18.118 DECOUPLING
LECTURE 12

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1. Multi-scale language

Last time we examined the non sharp decoupling theorem proven by Bourgain. In the following two lectures, we aim to prove the full decoupling theorem.

Theorem 1.1 (Bourgain-Demeter). *$D_{p,n}(R) \lesssim R^{\epsilon}$ for $2 \leq p \leq \frac{2(n+1)}{n-1}$.*

We begin by a diagonosis of the proof of non sharp decoupling theorem (for p up to $\frac{2n}{n-1}$).

- In the broad case, we used Multilinear Restriction.
- The multilinear Restriction theorem uses transversality at scale about 1.
- We need to use transversality at many angular scales even for broad case.
- We can do it by decoupling smaller caps $D_p(\tau = \sqcup \theta)$ which involves transversality at any scale below τ.

We define a notation that captures all the information that we are trying to pass from one scale to another.

$$M_{p.q}(r,\sigma) := \underset{B_r \subseteq B_R}{Avg} \Pi_{j=1}^{n} \Big(\sum_{\theta \subseteq \Omega_j : \sigma^{-1}-cap} \|f_{j,\theta}\|^2_{L^q_{avg}(B_r)} \Big)^{\frac{1}{2}\cdot\frac{1}{n}\cdot p} \tag{1}$$

Let's digest the notation. In Fourier space, Ω_j's are transversal pieces of paraboloids. We consider σ^{-1}–caps in each Ω_j. In physical space, for example when $\sigma = r^{1/2}$ (which is an important case for us), we divide B_R into finitely overlapping union of balls B_r, inside each B_r $f_{j,\theta}$ is roughly constant on $r^{1/2} \times r$–tubes pointing on the normal direction of θ.

The following two examples of $M_{p,q}(r,\sigma)$ are important special cases.

$$M_{p,q}(1,1) = \underset{B_1 \subseteq B_R}{Avg} \Pi_{j=1}^{n} \|f_j\|^{\frac{1}{n}\cdot p}_{L^q_{avg}(B_1)} \sim \fint_{B_R} \Pi_{j=1}^{n} |f_j|^{\frac{p}{n}} \tag{2}$$

By locally constant property, $|f_j|$ is about constant on B_1.

$$\Pi_{j=1}^n \|f_j\|_{L^q_{avg}(B_1)}^{\frac{1}{n}\cdot p} \sim \Pi_{j=1}|f_j|^{\frac{1}{n}\cdot p} \sim \fint_{B_1} \Pi_{j=1}|f_j|^{\frac{1}{n}\cdot p}.$$

$M_{p,q}(1,1)$ is the left-hand side of multilinear decoupling inequality.

$$M_{p,p}(R, R^{1/2}) = \Pi_{j=1}^n \Big(\sum_{\theta: R^{-1/2}-cap} \|f_{j,\theta}\|^2_{L^q_{avg}(B_R)} \Big)^{\frac{1}{2}\cdot\frac{1}{n}\cdot p} \tag{3}$$

We can drop the average since we only have one ball B_R, inside the product we sum over caps of radius $R^{-1/2}$, which is the right-hand side of multilinear decoupling inequality.

We would like to go from $M_{p,q}(1,1)$ to $M_{p,p}(R, R^{1/2})$ by increasing the r and σ.

2. Tools

We write the main tools in $M_{p,q}(r,\sigma)$ language and see how the tools enable us to grow r and σ.

- **O** = Orthogonality: If $\sigma \le r$, then $M_{p,2}(r,\sigma) \lesssim M_{p,2}(r,r)$.
- **MK** = Multilinear Kakeya: If $r \le R^{1/2}$, then $M_{p,2}(r,r) \lesssim r^{\epsilon} M_{p,2}(r^2, r)$ for $p = \frac{2n}{n-1}$.
- **H** = Hölder's inequality: If $q_1 \le q_2$, then $M_{p,q_1}(r,\sigma) \le M_{p,q_2}(r,\sigma)$.
- **H2** = Hölder's inequality 2: If $\|f_{j,\theta}\|_{L^q_{avg}(B_r)} \le \|f_{j,\theta}\|^{\alpha_1}_{L^{q_1}_{avg}(B_r)} \|f_{j,\theta}\|^{\alpha_2}_{L^{q_2}_{avg}(B_r)}$, then $M_{p,q}(r,\sigma) \le M_{p,q_1}(r,\sigma)^{\alpha_1} M_{p,q_2}(r,\sigma)^{\alpha_2}$.

Lemma 2.1.

$$M_{p,p}(r,\sigma) \le D(\frac{R}{\sigma^2})^p M_{p,p}(R, R^{1/2}).$$

Lemma 2.1 is a combination of parallel decoupling and parabolic rescaling which enables us to do induction on scale.

