Geometry of manifolds, Problem Set 3

Due on Monday, April 1 in class.

There are eight problems. Write up the answers for six of them. You should pick what seems educational and interesting to you. You should also read all of the problems. The last problems are longer and more open-ended, somewhere between problems and projects. For these problems, you don't have to give a complete solution – you can write up the things you figured out.

Part A. Calisthenics.

1a. Suppose that $Z^p \subset Y^n$ is a submanifold, and $z \in Z$. Prove that there is an open set $U \supset z$ in Y and a diffeomorphism $\phi : U \to V \subset \mathbb{R}^n$ so that $\phi(Z \cap U)$ is the intersection of V and a p-dimensional plane.

1b. Using 1a, prove the preimage lemma for transverse maps: If $f : X \to Y$ is transverse to Z, then $f^{-1}(Z)$ is a submanifold of X.

2. Defining the Hopf invariant using transversality. Recall from class the following theorem. Suppose that $F : S^{2n-1} \times [0,1] \to S^n$, and that y, z are regular values of F. (In other words, F is transverse to y and z.) Then the linking numbers $L(f_0^{-1}(y), f_0^{-1}(z))$ and $L(f_1^{-1}(y), f_1^{-1}(z))$ agree. Using this theorem and the transversality theorem, we will define the Hopf invariant.

Fix y, z in S^n. If $f : S^{2n-1} \to S^n$ is transverse to y, z, define $Hopf(f) = L(f^{-1}(y), f^{-1}(z))$. If f is not transverse to y and z, pick a homotopic map g and define $Hopf(f) = L(g^{-1}(y), g^{-1}(z))$.

a. Prove that $Hopf(f)$ does not depend on the choice of g (homotopic to f).

b. Prove that if f_1 is homotopic to f_2, then $Hopf(f_1) = Hopf(f_2)$.

c. Prove that the definition of the Hopf invariant does not depend on the choice of points $y, z \in S^n$. In this part, you may use the following lemma (compare the Homogeneity Lemma in Chap. 4 of *Topology from the Differentiable Viewpoint*): If $y_1 \neq z_1$ and $y_2 \neq z_2$ are points in a connected manifold N, then there is a diffeomorphism $\psi : N \to N$ isotopic to the identity so that $\psi(y_1) = y_2$ and $\psi(z_1) = z_2$.

Part B. On Euler numbers

3. Compute the Euler number of TS^n. To do this, find a smooth vector field $v : S^n \to TS^n$ which is transverse to the zero section, and add up the signs of the intersection points. (We sketched one such vector field v in class.)

Minor point: we need an orientation for TS^n. Suppose that $x \in S^n$ and $v \in T_x S^n$, so that $(x, v) \in TS^n$. The tangent bundle of TS^n at the point (x, v) is $T_x S^n \times T_x S^n$. If $e_1 ... e_n$ is a positive basis of $T_x S^n$, then $(e_1, 0), ..., (e_n, 0), (0, e_1), ..., (0, e_n)$ is a positive basis of $T_{(x,v)} TS^n$.

4. Prove that $TS^2 \times \mathbb{R}$ is diffeomorphic to $S^2 \times \mathbb{R}^3$. (In fact, each of these are 3-dimensional vector bundles over S^2, and they are isomorphic as vector bundles.) The key point is that the normal bundle of $S^2 \subset \mathbb{R}^3$ is trivial.

5. Prove that TS^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$. Here is a sketch.
a. Prove that $S^2 \times \mathbb{R}^2$ embeds into $\mathbb{R}^4$.
b. Using a., prove that if $X, Y \subset S^2 \times \mathbb{R}^2$ are compact 2-dimensional submanifolds (without boundary), then $I(X, Y) = 0$. (If X and Y are transverse, then $I(X, Y)$ is the sum of the intersection points of X and Y with signs. In general, if $i : X \to S^2 \times \mathbb{R}^2$ is the inclusion map, then $I(X, Y)$ means $I(i, Y)$.)
c. Using Problem 3 and part b., prove that TS^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.

