Hörmander's Topological Paley–Wiener Theorem

(Informal class notes, S. Helgason)

The space $\mathcal{D} = \mathcal{D}(\mathbf{R}^n) = \mathcal{C}_c^\infty(\mathbf{R}^n)$ is given the inductive limit topology of the spaces $\mathcal{D}_{\overline{B_j(0)}}$ of functions $\varphi \in \mathcal{D}$ with support in the ball $\overline{B_j(0)} = \{x \in \mathbf{R}^n = |x| \leq 1\}$. This topology can be characterized by the following result of Schwartz (Distributions, p. 67).

Theorem 1. *Given two monotonic sequences*

$$\{\epsilon\} : \epsilon_0\,,\, \epsilon_1\,, \ldots \qquad \epsilon_i \to 0$$

$$\{N\} : N_0\,,\, N_1\,, \ldots \qquad N_i \to \infty$$

let $V(\{\epsilon\}\{n\})$ denote the set of functions $\varphi \in \mathcal{D}$ satisfying for each $j \geq 0$ the conditions:

(1) $$|D^\alpha \varphi(x)| \leq \epsilon_j \text{ for } |\alpha| \leq N_j\,, \quad |x| \geq j\,.$$

Then the sets $V(\{\epsilon\}\,,\,\{N\})$ form a fundamental system of neighborhoods of 0 in $\mathcal{D}$.

Let $A \geq 0$ and $\mathcal{D}_A$ the space $\mathcal{D}_{\overline{B_A(0)}}$ topologized by the seminorms

(2) $$\|f\|_m = \sum_{|\alpha| \leq m} \sup_{|x|<A} |(D^\alpha f)(x)|\,.$$

Also let $\mathcal{H}_A = \mathcal{H}_A(\mathbf{C}^n)$ denote the space of holomorphic functions of exponential type A, that is the space of holomorphic functions φ such that for each $N \in \mathbf{Z}^+$

(3) $$|||\varphi|||_N = \sup_{\zeta \in \mathbf{C}^n} (1+|\zeta|)^N e^{-A|\operatorname{Im}\zeta|} |\varphi(\zeta)| < \infty\,.$$

$\operatorname{Im}\zeta$ denoting the imaginary part of ζ. We topologize $\mathcal{H}_A$ with the seminorms $|||\quad|||_N$.

Theorem 2. *The Fourier transform $f \to \widetilde{f}$ where*

$$\widetilde{f}(\zeta) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x,\zeta\rangle}\,dx\,, \quad \zeta \in \mathbf{C}^n$$

is a homeomorphism of $\mathcal{D}_A$ onto $\mathcal{H}_A$.

Proof:

The Paley–Wiener theorem states that

$$\widetilde{\mathcal{D}}_A = \mathcal{H}_A\,.$$

The continuity statements follow easily from the formulas

$$i^{|\beta|}\zeta^\beta \widetilde{f}(\zeta) = \int_{\mathbf{R}^n} (D^\beta f)(x) e^{-i\langle x,\zeta\rangle}\,dx \tag{4}$$

and the inversion

$$(\mathcal{D}^\alpha f)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} (i\zeta)^\alpha \widetilde{f}(\zeta) e^{i\langle x,\zeta\rangle}\,d\zeta\,. \tag{5}$$

The space $\mathcal{D}_A$ is complete. If $\widetilde{f}_i$ is a Cauchy sequence in $\mathcal{H}_A$, replacing f in (5) by $f_i - f_j$ we see that f_i is a Cauchy sequence in $\mathcal{D}_A$, hence convergent, whence $\mathcal{H}_A$ is complete. The map $f \to \widetilde{f}$ is a homeomorphism of $\mathcal{S}(\mathbf{R}^n)$ onto $\mathcal{S}(\mathbf{R}^n)$. The space $\mathcal{D}_A$ is closed in $\mathcal{S}$ and carries the relative topology. The same thus holds for $\mathcal{H}_A$.

