18.118 Decoupling
Lecture 1

Instructor: Larry Guth
Trans. : Sarah Tammen

September 7, 2017

Decoupling theory is a branch of Fourier analysis that is recent in origin and that has many applications to problems in both PDE and analytic number theory. The decoupling theorem of Jean Bourgain and Ciprian Demeter has given rise to solutions for many problems that previously seemed intractable. Before giving an overview of decoupling, we will discuss some motivating problems which exemplify applications of decoupling.

1 Motivating Problems

1.1 The Schrödinger Equation on $\mathbb{R}^d$ vs. $\mathbb{T}^d$

We consider the initial value problem

$$\begin{cases} \partial_t u & = i\Delta u \\ u(x,0) & = u_0(x) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$ or $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. It is a well-known fact that the quantity

$$\int_{\mathbb{R}^d} |u(x,t)|^2 \, dx \tag{2}$$

$\left(\text{resp. } \int_{\mathbb{T}^d} |u(x,t)|^2 \, dx\right)$ is conserved as time varies. Equation (1) can be used to model the behavior of a quantum mechanical particle: if one rescales so that the integral (2) is 1 , then for a fixed time t and measurable set $A \subset \mathbb{R}^d$ (resp.$\mathbb{T}^d$),

$$\int_A |u(x,t)|^2 \, dx$$

gives the probability that the particle is in the set A at time t.

One problem of interest is to determine how much the particle's motion can focus. The norm

$$\|u\|_{L^2_x} = \int |u(x,t)|^2 \, dx$$

is conserved, but at any given time, we may have spikes - small sets of x for which $|u(x,t)|$ is large - or the graph of $|u(.,t)|$ over $\mathbb{R}^d$ may be more spread out. The Strichartz inequality - stated below for $\mathbb{R}^d$ - allows us to deduce that the set of $x \in \mathbb{R}^d$ for which there is a spike of a given height must be small in relation to the L^2 norm of the initial data u_0.

Theorem 1.1. *If u satisfies* (1) *and*

$$p = \frac{2(d+2)}{d},$$

then

$$\|u\|_{L^p(\mathbb{R}^d\times\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)},$$

where the implied constant depends on d.

One can easily deduce (using Chebyshev's inequality, e.g.) the following corollary.

Corollary 1.2. *For any $\lambda > 0$, if*

$$\|u_0\|_{L^2(\mathbb{R}^d)} = 1,$$

then

$$|\{|u(x,t)| > \lambda\}| \lesssim \lambda^{-p}.$$

Analogous results are much more difficult to prove for the Schrödinger equation on $\mathbb{T}^d$. Intuitively, the increased difficulty is due to the interaction of wave packets - pieces of a solution that are localized in frequency space. Wave packets have more opportunity to interact on the torus than they do in $\mathbb{R}^d$. Nevertheless, using decoupling, Bourgain and Demeter proved Strichartz estimates for the Schödinger equation in the case that the Fourier transform of the initial data is supported in a ball. These periodic Strichartz estimates are stated in Theorem 1.3.

One can write initial data on the torus using its Fourier series

$$u_0(x) = \sum_{n\in\mathbb{Z}^d} a_n e^{in\cdot x}.$$

We will look at truncations

$$u_0(x) = \sum_{\substack{n\in\mathbb{Z}^d \\ |n|\leq N}} a_n e^{in\cdot x} \tag{3}$$

One can deduce by taking the Fourier transform of (1) that if u solves (1) on $\mathbb{T}^d$ and has initial data

$$u_0(x) = \sum_{\substack{n\in\mathbb{Z}^d \\ |n|\leq N}} a_n e^{in\cdot x},$$

then

$$u(x) = \sum_{\substack{n\in\mathbb{Z}^d \\ |n|\leq N}} a_n e^{i(n\cdot x+|n|^2 t)}. \tag{4}$$

Bourgain and Demeter proved the following using decoupling.

Theorem 1.3. *[Bourgain-Demeter] If u_0 has frequency $\leq N$ (i.e. u_0 is in the form (3)), u satisfies (1), and*

$$p = \frac{2(d+2)}{d},$$

then

$$\|u\|_{L^p(\mathbb{T}^d\times[0,1])} \lesssim N^\varepsilon \|u_0\|_{L^2(\mathbb{T}^d)}, \tag{5}$$

i.e. for all $\varepsilon > 0$, there exists a constant $C_{\varepsilon,d}$ so that

$$\|u\|_{L^p(\mathbb{T}^d\times[0,1])} \leq C_{\varepsilon,d} N^\varepsilon \|u_0\|_{L^2(\mathbb{T}^d)}.$$

Remark 1.4. *One should note the ε-loss in the periodic Strichartz estimates. One may also notice that our integration in time in the L^p norm of u is restricted. It does not make sense to study norms $L^p(\mathbb{T}^d\times\mathbb{R})$, because the periodic Schrödinger equation does not have the dispersion relation that the Schrödinger equation on $\mathbb{R}^d$ does.*

In the 1990s, Bourgain conjectured Theorem 1.3 and proved it for $d = 1, 2$. We provide a sketch of the proof for $d = 2$.

