim K^{O(1)} D_p(R^{1-2\delta})^{1/2} \cdots D_p(R^{1/2})^{1/2^{s-1}} R^{O(\delta)}.$$

Here, $\gamma = C\delta$, $\alpha_i = 1 - 2^i\delta$, $\beta_i = 1/2^i$, so

$$R^{\frac{\gamma}{1-\sum_i \alpha_i\beta_i}+\varepsilon} = R^{c/s+\varepsilon}.$$

By proposition 3.3, this quantity controls the decoupling constant $D_{p,n}(R)$.

For fixed ε, the proof is finished if one takes $s > 1/\varepsilon$. Hence, the number R_0 can be taken as

$$R_0 \geq 10^{2^s} \geq 10^{2^{\varepsilon^{-1}}}.$$