

Geometry of manifolds, Problem Set 1

Due on Friday, February 22 in class.

1. Small perturbations of the identity.

a.) Suppose that $F : \bar{B}^n \rightarrow \mathbb{R}^n$ is a smooth map from the closed unit n-ball to \mathbb{R}^n . Suppose that $|F(x) - x| \leq 1/2$ for all $x \in \bar{B}^n$. Prove that the image of F contains 0.

b.) (On your own, you don't have to turn it in.) Prove that the system of equations $x + 10^{-7}x^3y^5 + 10^{-3}y^6 = 0$ and $10^{-3}x^9 + 10^{-4}x^2y^5 + y = 0$ has a real solution $(x, y) \in \mathbb{R}^2$.

2. Here is a 'proof' of the Brouwer fixed point theorem in two dimensions. Write a referee report about this proof. Do you find it correct, or correct after some revisions, or seriously wrong?

Theorem 1. *If $f : [0, 1]^2 \rightarrow [0, 1]^2$ is a continuous map, then f has a fixed point.*

Proof. First we observe that the one-dimensional case of this theorem follows from the intermediate value theorem. We include this as a lemma.

Lemma 1. *If $g : [0, 1] \rightarrow [0, 1]$ is continuous, then g has a fixed point.*

Proof. If $g(0) = 0$ or $g(1) = 1$ we are done. So let us assume that $g(0) > 0$ and $g(1) < 1$. Define $h(x) = x - g(x)$. We see that $h(0) < 0$ and $h(1) > 0$. By the intermediate value theorem, there exists $c \in [0, 1]$ so that $h(c) = 0$. But then $g(c) = c$. \square

We will prove the two-dimensional case of the Brouwer fixed point theorem by using the one-dimensional version twice. More generally, we hope to prove the Brouwer fixed point theorem in all dimensions by using induction.

Write $f = (f_1, f_2)$. So f_1 is a function from $[0, 1]^2$ to $[0, 1]$. For each $y \in [0, 1]$, define $f_{1,y} : [0, 1] \rightarrow [0, 1]$ by $f_{1,y}(x) = f_1(x, y)$. By Lemma 1, each function $f_{1,y}$ has a fixed point, $c(y)$. Unwinding the definition, we see that $f_1(c(y), y) = c(y)$ for all $y \in [0, 1]$.

Now define the function $g : [0, 1] \rightarrow [0, 1]$ by $g(y) = f_2(c(y), y)$. Using Lemma 1 again, we see that g has a fixed point, say d . Unwinding the definition, we see that $f_2(c(d), d) = d$. But we also know from the last paragraph that $f_1(c(d), d) = c(d)$. Therefore, $f(c(d), d) = (c(d), d)$. \square

3. Write a 2-4 step outline of the main ideas of the proof of Sard's theorem.

4. Linking numbers.

Suppose that M^k and N^l are closed, connected manifolds of dimensions k and l . Suppose that $n = k + l + 1$. Suppose that $f : M \rightarrow \mathbb{R}^n$ and $g : N \rightarrow \mathbb{R}^n$ are smooth maps with disjoint images.

Define $h : M \times N \rightarrow S^{n-1}$ by

$$h(m, n) = \frac{f(m) - g(n)}{|f(m) - g(n)|}.$$

Define the mod 2 linking number of f and g by

$$Link_2(f, g) := Deg_2 h.$$

a.) Suppose that $F : M \times [0, 1] \rightarrow \mathbb{R}^n$ and $G : N \times [0, 1] \rightarrow \mathbb{R}^n$ are smooth homotopies. Define $F_t : M \rightarrow \mathbb{R}^n$ by $F_t(m) = F(m, t)$. Suppose that for each t , the images of F_t and G_t are disjoint. Prove that $Link_2(F_0, G_0) = Link_2(F_1, G_1)$.

b.) Construct two embeddings of S^1 into \mathbb{R}^3 so that the mod 2 linking number is non-zero.

c.) Suppose that the image of f and the image of g lie on opposite sides of a hyperplane. Prove that $Link_2(f, g) = 0$.

5. Winding numbers.

Suppose that M^{n-1} is a closed connected manifold. Suppose that $F : M \rightarrow \mathbb{R}^n$ is a smooth map. If $p \in \mathbb{R}^n$ is not in the image of F , then define the winding number $W_2(F, p)$ to be the mod 2 degree of the normalized map $\tilde{F} : M^{n-1} \rightarrow S^{n-1}$ given by the formula

$$\tilde{F}(m) = \frac{F(m) - p}{|F(m) - p|}.$$

Suppose that p, q are two points in the complement of the image of F . Suppose that $W_2(F, p) \neq W_2(F, q)$. Let γ be a smooth path from p to q . Prove that γ must cross the image of F .

6. On continuous and smooth maps.

Continuous maps can be approximated by smooth maps. Using such approximations, we can prove theorems about continuous maps using our knowledge of smooth maps. We state here two theorems about approximating continuous maps by smooth maps.

Smoothing theorem 1. *Suppose that M, N are smooth manifolds and $f : M \rightarrow N$ is a continuous map. Then there is a continuous homotopy $F_t : M \rightarrow N$ so that $F_0 = f$ and F_1 is a smooth map.*

Smoothing theorem 2. *Suppose that M is a manifold with boundary and N is a manifold. Suppose that $F : M \rightarrow N$ is a continuous map, and the restriction $F : \partial M \rightarrow N$ is a smooth map. Then there is a smooth map $G : M \rightarrow N$ so that F and G agree on ∂M .*

a.) You can use these theorems without proof on this problem. Do you think you know more or less how to prove them? Would it be interesting to you to discuss them in class?

b.) Using the smoothing theorems and our work on smooth maps, prove the following theorem. There is no continuous map f from the closed unit ball \bar{B}^n to S^{n-1} so that $f(x) = x$ for each $x \in \partial B^n = S^{n-1}$.

c.) Suppose that M, N are closed connected n -manifolds. If $f, g : M \rightarrow N$ are smooth maps that are continuous homotopic, prove that $\deg_2 f = \deg_2 g$.

d.) Prove that if $m < n$, then any continuous map $S^m \rightarrow S^n$ is contractible (continuous homotopic to a constant map).

e.) This last problem is a lot harder, but I think it's very interesting. If $m \neq n$, prove that \mathbb{R}^m and \mathbb{R}^n are not homeomorphic.