Proof of **O**. If τ is σ^{-1}–cap, and τ can be decomposed as disjoint union of r^{-1}–cap: $\tau = \sqcup \theta$, then

$$\|f_{j,\tau}\|^2_{L^2_{avg}(B_r)} \lesssim \sum_{\theta \subseteq \tau} \|f_{j,\theta}\|^2_{L^2_{avg}(B_r)}.$$

We plug the above orthogonality inequality into the definition of $M_{p,2}(r,\sigma)$. □

Proof of **MK**. Recall that if $g_j = \sum_a W_{j,a} T_{j,a}$, with $W_{j,a} \ge 0$ and $T_{j,a}$ being the characteristic function of cylinder of radius 1 pointing on a direction near e_j, then

$$\fint_{Q_S} \Pi_{j=1}^n g_j^{\frac{1}{n-1}} \lesssim S^\epsilon \Pi_{j=1}^n (\fint_{Q_S} g_j)^{\frac{1}{n-1}}.$$

We decompose the average over $B_r \subseteq B_R$ into two steps: $\underset{B_{r^2}\subseteq B_R}{Avg}\ \underset{B_r\subseteq B_{r^2}}{Avg}$. B_{r^2} is the scale where multilinear kakeya is going to happen. We take $g_j(x) = \sum_\theta \|f_{j,\theta}\|^2_{L^2_{avg}(B_r)}$ for $x \in B_r$. By locally constant property, $|f_{j,\theta}|$ is roughly constant on a $r \times r^2$–tube, so $g_j(x)$ is a sum of $r \times r^2$-tubes pointing on direction near the normal direction of Σ_j with positive weights (value of $|f_{j,\theta}|$).

$$\begin{aligned}
M_{p,2}(r,r) &= \underset{B_{r^2}\subseteq B_R}{Avg}\Big[\underset{B_r\subseteq B_{r^2}}{Avg}\ \Pi_{j=1}^n \big(\sum_\theta \|f_{j,\theta}\|^2_{L^2_{avg}(B_r)}\big)^{\frac{1}{2}\cdot\frac{1}{n}\cdot p}\Big] \\
&\lesssim \underset{B_{r^2}\subseteq B_R}{Avg}\ r^\epsilon \Pi_{j=1}^n \big(\fint_{B_{r^2}} g_j\big)^{\frac{1}{n-1}} \\
&\lesssim r^\epsilon \underset{B_{r^2}\subseteq B_R}{Avg}\ \Pi_{j=1}^n \big(\sum_\theta \|f_{j,\theta}\|^2_{L^2_{avg}(B_{r^2})}\big)^{\frac{1}{2}\cdot\frac{1}{n}\cdot p} \\
&= r^\epsilon M_{p,2}(r^2, r)
\end{aligned}$$

Here $p = \frac{2n}{n-1}$, we have $\frac{1}{2}\cdot\frac{1}{n}\cdot p = \frac{1}{n-1}$, which satisfies the condition of multilinear kakeya exponent. $\square$

3. Iteration

Let's rewrite the weak version of this multilinear decoupling inequality in $M_{p,q}(r,\sigma)$ language, then we think about how to improve it.

Proposition 3.1 (old). *$MD_{\frac{2n}{n-1}}(R) \lesssim R^\epsilon$.*

Proof. We can write the proposition as:

$$M_{\frac{2n}{n-1},q}(1,1) \lesssim R^\epsilon M_{\frac{2n}{n-1},\frac{2n}{n-1}}(R, R^{1/2}).$$

We start by trivial estimate $M_{p,q}(1,1) \lesssim r^{O(1)} M_{\frac{2n}{n-1},2}(r,1)$. There is a loss $r^{O(1)}$ which appears only once. The loss of $r^{O(1)}$ is OK since we choose r to be a constant of size bounded by R^δ for some small $\delta \ll \epsilon$.

$$\begin{aligned} M_{p,q}(1,1) &\lesssim r^{O(1)} M_{\frac{2n}{n-1},2}(r,1) \\ \mathbf{O} &\lesssim R^{O(\delta)} M_{\frac{2n}{n-1},2}(r,r) \\ \mathbf{MK} &\lesssim R^{O(\delta)} M_{\frac{2n}{n-1},2}(r,r) \\ \mathbf{O} &\lesssim \cdots \\ \mathbf{MK} &\lesssim R^{O(\delta)} M_{\frac{2n}{n-1},2}(R,R^{1/2}) \\ \mathbf{H} &\lesssim R^{O(\delta)} M_{\frac{2n}{n-1},\frac{2n}{n-1}}(R,R^{1/2}) \end{aligned}$$

□

Let us think about what goes wrong if we try to make p larger in order to prove the full range of decoupling theorem. One of our important tool multilinear kakeya is not as good. When $p > \frac{2n}{n-1}$, we can prove an estimate of the form

$$M_{p,2}(r,r) \lesssim r^\alpha M_{p,2}(r^2,r)$$

for some constant $\alpha = \alpha(n) > 0$. But losing powers of r at every stage, we will not be able to get the desired estimate. Instead we use the following variation of multilinear Kakeya:

Lemma 3.2. [**MK2**] *If $q = \frac{n-1}{n}p \geq 2$, $r \leq R^{1/2}$, then*

$$M_{p,q}(r,r) \lesssim r^\epsilon M_{p,q}(r^2,r).$$

In this case, we don't lose any power of r, but we need an exponent $q \geq 2$ on the right-hand side instead of 2. Increasing this q will also cause us an issue in our proof, but we will be able to deal with that issue by bringing into play decoupling at smaller angular scales. The proof of Lemma 3.2 is similar to the proof of inequality (MK) above, but a little trickier. We'll talk about it next class.