1.1.1 Proof sketch for d = 2

If $d = 2$, then $p = 4$. We observe that

$$\|u\|^4_{L^4_{x,t}} = \|u^2\|^2_{L^2_{x,t}}.$$

(This is useful because one can compute L^2 norms using Plancherel/Parseval. One can use a similar trick when computing L^p norms for any even p.) Using (4), we write

$$(u(x,t))^2 = \sum_{|m|,|n|\leq N} a_m a_n e^{i[(m+n)\cdot x+(|m|^2+|n|^2)t]}. \tag{6}$$

If the the frequencies appearing on the RHS were all distinct (i.e. if there were no points $m, n, m', n' \in \mathbb{Z}^2$ so that $|m|, |n|, |m'|, |n'| \leq N$ and $(m+n, |m|^2+|n|^2) = (m'+n', |m'|^2+|n'|^2)$), then the sum above would be the Fourier series of u^2. As it is, there may be some terms in the series that are repeated. However, the following lemma provides an upper bound on the number of repeats and thereby gives a bound for $\|u^2\|^2_{L^2}$.

Lemma 1.5 (Number Theory Lemma 1)**.** *For any* $\ell \in \mathbb{Z}^3$*, we have that*

$$\#\left\{ \begin{array}{c} (m,n) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \\ |m|, |n| \le N \end{array} : (m+n, |m|^2 + |n|^2) = \ell \right\} \lesssim N^{\varepsilon}.$$

Using this lemma along with (6), we have that

$$\begin{aligned} \|u^2\|^2_{L^2_{x,t}} &\lesssim N^{\varepsilon} \sum_{m,n} |a_m|^2 |a_n|^2 \\ &= N^{\varepsilon} \left(\sum_n |a_n|^2 \right)^2 \\ &= N^{\varepsilon} \|u_0\|^4_{L^2}. \end{aligned}$$

This concludes our proof of (5) for $d = 2$, modulo a proof of Lemma 1.5. We did not prove Lemma 1.5 in lecture; however, we did prove a simpler lemma in the same spirit.

Lemma 1.6 (Number Theory Lemma 2)**.** *Let* $M \in \mathbb{N}$*. Then*

$$\#\{(a_1, a_2) \in \mathbb{N}^2 : a_1^2 + a_2^2 = M\} \lesssim M^{\varepsilon}.$$

The proof of Lemma 1.6 uses unique factorization in $\mathbb{Z}[i]$: Supposing that $M = a_1^2 + a_2^2$, we factor M in $\mathbb{Z}[i]$ as

$$M = (a_1 + ia_2)(a_1 - ia_2). \tag{7}$$

M has a unique factorization

$$M = p_1^{e_1} \dots p_n^{e_n} \tag{8}$$

as a product of primes in $\mathbb{Z}[i]$. Comparing (7) and (8), we see that we must have

$$a_1 + ia_2 = p_1^{e_1'} \dots p_n^{e_n'}$$

for some exponents $0 \le e_i' \le e_i$. The number of such factorizations is $\lesssim M^{\varepsilon}$.

1.2 Number Theory Problems

In addition to its utility in studying partial differential equations like the Schrödinger equation, decoupling also has applications to many problems in number theory, especially the study of diophantine equations - polynomial equations for which integer solutions are sought.

Estimates for the number of solutions to some diophantine equations can be obtained from the periodic Strichartz estimates (Theorem 1.3) after recognizing that certain integrals count the number of integer solutions of these equations. For example, if $d = 1$ and

$$u_0 = \sum_{1 \le a \le N} e^{iax},$$

then the solution u to (1) satisfies

$$u(x,t) = \sum_{1 \le a \le N} e^{i(ax+a^2 t)}.$$

By the periodic Strichartz estimates,

$$\int |u|^6 \lesssim N^{3+\varepsilon}. \tag{9}$$

Expanding the integral $\int |u|^6$, we have

$$\begin{aligned} \int |u|^6 &= \int u^3 \overline{u}^3 \\ &= \int \sum_{\substack{1 \le a_1, a_2, a_3 \le N \\ 1 \le b_1, b_2, b_3 \le N}} e^{i[(a_1+a_2+a_3-b_1-b_2-b_3)x + (a_1^2+a_2^2+a_3^2-b_1^2-b_2^2-b_3^2)t]}\, dx\, dt. \end{aligned}$$

