18.118 DECOUPLING
LECTURE 10

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1. Proof Digestion

Let P be a truncated paraboloid in $\mathbb{R}^n$ and $\Omega = N_{1/R}P = \sqcup\theta$ where θ's are $R^{-1/2}$-cups. Last time we proved the following theorem.

Theorem 1.1 (Bourgain, 2011)**.** *If* $\operatorname{supp}\widehat{f} \subset \Omega$, *then for* $2 \leq p \leq \frac{2n}{n-1}$, *we have*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim R^{\epsilon}\Big(\sum_{\theta} \|f_\theta\|^2_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{2}},$$

i.e., $D_{p,n}(R) \lesssim R^{\epsilon}$.

Proof ingredients:

1. orthogonality;
2. locally constant property;
3. induction on scales and dimension
   (1) for the narrow part we used $(n-1)$-dimension theory on each $B = B_{K^2}$ and then used induction to analyse each coarse cap τ;
   (2) in Multilinear Kakeya problem, we reduced from $\frac{1}{100n}$-close to axis to δ-close to axis, and then used Loomis-Whitney's theorem at many scales, namely δ^{-1}, then δ^{-2},...;
   (3) In Multilinear restriction problem, we inducted on scales, relating Multilinear restriction at scale R to Multilinear restriction at scale $R^{1/2}$ via Multilinear Kakeya.
4. multilinearity.

For 3(2), apply Loomis-Whitney's theorem on each δ^{-1}-cube, the estimate on each such cube depends on the number of j-tubes that enter the cube, i.e. the number of δ^{-1}-radius tubes in direction j that cover the cube. This is exactly the original problem with δ^{-1}-thick tubes. (See figure 1.)

Trying to remember all the ways that multiscale analysis is used is one way to review and digest the proof. I think it's striking that induction on scales is used in three different tricky ways during the proof. When we do the stronger decoupling theorem of Bourgain and Demeter, we will see a fourth tricky way to use induction on scales. I think it would be interesting to find a more systematic way of taking advantage of multiple scales, instead of having four (or five or six...) separate tricks.

Proposition 1.2. *Theorem* 1.1 *is true if we replace* P *by* S^{n-1}.

Proof. The proof is exactly the same except for rescaling a cap. For a K^{-1}-cap τ on P, we took a linear change of variable L such that the image $L(\tau)$ is a paraboloid. For a $1/K$-cup of S^{n-1}, we apply L where L stretches the tangent direction by K and the normal direction by K^2. The image $L(\tau)$ is almost a paraboloid. (See figure 2.) □

In fact, same is true for a class of surfaces.

Theorem 1.3 (Bourgain)**.** *Let* S *be a surface with* C^3*-norm* ≤ 1 *such that the principal curvatures of* S *are in* $[1/10, 10]$ *(the number 10 is nothing special). The same estimate in Theorem* 1.1 *holds for* S.

Remark 1.4. *The estimate is false for planes or* $S^1 \times [0,1] \subset \mathbb{R}^3$.

Question 1.5. *What about surfaces having both positive and negative principal curvatures, for example, the graph* $\omega_3 = \omega_1^2 - \omega_2^2$, $|\omega| \lesssim 1$*?*

2. Application of Decoupling Theorem

We study the eigenfunction of the Laplacian operator. Write $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. $\{e^{2\pi i\omega\cdot x}\}_{\omega\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, which is also a basis of eigenfunctions of Δ since

$$\Delta e^{2\pi i\omega\cdot x} = -4\pi^2|\omega|^2 e^{2\pi i\omega\cdot x}.$$

If $\Delta f = -\lambda^2 f$, then we can write

$$f(x) = \sum_{\substack{\omega\in\mathbb{Z}^n \\ 4\pi^2|\omega|^2=\lambda^2}} a_\omega e^{2\pi i\omega\cdot x} \qquad \text{for some } a_\omega. \tag{1}$$

Question 2.1. *What can we say about*

$$\max\{\|f\|_{L^p(\mathbb{T}^n)} : \Delta f = -\lambda^2 f, \|f\|_{L^2(\mathbb{T}^n)} = 1\}?$$

We start with $p = \infty$ which is easier. Let $N(\lambda)$ denotes the dimension of the λ-eigenspace.