6. For each $D \in \mathbb{Z}$, we will define a 2-dimensional oriented vector bundle V_D over S^2. Write S^2 as two open sets $U_1 \cup U_2$, where U_1 is a neighborhood of the Northern hemisphere and U_2 is a neighborhood of the Southern hemisphere. Note that $U_1 \cap U_2$ is a neighborhood of the equator. On $U_1 \cap U_2$, we use (oriented) coordinates $(\theta, l) \in S^1 \times (-1, 1)$, where θ is the "longitude" and l is the latitude. The equator is given by $l = 0$, and the Southern hemisphere is given by $l \leq 0$. Let $R(\phi) \in GL_2(\mathbb{R})$ denote the rotation in the positive direction by angle ϕ. Now we glue $U_1 \times \mathbb{R}^2$ with $U_2 \times \mathbb{R}^2$ by the change-of-charts map

$$g_{12}(\theta, l) = R(D\theta).$$

This gluing of charts defines V_D. In other words, if $x = (\theta, l) \in U_1 \cap U_2$, and if (x, v) is in $U_1 \times \mathbb{R}^2$, and if we look at the corresponding point of V_D in the chart $U_2 \times \mathbb{R}^2$, we get the point $(x, R(D\theta)v) \in U_2 \times \mathbb{R}^2$.
Compute the Euler number of V_D.

Part C. More open-ended problems. These problems are not just applications of what we studied in class, but the things we learned in class are relevant.

7. Topology of complex algebraic curves. One of my goals during the course is to study the geometry and topology of complex algebraic varieties. This was a big topic in the development of differential geometry and topology, including Riemann's work in the 1850's and 60's and Lefshetz's work in the 1920's and 30's.
Let $Z \subset \mathbb{C}^2$ be the set $\{(z_1, z_2)$ such that $z_1^4 + z_2^4 = 1\}$.

Let $\Sigma \subset Z$ be the intersection of Z with a closed ball of radius 1000: $\Sigma = \{(z_1, z_2) \in Z$ such that $|z_1|^2 + |z_2|^2 \leq 1000^2\}$.

Both Z and Σ are two-dimensional surfaces in $\mathbb{C}^2$, which we identify with $\mathbb{R}^4$. More precisely, Z is a two-dimensional manifold and Σ is a two-dimensonal manifold with boundary. What can you figure out about the topology of Σ? Since Σ lies in $\mathbb{R}^4$, this task will require grappling with geometric thinking in four dimensions, where visualization is challenging.

Here are some particular questions you may look into.

Can you prove that Σ is a manifold with boundary and $\partial\Sigma$ is $\Sigma \cap S^3(1000)$? Can you prove that Σ is orientable? Can you prove that Σ is connected? Can you find the genus of Σ?

How many components does $\partial\Sigma$ have? Each component of $\partial\Sigma$ is a circle in the three-sphere $S^3(1000)$. How are the circles arranged? Are any two of them linked?

8. Challenge problem. Recall that a map between metric spaces has Lipschitz constant $\leq L$ if $dist(f(x), f(y)) \leq L dist(x, y)$. Suppose that f is a smooth map from S^3 to S^2 with Lipschitz constant L. How big can the Hopf invariant of f be?

(Here are a few ways of thinking about the Lipschitz constant. We use the unit sphere metrics on S^3 and S^2. That means that the distance between two points $x, y \in S^3$ is the length of the shortest path between them in S^3. A second way to think of the Lipschitz constant is that a smooth map $f : S^3 \to S^2$ has Lipschitz constant $\leq L$ if and only if $|df_x(v)| \leq L|v|$ for any $v \in T_x S^3$. A third point of view is that f has Lipschitz constant $\leq L$ if and only if, for every path γ in the domain, the length of $f(\gamma)$ is $\leq L length(\gamma)$. For any smooth map between Riemannian manifolds, the Lipschitz constant has these three interpretations, connecting to distances, the size of the derivative, and the lengths of paths.)

You can look for examples with large Hopf invariant. You can prove upper bounds for the Hopf invariant. And you can also look for simpler problems that seem related...