We note that the contribution of $(a_1, a_2, a_3, b_1, b_2, b_3)$ to the integrand is 1 if

$$\begin{cases} a_1 + a_2 + a_3 &= b_1 + b_2 + b_3 \\ a_1^2 + a_2^2 + a_3^2 &= b_1^2 + b_2^2 + b_3^2 \end{cases} \tag{10}$$

and that the contribution is 0 otherwise. Thus, $\int |u|^6$ counts the number of solutions to the diophantine system (10) in integer sextuples $(a_1, a_2, a_3, b_1, b_2, b_3)$ with $a_i, b_i \le N$, By (9),

$$\#\{(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{N}^6 : a_i, b_i \le N \text{ and (10) holds } \} \lesssim N^{3+\varepsilon}. \tag{11}$$

One may wonder if this is a 'good' estimate (i.e. an estimate that is close to being sharp). To address this question, we first observe that (10) has N^3 diagonal solutions (solutions in which $a_i = b_i$). Moreover, using probabilistic intuition, we observe that the probability of solving

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3$$

is on the order of $1/N$, since both sides have order of magnitude N. Similarly, the probability of solving

$$a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2$$

is on the order of $1/N^2$. If we were to assume that these events were independent, we would obtain an estimate $\sim 1/N^3$ for the probability of solving the system. Since there are N^6 choices for $(a_1, a_2, a_3, b_1, b_2, b_3)$, this would give $\sim N^3$ solutions to the system in the specified range, which corroborates our bound from the periodic Strichartz estimates.

This bound was long known to number theorists using unique factorization in the Gaussian integers, but it was surprising to see a proof based on such different tools. Some further development of the decoupling theory has led to new estimates about the number of solutions of diophantine systems. Before we describe one of the new results, we consider some related questions for context. The unique factorization trick allows to understand problems about squares. Analogous problems about cubes seem to be very difficult. For instance, one may consider the number of ways to write a large integer as a sum of three cubes. The following conjecture seems reasonable but is wide open.

Conjecture 1.7. *We have that*

$$\#\{(a_1, a_2, a_3) \in \mathbb{N}^3 : a_1^3 + a_2^3 + a_3^3 = M\} \lesssim M^{\varepsilon}$$

in the sense of Theorem 1.3.

We next consider a weaker conjecture regarding a related diophantine equation whose solutions can be counted by the L^6 norm of an exponential sum.

Conjecture 1.8. *We have that*

$$\#\{1 \leq a_i, b_i \leq N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\} \lesssim N^{3+\varepsilon}.$$

We remark that Conjecture 1.7 implies Conjecture 1.8 because after making one of N^3 possible choices for $b_1, b_2, b_3 \leq N$, one can apply Conjecture 1.7 to conclude that there are $\lesssim N^{\varepsilon}$ ways to choose $a_1, a_2, a_3 \leq N$ so that

$$a_1^3 + a_2^3 + a_3^3 \;=\; b_1^3 + b_2^3 + b_3^3. \tag{12}$$

We note that the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{a=1}^{N} e^{ia^3 x} \right|^6 \, dx$$

counts the number of solutions to (12).

Although the identification of the number of solutions with an integral seems promising, Conjecture 1.8 also remains unsolved. However, decoupling *has* been used to settle a conjecture of Vinogradov concerning sequences $\{J_{s,k}(N)\}$ defined as follows: for fixed positive integers s and k, let

$$J_{s,k}(N) \;=\; \#\left\{ \begin{array}{ll} a_1^j + \cdots + a_s^j, & j = 1, \ldots k \\ 1 \leq a_i, b_i \leq N & \end{array} \right\}. \tag{13}$$

In this notation, (11) reads as

$$J_{3,2}(N) \lesssim N^{3+\varepsilon}.$$

Vinogradov conjectured the following concerning the sequences $J_{s,k}(N)$.

Conjecture 1.9. *If $J_{s,k}(N)$ is defined as in* (13)*, then*

$$J_{s,k}(N) \lesssim N^{\varepsilon}\left(N^s + N^{2s-k(k+1)/2}\right).$$

We remark that the first factor multiplied by N^{ε} represents diagonal solutions to Vinogradov's system, whereas the second is derived form probabilistic intuition as in our remarks following (11). The problem is most interesting in the case that $s = k(k+1)/2$. One member of our class commented that there is some connection between this value of s and the threshold at which a variety becomes rationally connected. [Scribe's note: I did not quite catch the details of this remark, but I would be interested to learn more about the connection.]

