Geometry of manifolds, Problem Set 4

Due on Friday April 12 in class.

1. Recall that for a matrix M, $\sigma_k M$ is the polynomial given by the expansion

$$det(I + tM) = \sum_k t^k \sigma_k M.$$

Also, σ_k is the k^{th} symmetric function of the eigenvalues of M. For instance if M is a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then $\sigma_2(M) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$.

Let E be a complex vector bundle over a base B, equipped with a connection ∇ with curvature F. We proved in class that $\sigma_k F$ is a well-defined 2k-form on B (with complex values). We proved that $\sigma_k F$ is closed, and that the de Rham cohomology class of $\sigma_k F$ is independent of the choice of ∇, F.

The k^{th} Chern class is defined to be the de Rham cohomology class of $(2\pi i)^{-k}\sigma_k F$, lying in $H^{2k}_{deRham}(B, \mathbb{C})$. (It turns out to be a real cohomology class, but we didn't prove it yet.) The k^{th} Chern class is written $c_k(E)$.

a.) Prove the following nice formula for the Chern class of the direct sum of two vector bundles.

$$c_k(E \oplus E') = \sum_{j=0}^{k} c_j(E) \cup c_{k-j}(E').$$

(Hint: Choose a connection on E and on E' individually, and then combine them to give a connection on $E \oplus E'$.)

b.) Suppose that E is an n-dimensional complex bundle. Suppose that there are $n-k+1$ sections $s_1, ..., s_{n-k+1}$ which are linearly independent at every point $b \in B$. Prove that $c_k(E) = 0$.

c.) Optional: does this suggest any way to find $c_k(E)$ by studying the behavior of $n-k+1$ generic sections? If you like, try to make a conjecture.

2. Variations on the Poincaré Lemma.

Suppose that β is a closed k-form on $[0,1]^m \times \mathbb{R}^n$ which vanishes in a neighborhood of $(\partial[0,1]^m) \times \mathbb{R}^n$. If $m = k$, we also assume that $\int_{[0,1]^m \times \{y_0\}} \beta = 0$ for some $y_0 \in \mathbb{R}^n$. (Note that the integral does not depend on y_0 because β is closed.)

With these assumptions, prove that $\beta = d\alpha$ for a (k-1)-form α which vanishes in a neighborhood of $(\partial[0,1]^m) \times \mathbb{R}^n$.

This result is the main ingredient in a proof of the de Rham theorem. The two most fundamental cases are $n = 0$ (we did in class), and $m = 0$. But the mixed case is also useful.

3. The de Rham cohomology of S^n.

a.) Using problem 2, compute the de Rham cohomology of S^n. Suppose β is a closed k-form on S^n. If $1 \le k \le n-1$, prove that β is exact. If $k = n$, prove that β is exact if and only if $\int_{S^n} \beta = 0$.

Hint: Cover S^n with two charts. Use versions of the Poincaré lemma on the charts.

b.) (Optional) With almost the same argument, prove the same result for $S^n \times \mathbb{R}^m$ for any $m \ge 0$. You can turn in just the answer to b. or just the answer to a.

4. The Hopf invariant and differential forms.

Let $F : S^3 \to S^2$ be a smooth map. Let ω be a 2-form on S^2 with $\int_{S^2} \omega = 1$.

The pullback $F^*\omega$ is a closed 2-form on S^3. By Problem 3, there exists a 1-form α with $d\alpha = F^*\omega$.

Define $H(F) = \int_{S^3} \alpha \wedge F^*\omega$. Prove the following results – you can use any of the previous problems, including 3a and 3b.

a.) Prove that $H(F)$ does not depend on the choice of α.

b.) Prove that $H(F)$ is a homotopy invariant.

c.) Let ω_1, ω_2 be two 2-forms on S^2 with integral 1. Let α_i be a primitive of $F^*\omega_i$. Prove that $\int_{S^3} \alpha_2 \wedge F^*\omega_2 = \int_{S^3} \alpha_1 \wedge F^*\omega_2$. As a corollary, show that $H(F)$ does not depend on the choice of ω (so long as $\int \omega = 1$).

The invariant $H(F)$ is the Hopf invariant. As an optional extra credit problem, you can try to prove that our two definitions agree. Let y_1 and y_2 be regular values of F. Let ω_i be a 2-form with integral 1 concentrated in a small neighborhood of y_i.

d.) (optional) Using that $H(F) = \int_{S^3} \alpha_1 \wedge F^*\omega_2$, try to prove that $H(F)$ is the linking number of $F^{-1}(y_1)$ and $F^{-1}(y_2)$).

e.) (optional) What happens if we take ω highly concentrated near a single regular value and look at $\int \alpha \wedge F^*\omega$?