Proposition 2.2. *We have*

$$\max\{\|f\|_{L^\infty(\mathbb{T}^n)} : \Delta f = -\lambda^2 f, \|f\|_{L^2(\mathbb{T}^n)} = 1\} = N(\lambda)^{\frac{1}{2}}.$$

Proof. Take $a_\omega = N(\lambda)^{-1/2}$ for every ω in (1). Then $\|f\|_{L^2(\mathbb{T}^n)} = 1$ and

$$f(0) = \sum_\omega a_\omega = N(\lambda) \cdot \frac{1}{N(\lambda)^{\frac{1}{2}}} = N(\lambda)^{\frac{1}{2}}.$$

Also note that for all g in the λ-eigenspace with $\|g\|_{L^2(\mathbb{T}^n)} = 1$, we have

$$\begin{aligned} |g(x)| &\leq \sum_\omega |a_\omega| \cdot 1 \\ &\leq (\sum_\omega |a_\omega|^2)^{\frac{1}{2}} N(\lambda)^{\frac{1}{2}} \qquad \text{by Cauchy-Schwarz inequality} \\ &= \|g\|_{L^2(\mathbb{T}^n)} N(\lambda)^{\frac{1}{2}} = N(\lambda)^{\frac{1}{2}}. \end{aligned}$$

Hence the maximum is attained by f and the value is $N(\lambda)^{1/2}$. $\square$

In general, if

$$f(x) = \sum_{|\omega| = \frac{|\lambda|}{2\pi}, \omega \in \mathbb{Z}^n} a_\omega e^{2\pi i \omega \cdot x},$$

then

$$\operatorname{supp} \widehat{f} \subset S^{n-1}(\frac{\lambda}{2\pi}) \subset B^n(\lambda).$$

Moreover, $|f| \approx$ constant on $B^n(\frac{1}{\lambda})$ with value $\approx N(\lambda)^{1/2}$. Hence

$$\|f\|_{L^p(\mathbb{T}^n)} \gtrsim N(\lambda)^{\frac{1}{2}} \lambda^{-\frac{n}{p}}.$$

We may conjecture that this is the worst case (but the following conjecture is not in the literature).

Conjecture 2.3 (Naive)**.**

$$\begin{aligned} &\max\{\|f\|_{L^p(\mathbb{T}^n)} : \Delta f = -\lambda^2 f, \|f\|_{L^2(\mathbb{T}^n)} = 1\} \\ &\qquad \leq C(n) \max\{N(\lambda)^{\frac{1}{2}} \lambda^{-\frac{n}{p}}, 1\}. \end{aligned}$$

Perhaps we need to replace $C(n)$ by $C(n, p, \epsilon)\lambda^\epsilon$. $N(\lambda)^{\frac{1}{2}}\lambda^{-\frac{n}{p}}$ corresponds to functions very concentrated and 1 corresponds to functions evenly spread.

Then we want to know how big $N(\lambda)$ is. Observe that $N(\lambda)$ is the number of integer points on the sphere $S^{n-1}(\frac{\lambda}{2\pi})$. In number theory language, for $A \in \mathbb{N}$, let $r_n(A)$ be the number of integer solutions to $a_1^2 + \cdots + a_n^2 = A$, i.e. the number of lattice points on $S^{n-1}(\sqrt{A})$. So $N(\lambda) = r_n(\frac{\lambda^2}{4\pi^2})$. Recall that we sketched $r_2(A) \lesssim A^\epsilon$ using unique factorization. We have the following few results from number theory.