The case $k = 2$ can be solved (relatively) easily using unique factorization. Vinogradov proved the conjecture for $s \geq 10k^2 \log k$, but was unable to furnish a proof for small s. The conjecture was settled only very recently. Bourgain, Demeter, and Guth settled the conjecture in full in 2015 using a decoupling inequality. The results that made their proof possible concern estimating norms of a sum of functions at different frequencies, as discussed in the next section. At around the same time, Trevor Wooley proved the full conjecture using techniques from number theory.

2 Overview of Decoupling

Let Ω be a region in $\mathbb{R}^n$, viewed as Fourier space. We write Ω as the union

$$\Omega = \bigsqcup \theta$$

for some disjoint subregions θ. For a sufficiently regular function f with $\operatorname{supp} \hat{f} \subset \Omega$, we can decompose f as the sum

$$f = \sum_{\theta} f_{\theta}, \tag{14}$$

where

$$f_{\theta} = \int_{\theta} \hat{f}(\omega) e^{i\omega \cdot x}\, d\omega. \tag{15}$$

Having defined f_{θ} this way, the identity (14) follows from Fourier inversion. Given an exponent p, one might ask how $\|f\|_{L^p}$ relates to the norms $\|f_{\theta}\|_{L^p}$. Under appropriate hypotheses, decoupling theory asserts the existence of constants $D_p(\Omega = \sqcup\theta)$ so that

$$\|f\|_{L^p(\mathbb{R}^n)} \leq D_p\left(\Omega = \bigsqcup \theta\right)\left(\sum_{\theta} \|f_{\theta}\|^2_{L^p(\mathbb{R}^n)}\right)^{1/2}. \tag{16}$$

Specifically, given a disjoint union

$$\Omega = \bigsqcup \theta,$$

if there exists a constant C with

$$\|f\|_{L^p(\mathbb{R}^n)} \le C \left(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \right)^{1/2},$$

we define $D_p(\Omega = \bigsqcup \theta)$ to be the smallest such constant. Inequalities of the form (16) are useful for estimating exponential sums

$$\sum_j a_j e^{i\omega_j \cdot x}$$

with $\omega_j \in \theta_j$ for each j. In particular, we have the following result regarding exponential sums.

Proposition 2.1. *Let $R > 0$, let*

$$\Omega = \bigsqcup \theta_j,$$

and let

$$g = \sum_j a_j e^{i\omega_j \cdot x}.$$

If $B_{R^{-1}}(\omega_j) \subset \theta_j$ for all j, then for any ball B_R of radius R,

$$\|g\|_{L^p(B_R)} \lesssim D_p \left(\Omega = \bigsqcup \theta_j \right) \left(\sum |a_j|^2 \right)^{1/2} R^{1/p}.$$

Proof. Consider a fixed ball B_R. Let $f = \eta g$ for a function η so that

$$\operatorname{supp} \hat{\eta} \subset B_{R^{-1}}$$

and

$$|\eta| \sim 1$$

on B_R but decays rapidly outside B_R. We claim that

$$f_{\theta j} = \eta\, a_j e^{i\omega_j \cdot x}. \tag{17}$$

One can prove the claim by taking the Fourier transform of f to give

$$\hat{f} = \hat{\eta} * \hat{g}$$

then using (15) along with the fact that $\operatorname{supp} \hat{\eta} \subset B_{R^{-1}}$. After proving (17), it follows that

$$\begin{aligned} \|g\|_{L^p(B_R)} &\lesssim \|f\|_{L^p(\mathbb{R}^n)} \\ &\le D_p \left(\Omega = \bigsqcup \theta_j \right) \left(\sum_j \|\eta a_j e^{i\omega_j \cdot x}\|^2_{L^p(\mathbb{R}^n)} \right)^{1/2} \\ &\sim \left(\sum |a_j|^2 \right)^{1/2} D_p\, R^{1/p}. \end{aligned}$$

(Here, the factor of $R^{1/p}$ comes from η.) □

Remark 2.2. *This result is sharp if*

$$|g(x)| \sim \left(\sum |a_j|^2\right)^{1/2} \tag{18}$$

on most of B_R. *This can easily happen; for example, if* $a_j \mapsto \pm a_j$ *with i.i.d. signs then* (18) *occurs with high probability.*

One may ask, if we only want to prove L^p estimates for exponential sums, then why prove the decoupling inequality, which is more general. The answer has to do with induction. The proof of decoupling uses induction very heavily. When we prove something by induction, it often happens that it is easier to prove a more general theorem, because then we get to use a more general inductive hypothesis. The proof of decoupling is based on induction on scales. The decoupling inequality was first suggested by Tom Wolff in his work on the local smoothing conjecture for the wave equation – we will discuss this problem later in the course. Wolff realized that the decoupling inequality is very well designed to study using induction on scales. We will say much more about this induction on scales in lectures to come.