

Geometry of manifolds, Problem Set 2

Due on Friday, March 8 in class.

1. (Processing the proof of Sard's theorem) Suppose $f : [0, 1]^3 \rightarrow \mathbb{R}^2$ is a C^1 map, but f may not have any more regularity. Suppose that df_x has rank ≤ 1 for every x - in other words, every point of the domain is a critical point. Recall that $C = [0, 1]^3$ is the set of critical points of f and that $C_1 \subset [0, 1]^3$ is the set of points where $df_x = 0$. Prove that $f(C \setminus C_1)$ has measure 0. (You can use the inverse function theorem for C^1 maps.)

2. Nice fiber lemma for manifolds with boundary: a referee report. Here is a 'proof' of Lemma 4 in Chapter 2 of *Topology from the Differentiable Viewpoint*. Do you find it correct, or correct after some revisions, or seriously wrong? You can compare it with the proof in the book.

Lemma 1. *Suppose that M is a manifold with boundary of dimension m and N is a manifold of dimension n , and $f : M \rightarrow N$ is a smooth map. If y is a regular value of f , then $f^{-1}(y)$ is an $(m-n)$ -dimensional manifold with boundary, and $\partial f^{-1}(y)$ is exactly $f^{-1}(y) \cap \partial M$.*

Proof. Let $\text{Int}(M)$ be the interior of M . We already proved that $f^{-1}(y) \cap \text{Int}(M)$ is an $(m-n)$ -dimensional manifold. We only have to worry about the points of $f^{-1}(y)$ in ∂M . Recall that y is a regular value of f iff y is a regular value of $f : \text{int}(M) \rightarrow N$ and y is a regular value of $f : \partial M \rightarrow N$. Suppose that $x \in f^{-1}(y) \cap \partial M$. We have to check that a neighborhood of $f^{-1}(y)$ near x is diffeomorphic to an open set in H^{n-m} , where the diffeomorphism takes x to a boundary point of H^{n-m} .

By choosing charts, we can reduce to the case of a map $f : H^m \rightarrow \mathbb{R}^n$. Suppose that $p \in \partial H^m$, and $f(p) = y$ and p a regular point of f . Let $\mathbb{R}^{m-1} = \partial H^m$. So $df_p : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^n$ is surjective. By making a linear change of variables in the first $m-1$ variables of the domain and a linear change of variables in the range, we can reduce to the case that $df_p(e_j) = e_j$ for $j = 1, \dots, n$. (Here e_j is the j^{th} basis vector: all 0's except a 1 in the j^{th} component.)

Let $F : H^m \rightarrow \mathbb{R}^m$ be given by $F(x) = (f(x), x_{n+1}, \dots, x_m)$. Now dF_p is an isomorphism. By the inverse function theorem, F is a diffeomorphism on a domain containing p . The map F also maps ∂H^m to ∂H^m and the interior of H^m to the interior of H^m . It gives a diffeomorphism from a neighborhood of p in $f^{-1}(y) \cap H^m$ to a neighborhood of a boundary point in H^{n-m} . \square

3. Zip code problem. The postal office on planet X wants you to help them redesign their zip code system to be more logical. The one country on planet X has

been divided into an $N \times N$ grid of squares. Here N is a large number, say $N = 10^6$. Each square is assigned a zip code, which is a number from 1 to N^2 . The post office director has complained that the current system is illogical. Two adjacent squares sometimes have radically different zipcodes. Other times, two squares on opposite sides of the country have similar zip codes.

The different squares in the system form an $N \times N$ grid, and we refer to them by vectors $\vec{x} = (x_1, x_2)$ where x_i are integers between 1 and N . The “distance” between two squares x and y will be $|x_1 - y_1| + |x_2 - y_2|$, denoted $D(x, y)$. The zipcode of x is denoted $Z(x)$. The post office director would like a set of zip codes with the following properties:

- If $D(x, y) \leq 1$, then $|Z(x) - Z(y)| \leq N^{4/3}$.
- If $D(x, y) \geq N/3$, then $|Z(x) - Z(y)| \geq N^{5/3}$.

Your job is to design such a zipcode system, or else explain to the director why it is impossible.

4. The ham sandwich theorem with circles and lines in the plane. Suppose that U_1, U_2, U_3 are bounded open sets in \mathbb{R}^2 . Prove that there is either a circle or a line that bisects all three sets. (The circle may have any radius.)

5. (Degrees of maps) Let Σ_G denote a closed oriented surface of genus G . (So Σ_1 is a torus, and Σ_0 is a 2-sphere.) Look for maps from Σ_3 to Σ_2 of many different degrees. What are the degrees of the maps that you can construct? Briefly describe the maps, but you don't have to give careful proofs. (You can also draw pictures.)

6. Suppose that $h : S^3 \rightarrow S^2$ is a map of Hopf invariant H . Suppose that $g : S^2 \rightarrow S^2$ is a map of degree D . What is the Hopf invariant of $g \circ h$?

Now suppose that $f : S^3 \rightarrow S^3$ is a map of degree D . What is the Hopf invariant of $h \circ f$?

7. (optional) (Turning pants inside out) First we describe the situation in everyday terms. You can imagine it or even try it out at home. Take a pair of pants and glue together the bottoms of the two legs. The result is a surface of genus 1 with one boundary component (the waist). Now forget about the gluing and start again. Take the pants, turn them inside out, and then glue together the bottoms of the two legs. The result is the same surface of genus 1 with one boundary component. Now it is embedded in \mathbb{R}^3 in a new position - it is “inside out”. Without undoing the glue, can you get the pants back to the original position?

Next we describe the problem mathematically. Let T be a torus of revolution. Start with the circle of radius 1 around the point $(2, 0)$ in the xz -plane, and rotate it around the z axis. Let $\Sigma \subset T$ be $\{(x, y, z) \in T \mid z \leq 2\}$. The surface Σ is a

2-dimensional compact manifold with boundary, with genus 1 and one boundary component. The map I defined by $I(x, y, z) = (x, -y, z)$ is a diffeomorphism from Σ to itself. In particular, I is an embedding from Σ into \mathbb{R}^3 . Is I isotopic to the identity embedding?

Recall that a smooth map $I : M \rightarrow N$ is called an embedding if it is a diffeomorphism onto its image. Two embeddings $I_1, I_2 : M \rightarrow \mathbb{R}^n$ are called isotopic if there is a smooth homotopy I_t so that for each t , I_t is an embedding.

You can turn in either a mathematically rigorous answer or some intuition. Can you get this problem to connect with the topology we have learned in class?

(Here is a good warmup question. Instead of a torus, start with a sphere. Let $\Sigma_0 := \{(x, y, z) \in S^2 \mid z \leq 1/2\}$. Let $I : \Sigma_0 \rightarrow \mathbb{R}^3$ be the same map $I(x, y, z) = (x, -y, z)$. In this situation, is I isotopic to the identity?)