Geometry of manifolds, Problem Set 5

Due on Friday May 10 in class.

1. (Finishing the proof of the de Rham theorem)

Suppose that M is a closed manifold with a triangulation Tri. The triangulation leads to a cochain complex, which we write as $C^j_{Tri}(M, \mathbb{R})$. We will usually leave out the Tri and the $\mathbb{R}$ and just write $C^j(M)$. Integration gives a map $I : \Omega^j(M) \to C^j(M)$. By Stokes's theorem, this is a cochain map. In other words, if $d_\Delta : C^j(M) \to C^{j+1}(M)$ is the coboundary map in the complex of simplicial cochains, then

$$I(d\beta) = d_\Delta(I\beta)$$

for any $\beta \in \Omega^j(M)$. Therefore, I induces a map $I_* : H^k_{deRham}(M) \to H^k_{Tri}(M)$. The de Rham theorem says that this map is an isomorphism. As a corollary, one gets a finite algorithm to compute the de Rham cohomology of any closed triangulated manifold.

We proved in class that the map I_* is injective.

Prove that I_* is surjective.

2. The geometry and topology of $\mathbb{CP}^n$.

a.) Covering $\mathbb{CP}^n$ with charts. Suppose that $\mathbb{CP}^n$ is covered with charts U_i, each diffeomorphic to a subset of $\mathbb{R}^{2n}$. The covering we have been using in class involves $n+1$ charts. We will prove that we cannot cover $\mathbb{CP}^n$ with fewer charts.

Here is an outline. Suppose that $\mathbb{CP}^n$ is covered with charts $U_1, ..., U_n$. Choose compact $K_j \subset U_j$ so that $\cup K_j = \mathbb{CP}^n$. Next, let ω denote the Fubini-Study form. Find 2-forms $\beta_j = \omega + d\alpha_j$ so that β_j vanishes on K_j. Conclude that $\int_{\mathbb{CP}^n} \beta_1 \wedge ... \wedge \beta_n = 0$. But check that this integral $= \int_{\mathbb{CP}^n} \omega^n = Vol_{2n}(\mathbb{CP}^n, g_{FS}) > 0$.

b.) Suppose that $f : \mathbb{CP}^2 \to S^2$ is a smooth map. Consider the restriction of f to $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Prove that this restriction has degree zero.

c.) Is there an orientation-reversing diffeomorphism of $\mathbb{CP}^2$?

3. Hopf-type invariants of maps to $\mathbb{CP}^2$.

Let ω be the Fubini-Study form on $\mathbb{CP}^2$ renormalized so that $\int_{\mathbb{CP}^1} \omega = 1$, where $\mathbb{CP}^1 \subset \mathbb{CP}^2$ in the canonical way ($[z_0, z_1] \to [z_0, z_1, z_2]$). In particular, ω is a closed form.

Let $f : S^3 \to \mathbb{CP}^2$. Since $f^*(\omega)$ is a closed 2-form on S^3, it is equal to $d\alpha$ for some 1-form α. Let $H_1(f) := \int_{S^3} \alpha \wedge f^*\omega$. Does $H_1(f)$ depend on the choice of α? Is it a homotopy invariant?

Let $g : S^5 \to \mathbb{CP}^2$. Since $g^*(\omega)$ is a closed 2-form on S^5, it is equal to $d\alpha$ for some 1-form α. Let $H_2(g) := \int_{S^5} \alpha \wedge g^*\omega \wedge g^*\omega$. Does $H_2(g)$ depend on the choice of α? Is $H_2(g)$ homotopy invariant?

Extra credit: Can you describe H_1 and/or H_2 in terms of regular values, transversality, linking...

4. As we come to the end of class, do you have any questions for me – about topics we discussed in class, or about differential geometry in general?