Course Description for Math 103: Fourier Analysis

Fourier analysis is a tool that takes a complicated function f and breaks it into a sum of simple functions. It takes a basically arbitrary function $f(x)$ and writes it as a linear combination of complex exponentials – functions of the form e^{inx}. Equivalently, we can write $f(x)$ as a linear combination of sine and cosine waves with different frequencies, functions of the form $\sin(nx)$ or $\cos(nx)$.

This decomposition is a useful tool in a remarkable variety of problems. The original application was partial differential equations that appear in physics, like the heat equation and the wave equation. A century later, Fourier analysis proved useful for studying the equations of quantum mechanics. It has applications in number theory, combinatorics, and geometry. It's also a fundamental tool in applied math. For example, it is used in efficient algorithms, and it can be helpful to process the information in medical imaging.

In this course, we will study the fundamental results of Fourier analysis. We will study applications of Fourier analysis to several areas of pure math, and a couple of small applications in applied math. We will also learn Lebesgue integration and use it to study subtle questions of convergence that come up in Fourier analysis and its applications.

This is a fairly detailed description of the course, designed to be read by a student thinking of taking the course.

1. The fundamental theorem of Fourier analysis

A 2π-periodic function is a function $f : \mathbb{R} \to \mathbb{C}$ so that $f(x + 2\pi) = f(x)$ for any x. Fundamental examples of periodic functions are sines, cosines, or complex exponetials. More precisely, for any integer n, the functions $\sin(nx)$, $\cos(nx)$, and e^{inx} are 2π-periodic. Recall that by Euler's formula, $e^{inx} = \cos(nx) + i \sin(nx)$.

The fundamental insight of Fourier is that complex exponentials are basic building blocks for periodic functions, and any periodic function can be written as a combination of them. We state this first as a vague theorem.

Vague Theorem 1. *If $f : \mathbb{R} \to \mathbb{C}$ is a "reasonable" 2π-periodic function, then there is a sequence of numbers $a_n \in \mathbb{C}$, for $n \in \mathbb{Z}$, so that*

$$\sum_{n=-\infty}^{\infty} a_n e^{inx} \text{ converges in some sense to } f(x).$$

Fourier was the first person to believe that this is true. He used this insight to study heat flow, as we will explain below. But he was not able to prove it or even to formulate it in a precise way. Here is one precise theorem about this question.

Recall that a function f is C^k if it can be differentiated k times and its k^{th} derivative is continuous.

Theorem 1. *If $f : \mathbb{R} \to \mathbb{C}$ is a C^2 2π-periodic function, then there is a sequence of numbers $a_n \in \mathbb{C}$ so that at each point x, the infinite series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ converges to $f(x)$.*

Moreover, there is a formula for finding the coefficients a_n. This formula is so fundamental that we motivate it in this course description. Suppose for a moment that f was a finite series $f(x) = \sum_{n=-N}^{N} a_n e^{inx}$. Suppose we knew the value of f at each point x, and we would like to recover the coefficient a_n. There is a nice way to recover the coefficients based on the following observation: if n is a non-zero integer, then

$$\int_{-\pi}^{\pi} e^{inx} dx = 0,$$

whereas, if $n = 0$, then

$$\int_{-\pi}^{\pi} e^{inx} dx = 2\pi.$$

Therefore, if $f(x) = \sum_{n=-N}^{N} a_n e^{inx}$, then

$$\int_{-\pi}^{\pi} f(x) e^{-imx} dx = \sum_n a_n \int_{-\pi}^{\pi} e^{i(n-m)x} dx = 2\pi a_m.$$

Based on this example, we define the n^{th} Fourier coefficient $\hat{f}(n)$ by

$$\hat{f}(n) := (1/2\pi) \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \tag{1}$$

We can now state a better version of Theorem 1:

Theorem 2. *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a C^2 2π-periodic function, and let $\hat{f}(n)$ be defined by Equation 1. Then at each point x, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$ converges to $f(x)$.*

This theorem tells us how an arbitrary (C^2) periodic function can be written as a combination of complex exponentials with different frequencies.

Fourier analysis uses this fundamental result to study a variety of problems. Wherever a problem involves a periodic function, we can try to study the problem by writing the function as a sum of complex exponentials. In the next sections, we will give examples to explain how this is useful.