If we give $\mathcal{H} = \bigcup_{i\in\mathbf{Z}^+} \mathcal{H}_i$ the inductive limit topology the Fourier transform $f \to \widetilde{f}$ is a homeomorphism of $\mathcal{D}$ onto $\mathcal{H}$. This is of little use unless we can describe the topology of $\mathcal{H}$ more directly. This is accomplished in the following

Theorem 3 (Hörmander)**.** *A convex set V is a neighborhood of 0 in $\mathcal{D}$ if and only if there exist sequences*

$$\{\delta\} = \delta_0, \delta_1\,, \ldots \quad \{M\} = M_0, M_1\,, \quad \delta_i, M_i > 0$$

such that V contains all $u \in \mathcal{D}$ satisfying

$$|\widetilde{u}(\zeta)| \leq \sum_0^\infty \delta_k \frac{1}{(1+|\zeta|)^{M_k}} e^{k|\operatorname{Im}\zeta|}\,, \quad \zeta \in \mathbf{C}^n\,. \tag{6}$$

This theorem (15.4.2 from Hörmander's Analysis of Linear Partial Differential Operators, Vol. II) is a basic result in distribution theory and has very important consequences. Hörmander's result is more general and deals with convex sets instead of balls. The proof below is therefore a slight simplification of Hörmander's. It is also much more pedestrian in that the initial steps in the inductive proof are spelled out in detail because this seems to me to reveal the subtle ideas in the proof most clearly. The idea of a proof of this nature involving a contour like Γ_m below appears already in Ehrenpreis' paper in *Amer. J. Math.* 1956, 685–715 although not correctly carried out in details.

Proof:

Let $W(\{\delta\}, \{M\})$ denote the set of $u \in \mathcal{D}$ satisfying (6). Given $k \in \mathbf{Z}^+$ the set

$$W_k = \{u \in \mathcal{D}_{\overline{B_k(0)}} : |\widetilde{u}(\zeta)| \le \delta_k \frac{1}{(1+|\zeta|)^{M_k}} e^{k|\operatorname{Im}|}\}$$

is by Theorem 2 a neighborhood of 0 in $\mathcal{D}_{\overline{B_k(0)}}$ and is clearly contained in $W(\{0\}\{M\})$. If V is a convex set containing $W(\{\delta\}, \{M\})$ then $V \cap \mathcal{D}_{\overline{B_k}(0)}$ contains the neighborhood W_k of 0 in $\mathcal{D}_{\overline{B_k(0)}}$ so by the definition of inductive limit V is a neighborhood of 0 in $\mathcal{D}$.

Proving the converse amounts to proving that given $V(\{\epsilon\}, \{N\})$ there exist sequences $\{\delta\}$ $\{M\}$ such that

$$W(\{\delta\}\{M\}) \subset V(\{\epsilon\}, \{N\}).$$

For this we shift the path of integration in

$$u(x) = (2\pi)^{-n} \int\limits_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi \tag{7}$$

to another one, in which the two weight factors in (3) are comparable. We write

$$\begin{aligned} &x = (x_1, \ldots, x_n), \quad && x' = (x_1, \ldots, x_{n-1}) \\ &\zeta = (\zeta_1, \ldots, \zeta_n), \quad && \zeta' = (\zeta_1, \ldots, \zeta_{n-1}) \\ &\xi = (\xi_1, \ldots, \xi_n), \quad && \xi' = (\xi_1, \ldots, \xi_{n-1}) \\ &\zeta = \xi + i\eta && \xi, \eta \in \mathbf{R}^n. \end{aligned}$$

Then

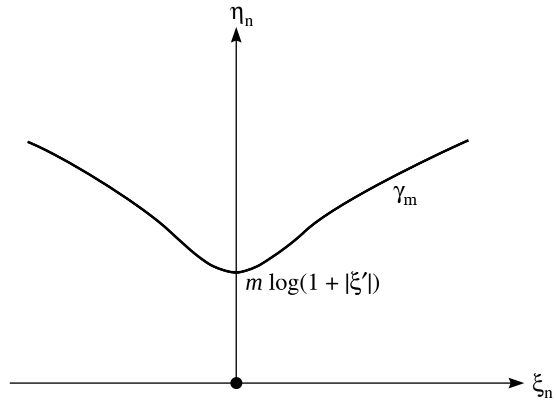
$$\int\limits_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int\limits_{\mathbf{R}^{n-1}} e^{i\langle x',\xi'\rangle}\, d\xi' \int\limits_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi', \xi_n)\, d\xi_n. \tag{8}$$