Lemma 2.4.

$$\text{Average}_{A\le B\le 2A} r_n(B) \sim_n A^{\frac{n-2}{2}}.$$

Proof.

$$\frac{1}{A}\sum_{A\le B\le 2A} r_n(B) = \frac{1}{A}\cdot \#\,\text{lattice points in Annulus}(A^{\frac{1}{2}},(2A)^{\frac{1}{2}})$$
$$\sim \frac{1}{A}\cdot A^{\frac{n}{2}} = A^{\frac{n-2}{2}}$$

since the volume of Annulus$(A^{\frac{1}{2}},(2A)^{\frac{1}{2}}) \sim A^{\frac{n}{2}}$. □

Theorem 2.5 (Hardy-Littlewood)**.** *If $n\ge 5$, then $r_n(A)\sim_n A^{\frac{n-2}{2}}$ for all $A\in\mathbb{N}$.*

Corollary 2.6. *If $n\ge 5$, then $N(\lambda)\sim\lambda^{n-2}$.*

Remark 2.7. *There exist exact formulas for $r_n(A)$ in terms of factors of A when $n=2$ and 4. For example, a theorem of Jacobi says that if A is odd, then $r_4(A)=8\sum_{d|A} d$. If A is prime, then $r_4(A)\sim A$, but it can be shown that $r_4(A)\ge A(\log A)^{1/2}$ for infinitely many A.*

We may combine our naive conjecture with Corollary 2.6.

Conjecture 2.8 (Naive)**.**

$$\max\{\|f\|_{L^p(\mathbb{T}^n)} : \Delta f = -\lambda^2 f, \|f\|_{L^2(\mathbb{T}^n)}=1\}$$
$$\le C(n)\max\{\lambda^{\frac{n-2}{2}-\frac{n}{p}},1\}.$$

Note that if $p<\frac{2n}{n-2}$, then $\lambda^{\frac{n-2}{2}-\frac{n}{p}}\le 1$. There is indeed a similar conjecture in the literature.

Conjecture 2.9. *If $2\le p<\frac{2n}{n-2}$, then*

$$\max\{\|f\|_{L^p(\mathbb{T}^n)} : \Delta f = -\lambda^2 f, \|f\|_{L^2(\mathbb{T}^n)}=1\}\le C(n,p).$$

We have a classical result.

Theorem 2.10 (Zygmund-Cook)**.** *If $n=2$, then $\|f\|_{L^p(\mathbb{T}^n)}\le C(p)\|f\|_{L^2(\mathbb{T}^n)}$ for $2\le p\le 4$.*

The following result is obtained by unique factorization trick.

Theorem 2.11 (Bourgain-Rudnick-Sarnak)**.** *If $n=3$, then $\|f\|_{L^p(\mathbb{T}^n)}\lesssim \lambda^\epsilon\|f\|_{L^2(\mathbb{T}^n)}$ for $2\le p\le 4$.*

But Decoupling theorem tells us more.

Corollary 2.12 (of decoupling)**.** *For every n, $\|f\|_{L^p(\mathbb{T}^n)}\le\lambda^\epsilon\|f\|_{L^2(\mathbb{T}^n)}$ for $2\le p\le\frac{2n}{n-1}$.*

Proof. Decompose $S^{n-1}(\frac{\lambda}{2\pi}) = \sqcup\theta$ where θ are 1-caps, i.e. $1/\lambda$-angular caps (see figure 3). Warning: lattice points are NOT a 1-net since there are some gaps. Nevertheless, we can rescale Decoupling theorem (Proposition 1.2) for $S^{n-1}(\frac{\lambda}{2\pi})$, which becomes

$$\|g\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^\epsilon \Big(\sum_\theta \|g_\theta\|^2_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{2}} \qquad \text{for } 2 \le p \le \frac{2n}{n-1}.$$

We want to apply (2) to λ-eigenfunction

$$f(x) = \sum_{|\omega| = \frac{|\lambda|}{2\pi}, \omega \in \mathbb{Z}^n} a_\omega e^{2\pi i \omega\cdot x} = \sum_\theta f_\theta(x),$$

where

$$f_\theta(x) = \begin{cases} a_\omega e^{2\pi i\omega\cdot x} & \text{if } \omega\in\theta \\ 0 & \text{if no such } \omega\in\theta \end{cases}.$$