So far we discussed Fourier analysis for a periodic function on $\mathbb{R}$. We quickly remark that there is also a version of the story for an arbitrary (non-periodic) function on $\mathbb{R}$, and a version for functions on higher-dimensional spaces $\mathbb{R}^d$. We will cover all these versions in the class.

2. Applications to partial differential equations

Fourier came to his idea studying heat flow. Suppose as a model problem that we consider heat on a long thin metal rod. We idealize the problem so that the rod is an infinitely thin line, and we let $u(x,t)$ denote the temperature at point x in the rod at time t. The heat equation is a partial differential equation that describes how the temperature in the rod evolves over time:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

In the class, we will give some motivation for this equation, but for now we focus on how Fourier analysis helps to understand it. One problem Fourier solved was the initial value problem: given the initial temperature distribution $u(x,0)$, find the temperature at some later time $u(x,t)$. In physical applications, the temperature $u(x,t)$ is of course a real number. But the equation also makes sense for complex numbers. Moreover, if $u(x,0) = e^{inx}$, the equation happens to have a simple solution: $e^{-n^2 t} e^{inx}$.

If the initial data $u(x,0)$ happens to be 2π-periodic, then we can write it as a series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$. We know how to solve the heat equation with initial data e^{inx}. Moreover, the sum of two solutions to the heat equation solves the heat equation. Based on this, Fourier guessed that the solution to the initial value problem should be

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-n^2 t} e^{inx}. \tag{3}$$

Starting from these ideas, Fourier was able to figure out many things about heat flow. I'm not sure if this is true, but I once heard that Fourier used this work to calculate the ideal depth for a wine cellar - the depth at which the temperature stays essentially constant all year long. Here is another example problem. We take a frozen chicken which we model as a ball of radius R and leave it on the counter to thaw. How does the time that it takes to thaw depend on the radius?

A formula like Equation 3 is interesting, but it takes substantial further work to get from this type of formula to be able to answer questions about heat flow like the ones in the paragraph above.

Fourier analysis is useful for studying many partial differential equations. In this class, we will focus on two equations: the heat equation and the wave equation. These two equations behave very differently from each other, but Fourier analysis is useful for both. In each case, we begin by finding a formula for the solution as a combination of complex exponentials. Then we will use these different formulas to study the different behaviors of the two equations.

There are a number of subtle issues about convergence that come up when we use Fourier analysis to study PDE. For example, from the formula for the heat equation, Equation 2, it's easy to check that the sum of two solutions to the heat equation is also a solution to the heat equation. But it's not clear whether the sum of an infinite number of solutions to the heat equation is also a solution to the heat equation – in particular, it's not clear that the formula in Equation 3 is a solution to the heat equation. We will study these issues carefully in the course.

3. Other applications

Fourier analysis has a remarkable number of applications in both pure math and applied math. Here is a sample of other applications that we will discuss.

- **Weyl's work on equidistribution.** If x is a real number, we let $[x] \in [0,1)$ denote the fractional part of x. Take an irrational number θ, and consider the sequence $[n\theta]$ where $n = 1, 2, \ldots$. Weyl proved that this sequence of numbers is evenly distributed in the unit interval as $n \to \infty$. More precisely, for any interval $I \subset [0,1)$ of length $|I|$,

$$\lim_{N \to \infty} \frac{\#\{1 \leq n \leq N \text{ so that } [n\theta] \in I\}}{N} = |I|.$$

  More remarkably, he proved that the same equidistribution holds for the much more confusing sequence $[n^2\theta]$.

- **Medical imaging.** Suppose that f is a continuous compactly supported function on $\mathbb{R}^3$. For any line $l \subset \mathbb{R}^3$, consider the integral $\int_l f$. Suppose that we know this integral for every line $l \subset \mathbb{R}^3$. From this information, can we recover the function f? This type of problem can come up in medical imaging. Suppose we inject a dye into the patient that clings to something we would like to find, such as a tumor. We let $f(x)$ be the density of the dye. Then we image the patient. We shine some type of ray throught the patient on one side and we measure the ray on the other side. By measuring the intensity of the ray coming through the patient, we can infer how much dye the ray passed through. In this way, we can measure $\int_l f$ for some line l. By moving around the machine, we can measure $\int_l f$ for many different lines. Based on this information, we want to reconstruct the function f, which will tell us where the tumor is. This is possible to do, and one method involves Fourier analysis.