In the last integral we shift from $\mathbf{R}$ to the contour in $\mathbf{C}$ given by

$$\gamma_m : \zeta_n = \xi_n + im \log(1 + [|\xi'|^2 + \xi_n^2]^{1/2}), \tag{9}$$

m being arbitrary. We claim that, by Cauchy's theorem

$$\int\limits_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi', \xi_n)\, d\xi_n = \int\limits_{\gamma_m} e^{ix_n\zeta_n} \widetilde{u}(\xi', \zeta_n)\, d\zeta_n. \tag{10}$$



For this we must estimate the right integrand in the "strip" between the ξ_n-axis and the curve γ_m.

The function $\zeta_n \to \widetilde{u}(\xi', \zeta_n)$ satisfies

$$(11) \qquad |\widetilde{u}(\xi', \zeta_n)| \leq C_N \frac{e^{A|\operatorname{Im} \zeta_n|}}{(1 + |\zeta_n|)^N}$$

for some A, all N, the constant C_N depending only on N . On the vertical line joining $(\xi_n, 0)$ to (ξ_n, η_n), $\widetilde{u}(\xi', \zeta_n)$ (with ξ' fixed) decays faster than any power of $|\zeta_n|^{-1}$. Secondly,

$$|e^{ix_n\zeta_n}| \leq e^{|x_n||\eta_n|} ,$$

which is bounded by a polynomial in $|\zeta_n|$. Also on γ_m

$$(12) \qquad \left|\frac{d\zeta_n}{d\xi_n}\right| = \left|1 + im\frac{1}{1 + |\xi|}\;\frac{\partial(|\xi|)}{\partial \xi_n}\right| \leq 1 + m \quad (m > 0)$$

thus (10) follows from Cauchy's theorem in *one* variable. Putting

$$\Gamma_m = \{\zeta \in \mathbf{C}^n | \zeta' \in \mathbf{R}^{n-1} , \, \zeta_n \in \gamma_m\}$$

and $d\zeta = d\xi_1 \ldots d\xi_{n-1}\, d\zeta_n$ we thus have for each $m > 0$

$$(13) \qquad u(x) = (2\pi)^{-n} \int\limits_{\Gamma_m} \widetilde{u}(\zeta) e^{i\langle x, \zeta\rangle}\, d\zeta \, .$$

Now suppose the sequences $\{\epsilon\}$, $\{N\}$ and $V(\{\epsilon\}, \{N\})$ are given as in Theorem 1. We have to construct sequences $\{\delta\}$ $\{M\}$ such that (6) implies

(1). By rotational invariance we may assume $x = (0, \ldots, 0, x_n)$ with $x_n > 0$. For each n-tuple α we have

$$(D^\alpha u)(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta)(i\zeta)^\alpha e^{i\langle x,\zeta\rangle}\, d\zeta\,. \tag{14}$$

Starting with positive sequences $\{\delta\}$, $\{M\}$ we shall modify them successively such that (6) $\Rightarrow$ (1). Note that for $\zeta \in \Gamma_m$

$$e^{k|\,\mathrm{Im}\,\zeta|} \le (1+|\xi|)^{km} \tag{15}$$

$$|\zeta^\alpha| \le |\zeta|^{|\alpha|} \le ([|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{|\alpha|}\,. \tag{16}$$

For (1) with $j = 0$ we take $x_n = |x| \ge 0$, $|\alpha| \le N_0$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\mathrm{Im}\,\zeta|} \le 1 \quad \text{for } \zeta \in \Gamma_m\,. \tag{17}$$

Thus if u satisfies (6) we have by (12), (15), (16)

$$\text{(18)}\quad |(D^\alpha u)(x)| \le \sum_0^\infty \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_0 - M_k}(1+|\xi|)^{km}(1+m)\, d\xi\,.$$

We can choose sequences $\{\delta\}$, $\{M\}$ (all δ_k, $M_k > 0$) such that this expression is $\le \epsilon_0$. This then verifies (1) for $j = 0$. We now fix δ_0 and M_0. Next we want to prove (1) for $j = 1$ by shrinking the terms in $\delta_1, \delta_2, \ldots$ and increasing the terms in $M_1, M_2, \ldots$ (δ_0, M_0 having been fixed).