Since $\|f_\theta\|_{L^p(\mathbb{R}^n)} = \infty$ in the first case, we need to use a cutoff function η_R such that $\operatorname{supp}\widehat{\eta_R} \subset B_{1/R}$ for some $R > \lambda$. Let $g = \eta_R f$, then $\widehat{g} = \widehat{\eta_R} * \widehat{f}$, so $g_\theta(x) = \eta_R(x) a_\omega e^{2\pi i\omega x}$ or 0. Since f is periodic,

$$\|g\|_{L^p(\mathbb{R}^n)} \sim R^{\frac{n}{p}} \|f\|_{L^p(\mathbb{T}^n)},$$
$$\|g_\theta\|_{L^p(\mathbb{R}^n)} \sim R^{\frac{n}{p}} |a_\omega|$$

(see figure 4). Hence,

$$\begin{aligned}\|f\|_{L^p(\mathbb{T}^n)} &= R^{-\frac{n}{p}} \|g\|_{L^p(\mathbb{R}^n)} \\ &\lesssim R^{-\frac{n}{p}} \lambda^\epsilon \Big(\sum_\omega |R^{\frac{n}{p}} a_\omega|^2\Big)\frac{1}{2} = \lambda^\epsilon \|f\|_{L^2(\mathbb{T}^n)}.\end{aligned}$$

□

In general, let $\Omega \subset \mathbb{Z}^n$, define

$$\Lambda_p(\Omega) := \max\Big\{\frac{\|f\|_{L^p(\mathbb{T}^n)}}{\|f\|_{L^2(\mathbb{T}^n)}} : f(x) = \sum_{\omega\in\Omega} a_\omega e^{2\pi i\omega\cdot x}\Big\}.$$

If we write $\Omega_\lambda := \{\omega \in \mathbb{Z}^n : |\omega| = \frac{|\lambda|}{2\pi}\}$, then Corollary 2.12 says that $\Lambda_p(\Omega_\lambda) \lesssim \lambda^\epsilon$ for $2 \le p \le \frac{2n}{n-1}$.

Question 2.13. *Do there exist sets $\Omega \subset \mathbb{Z}^n$ such that $|\Omega|$ is arbitrarily large and yet $\Lambda_3(\Omega)$ is uniformly bounded?*

The answer is Yes. We saw an example on the first problem set which is close to answering this question, although it doesn't quite work.

Example 2.14 (from Problem Set 1)**.** *$n = 1$, $\Omega = \{\text{squares}\} \subset \mathbb{Z}$. Write $S_N := \{1^2, 2^2, ..., N^2\}$. We were asked to show that $\Lambda_4(S_N) \lesssim N^\epsilon$. Note that since $\|f\|_{L^p(\mathbb{T}^n)}$ is an increasing function of p, $\Lambda_p(\Omega)$ is also an increasing function of p. Hence*

$$\Lambda_3(S_n) \lesssim \Lambda_4(S_N) \lesssim N^\epsilon.$$

The proof boiled down to checking that a number M can be written as a sum of two squares in $\lesssim M^\epsilon$ different ways. We can construct other sets Ω so that a number M can be written as a sum of two elements of Ω in $\lesssim 1$ different ways. For such a set $\Lambda_4(\Omega) \lesssim 1$. For instance, we have the following example:

Example 2.15. *Again $n = 1$, $\Omega = \{2^j\}_{j \in \mathbb{N}}$. Then $\Lambda_3(\Omega) \leq \Lambda_4(\Omega) \lesssim 1$, and yet Ω is infinite.*

Question 2.16 (old)**.** *Can we have $\Lambda_3(\Omega) \lesssim 1$ but $\Lambda_4(\Omega) = \infty$.*

The answer is Yes, but the problem is actually very difficult. It was open for many years before Bourgain resolved it in the late 1980's using randomly constructed Ω. As far as I know, it remains an open problem to prove such an estimate for an explicit set Ω. Corollaries 2.6 and 2.12 give something in this spirit for the explicit sets Ω_λ. For instance, we can see from Corollary 2.6 and 2.12 that when $n = 5$, $\Lambda_{2.5}(\Omega_\lambda) \lesssim \lambda^\epsilon$ and $\Lambda_4(\Omega_\lambda) \geq \lambda^{1/4}$.