We set $p = \frac{2(n+1)}{n-1}$ and try to imitate the proof of Proposition 3.1.

$$\begin{aligned} M_{p,2}(1,1) &\lesssim R^{O(\delta)} M_{p,2}(r,1) \\ \mathbf{O} &\lesssim R^{O(\delta)} M_{p,2}(r,r) \\ \mathbf{H} &\lesssim R^{O(\delta)} M_{p,\frac{(n-1)p}{n}}(r,r) \\ \mathbf{MK2} &\lesssim R^{O(\delta)} M_{p,\frac{(n-1)p}{n}}(r^2,r) \\ \mathbf{H_2} &\lesssim R^{O(\delta)} M_{p,2}(r^2,r)^{1/2} M_{p,p}(r^2,r)^{1/2} \end{aligned}$$

To analyze the factor $M_{p,2}(r^2,r)$, we can repeat the procedure we just ran. To analyze $M_{p,p}(r^2,r)$ we will use induction on angular scale. By

Hölder's inequality,

$$M_{p,p}(r,\sigma) = \underset{B_r\subseteq B_R}{Avg}\,\Pi\Big(\sum_{\tau:\sigma^{-1}-cap}\|f_j,\tau\|^2_{L^p_{avg}(B_r)}\Big)^{\frac{1}{2}\cdot\frac{1}{n}\cdot p}$$

$$\mathbf{H} \le \Pi\Big(\underset{B_r\subseteq B_R}{Avg}\Big(\sum_{\tau}\|f_{j,\tau}\|^2_{L^p_{avg}(B_r)}\Big)^{\frac{1}{2}\cdot p}\Big)^{\frac{1}{n}}$$

$$\mathbf{Minkowski} \le \Pi\Big(\sum_{\tau}\|f_{j,\tau}\|^2_{L^p_{avg}(B_R)}\Big)^{\frac{1}{2}\cdot\frac{1}{n}}$$

$$\le D_p(\tau=\sqcup\theta)^p M_{p,p}(R,R^{1/2}) = D_p(\frac{R}{\sigma^2})M_{p,p}(R,R^{1/2})$$

The last step matches our diagnosis: decoupling smaller caps τ into caps θ of radius $R^{-1/2}$, which involves transversality at the scale of τ.

4. Summary

The key step we have made is

$$M_{p,2}(r,r^{1/2}) \lesssim r^{\epsilon}M_{p,2}(r^2,r)^{1/2}D_p(\frac{R}{r^2})^{\frac{p}{2}}M_{p,p}(R,R^{1/2})^{1/2} \tag{4}$$

for any $r\le R^{1/2}$. We iterate 4 with $r=R^{\delta}$ and $\delta=2^{-s}$.

$$\begin{aligned} M_{p,2}(1,1) &\lesssim R^{O(\delta)}M_{p,2}(r,r^{1/2}) \\ &\lesssim R^{O(\delta)}D_p(R^{1-2\delta})^{\frac{1}{2}\cdot p}M_{p,2}(r^2,r)^{1/2}M_{p,p}(R,R^{1/2})^{1/2} \\ &\lesssim R^{O(\delta)}D_p(R^{1-2\delta})^{\frac{1}{2}\cdot p}D_p(R^{1-4\delta})^{\frac{1}{4}\cdot p}M_{p,2}(r^4,r^2)^{\frac{1}{4}}M_{p,p}(R,R^{1/2})^{\frac{3}{4}} \lesssim \cdots \end{aligned}$$

We have the following two intertwining estimates for decoupling constant $D_{p,n}(R)$ and multilinear decoupling constant $MD_{p,n}(R)$ for paraboloid.

(5)

$$MD_{p,n}(R) \lesssim R^{O(\delta)}D_{p,n}(R^{1-2\delta})^{1/2}D_{p,n}(R^{1-4\delta})^{1/4}\cdots D_{p,n}(R^{1/2})^{2^{-(s-1)}}$$

$$D_{p,n}(R) \lesssim K^{O(1)}MD_{p,n}(R) + D_{p,n-1}(K^2)D_{p,n}(\frac{R}{K^2}) \tag{6}$$

We plug in estimate 5 to get iterative estimate for $D_{p,n}$ which we discuss in detail next class.

Here is a warm-up excercise: if $F(R)\le F(R^{1/2})^{3/2}$, how fast does F grow in terms of R?

Hint: consider in log.