- **Fast multiplication.** Suppose that A and B are N-digit numbers, where N is large. We want to compute the product $A \cdot B$. The algorithm that we learned in elementary school takes on the order of N^2 operations, because we have to multiply each digit of A by each digit of B. Remarkably, there is a fast multiplication algorithm that requires only on the order of $N \log N$ operations. This algorithm involves (a version of) Fourier series.
- **The Heisenberg uncertainty principle.** In quantum mechanics, Heisenberg formulated a principle saying that we cannot simultaneously know the position and the momentum of a particle beyond a certain accuracy. One way to look at quantum mechanics is Schrodinger's point of view, using wave functions and the Schrodinger equation. In this approach, a particle in 3-dimensional space $\mathbb{R}^3$ is described by a complex-valued wave function $\psi : \mathbb{R}^3 \to \mathbb{C}$ normalized so that $\int |\psi|^2 = 1$. The probability that the particle lies in a certain region $\Omega \subset \mathbb{R}^3$ is $\int_\Omega |\psi^2|$. The momentum of the particle is described by the Fourier transform of ψ. Once the position and momentum of a particle are described in terms of Fourier analysis, the uncertainty principle can be formulated as a statement about the Fourier transform, and we will prove it.

4. Fine points of convergence and Lebesgue theory

The story of the convergence of Fourier series has many subtleties. We stated above that if $f(x)$ is a C^2 periodic function, then its Fourier series converges pointwise. But if f is just a continuous periodic function, then this is not true. Let $S_N f$ denote the partial sum of the Fourier series:

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

If f is C^2 periodic, then $S_N f(x) \to f(x)$ for each point x, but if f is only continuous, this may not be true. We will see an example of a continuous periodic f and a point x so that $S_N f(x)$ diverges.

Nevertheless, if f is a continuous function, the partial sums $S_N f$ do get close to f in a different sense - a more averaged sense.

Proposition 1. *If f is a 2π-periodic continuous function, then*

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} |S_N f - f|^2 dx = 0.$$

We say that a sequence of functions g_N "converges to g in the L^2 sense" if

$$\lim_{N \to \infty} \int |g_N - g|^2 = 0. \tag{4}$$

This notion of convergence comes up very naturally in Fourier analysis, and we will study it in the class. There is a similar notion of a Cauchy sequence in the L^2 sense. We say that a sequence of functions g_N is Cauchy in the L^2 sense if for any $\epsilon > 0$, there is some N so that for any $N_1, N_2 \geq N$,

$$\int |g_{N_1} - g_{N_2}|^2 < \epsilon.$$

If $\{g_N\}$ is a sequence of continuous functions that is Cauchy in the L^2 sense, is there always a limit function g? If so, how badly behaved can g be?

From the point of view of a first course in analysis, an L^2-Cauchy sequence can behave rather wildly. For instance, if $g_N(x)$ is an L^2-Cauchy sequence, it may happen that $g_N(x)$ fails to converge for every value of x. If $g_N(x)$ does converge at each point x to a limit $g(x)$, the limiting function $g(x)$ may behave in a rather wild way. Even if each function $g_N(x)$ is continuous, the limit $g(x)$ may be discontinuous at every point x, and it may fail to be Riemann integrable.

The theory of Lebesgue integration gives a different point of view about L^2-Cauchy sequences, and from the point of view of Lebesgue integration they behave in a much nicer way. First of all, we will define a notion of limit using Lebesgue integration, and in with this definition, every L^2-Cauchy sequence g_N has a limit g. It is still true that the limit function g may be discontinuous at every point. However, the limit function g is not a hopelessly complicated object, and using the Lebesgue theory of integration we will be able to define integrals of the form $\int_a^b g(x)dx$.

The theory of Lebesgue integration also helps with other delicate issues about infinite sums. For example, suppose that u_n are solutions to the heat equation and we consider an infinite sum $\sum_{n=1}^{\infty} u_n$. When is this infinite sum also a solution to the heat equation? Using Lebesgue integration helps to prove simple criteria when this happens.