Now we have $x_n = |x| \ge 1$ so (17) is replaced by

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\mathrm{Im}\,\zeta\rangle} \le (1+|\xi|)^{-m} \text{ for } \zeta \in \Gamma_m \tag{19}$$

so in the integrals in (18) the factor $(1+|\xi|)^{km}$ is replaced by $(1+|\xi|)^{(k-1)m}$.

In the sum we separate out the term with $k = 0$. Here M_0 has been fixed but now we have the factor $(1+|\xi|)^{-m}$ which assures that this $k = 0$ term is $< \frac{\epsilon_1}{2}$ for a sufficiently large m which we now fix. In the remaining terms in (18) (for $k > 0$) we can now increase $1/\delta_k$ and M_k such that the sum is $< \epsilon_1/2$. Thus (1) holds for $j = 1$ and it will remain valid for $j = 0$. We now fix this choice of δ_1 and M_1.

Now the inductive process is clear. We assume $\delta_0, \delta_1, \ldots, \delta_{j-1}$ and $M_0, M_1, \ldots, M_{j-1}$ having been fixed by this shrinking of the δ_i and enlarging of the M_i.

We wish to prove (1) for this j by increasing $1/\delta_k$, M_k for $k \geq j$. Now we have $x_n = |x| \geq j$ and (19) is replaced by

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\mathrm{Im}\ \zeta\rangle} \leq 1 + |\xi|^{-jm} \tag{20}$$

and since $|\alpha| \leq N_j$, (18) is replaced by

$$\begin{aligned} &(21) \qquad |(D^\alpha f)(x)| \\ &\leq \sum_{k=0}^{j-1} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi \\ &+ \sum_{k \geq j} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi\,. \end{aligned}$$

In the first sum the M_k have been fixed but the factor $(1+|\xi|)^{(k-j)m}$ decays exponentially. Thus we can fix m such that the first sum is $< \frac{\epsilon_j}{2}$.

In the latter sum the $1/\delta_k$ and the M_k can be increased so that the total sum is $< \frac{\epsilon_j}{2}$. This implies the validity of (1) for this particular j and it remains valid for $0, 1, \ldots j-1$. Now we fix δ_j and M_j.

This completes the induction. With this construction of $\{\delta\}$, $\{M\}$ we have proved that $W(\{\delta\},\{M\}) \subset V(\{\epsilon\},\{N\})$. This proves Theorem 3.

Differential Operators with Constant Coefficients

The description of the topology of $\mathcal{D}$ in terms of the range $\widetilde{\mathcal{D}}$ given in Theorem 3 has important consequences for solvability of differential equations on $\mathbf{R}^n$ with constant coefficients.

Theorem 4. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant coefficients. Then the mapping $f \to \mathcal{D}f$ is a homeomorphism of $\mathcal{D}$ onto $D\mathcal{D}$.*

Proof: This proof was shown to me by Hörmander in 1972. A related proof appears in Ehrenpries, *loc. cit.*

It is clear from Theorem 2 that the mapping $f \to Df$ is injective on $\mathcal{D}$. The continuity is also obvious.

For the continuity of the inverse we need the following simple lemma.

Lemma 5. *Let $P \neq 0$ be a polynomial of degree m, F an entire function on $\mathbf{C}^n$ and $G = PF$. Then*

$$|F(\zeta)| \leq C \sup_{|z|\leq 1} |G(z+\zeta)|, \quad \zeta \in \mathbf{C}^n\,,$$

where C is a constant.

Proof: Suppose first $n = 1$ and that $P(z) = \sum_0^m a_k z^k (a_m \neq 0)$. Let $Q(z) = z^m \sum_0^m \overline{a}_k z^{-k}$. Then, by the maximum principle,

$$|a_m F(0)| = |Q(0)F(0)| \leq \max_{|z|=1} |Q(z)F(z)| = \max_{|z|=1} |P(z)F(z)| . \tag{22}$$

For general n let A be an $n \times n$ complex matrix, mapping the ball $|\zeta| < 1$ in $\mathbf{C}^n$ into itself and such that

$$P(A\zeta) = a\zeta_1^m + \sum_0^{m-1} P_k(\zeta_2, \dots, \zeta_n)\zeta_1^k, \quad a \neq 0 .$$

Let

$$F_1(\zeta) = F(A\zeta), \quad G_1(\zeta) = G(A\zeta), \quad P_1(\zeta) = P(A\zeta) .$$

Then

$$G_1(\zeta_1 + z, \zeta_2, \dots, \zeta_n) = F_1(\zeta_1 + z, \zeta_2, \dots, \zeta_n) P_1(\zeta_1 + z, \zeta_2, \dots, \zeta_n)$$

and the polynomial

$$z \to P_1(\zeta_1 + z, \dots, \zeta_n)$$

has leading coefficient a. Thus by (22)

$$|aF_1(\zeta)| \leq \max_{|z|=1} |G_1(\zeta_1 + z, \zeta_2, \dots, \zeta_n)| \leq \max_{\substack{z \in \mathbf{C}^n \\ |z| \leq 1}} |G_1(\zeta + z)| .$$

Hence by the choice of A

$$|aF(\zeta)| \leq \sup_{\substack{z \in \mathbf{C}^n \\ |z| \leq 1}} |G(\zeta + z)|$$

proving the lemma.

For Theorem 4 it remains to prove that if V is a convex neighborhood of 0 in $\mathcal{D}$ then there exists a convex neighborhood W of 0 in $\mathcal{D}$ such that

$$f \in \mathcal{D},\ Df \in W \Rightarrow f \in V . \tag{23}$$

We take V as the neighborhood $W(\{\delta\}, \{M\})$. We shall show that if $W = W(\{\epsilon\}, \{M\})$ (same $\{M\}$) then (26) holds provided the ϵ_j in $\{\epsilon\}$ are small enough. We write $u = Df$ so $\widetilde{u}(\zeta) = P(\zeta)\widetilde{f}(\zeta)$ where P is a polynomial. By Lemma 5

$$|\widetilde{f}(\zeta)| \leq C \sup_{|z| \leq 1} |\widetilde{u}(\zeta + z)| . \tag{24}$$

But $|z| \leq 1$ implies

$$(1 + |z + \zeta|)^{-M_j} \leq 2^{M_j}(1 + |\zeta|)^{-M_j}, \quad |\operatorname{Im}(z + \zeta)| \leq |\operatorname{Im} \zeta| + 1 ,$$

so if $C 2^{M_j} e^j \epsilon_j \leq \delta_j$ then (23) holds.

Q.e.d.

Corollary 6. *Let* $D \neq 0$ *be a differential operator on* $\mathbf{R}^n$ *with constant (complex) coefficients. Then*

$$D\,\mathcal{D}' = \mathcal{D}'\,. \tag{25}$$

In particular, there exists a distribution T *on* $\mathbf{R}^n$ *such that*

$$DT = \delta\,. \tag{26}$$

Definition A distribution T satisfying (26) is called a *fundamental solution* for D.

To verify (25) let $L \in \mathcal{D}'$ and consider the functional $D^*u \to L(u)$ on $D^*\mathcal{D}$ ($*$ denoting adjoint). Because of Theorem 2 this functional is well defined and by Theorem 4 it is continuous. By the Hahn-Banach theorem it extends to a distribution $S \in \mathcal{D}'$. Thus $S(D^*u) = Lu$ so $DS = L$, as claimed.

Corollary 7. *Given* $f \in \mathcal{D}$ *there exists a smooth function* u *on* $\mathbf{R}^n$ *such that*

$$Du = f\,.$$

In fact, if T is a fundamental solution one can put $u = f * T$.