LINEAR ALGEBRA
FOR
EVERYONE

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WELLESLEY - CAMBRIDGE PRESS
Box 812060 Wellesley MA 02482

Linear Algebra for Everyone

ISBN 978-1-7331466-3-0

LaTeX typesetting by Ashley C. Fernandes (info@problemsolvingpathway.com)
Printed in the United States of America

9 8 7 6 5 4 3 2 1

QA184.2 .S773 2020 | DDC 512/.5–dc23

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1.3 Matrices and Column Spaces

1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a **3 by 2 matrix**: $m = 3$ rows and $n = 2$ columns. Rank 2.

2 The 3 components of $A\boldsymbol{x}$ are dot products of the 3 rows of A with the vector $\boldsymbol{x}$:

Row at a time
A* times *x

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 8 \\ 3 \cdot 7 + 4 \cdot 8 \\ 5 \cdot 7 + 6 \cdot 8 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}.$$

3 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ is also a **combination of the columns** $\quad A\boldsymbol{x} = \mathbf{7} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \mathbf{8} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

4 The **column space of *A*** contains **all combinations** $A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ of the columns.

5 **Rank one matrices**: All columns of A (and all combinations $A\boldsymbol{x}$) lie on one line.

Sections 1.1 and 1.2 explained the mechanics of vectors—linear combinations, dot products, lengths, and angles. We have vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ and every $\mathbf{R}^n$.

Section 1.3 begins the algebra of m by n matrices: our true goal. A typical matrix A is a rectangle of m times n numbers—$\boldsymbol{m}$ **rows and** $\boldsymbol{n}$ **columns**. If m equals n then A is a "square matrix". The examples below are 3 by 3 matrices.

$$\begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{2} & 0 & 0 \\ 0 & \mathbf{4} & 0 \\ 0 & 0 & \mathbf{5} \end{bmatrix} \qquad \begin{bmatrix} 2 & \mathbf{1} & \mathbf{-3} \\ \mathbf{0} & 4 & 7 \\ \mathbf{0} & \mathbf{0} & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & \mathbf{1} & \mathbf{-3} \\ \mathbf{1} & 4 & \mathbf{7} \\ \mathbf{-3} & \mathbf{7} & 5 \end{bmatrix}$$

Identity matrix — **Diagonal** matrix — **Triangular** matrix — **Symmetric** matrix

We often think of the columns of A as vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. Each of those n vectors is in m-dimensional space. In this example the $\boldsymbol{a}$'s have $m = 3$ components each:

$\boldsymbol{m} = 3$ rows
$\boldsymbol{n} = 4$ columns
3 by 4 matrix

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{-1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

This example is a "difference matrix" because multiplying A times $\boldsymbol{x}$ produces a vector $A\boldsymbol{x}$ of differences. *How does an m by n matrix A multiply an n by 1 vector $\boldsymbol{x}$?* There are two ways to the same answer—we work with the rows of A or we work with the columns.

The **row picture** of $A\boldsymbol{x}$ will come from **dot products** of $\boldsymbol{x}$ with the rows of A.
The **column picture** will come from **linear combinations** of the columns of A.

Row picture of $A\boldsymbol{x}$ Each row of A multiplies the column vector $\boldsymbol{x}$. Those multiplications *row times column* are dot products! The first dot product comes from row 1 of A:

$$(\textbf{row 1}) \cdot \boldsymbol{x} = (-1, 1, 0, 0) \cdot (x_1, x_2, x_3, x_4) = \boldsymbol{x_2} - \boldsymbol{x_1}.$$

It takes m times n small multiplications to find the $m = 3$ dot products that go into $A\boldsymbol{x}$.

Three dot products
$$A\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \text{row } 1 \cdot \boldsymbol{x} \\ \text{row } 2 \cdot \boldsymbol{x} \\ \text{row } 3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x_2} - \boldsymbol{x_1} \\ \boldsymbol{x_3} - \boldsymbol{x_2} \\ \boldsymbol{x_4} - \boldsymbol{x_3} \end{bmatrix} \quad (1)$$

Notice well that each row of A has the same number of components as the vector $\boldsymbol{x}$. Four columns multiply x_1 to x_4. Otherwise multiplying $A\boldsymbol{x}$ would be impossible.

Column picture of $A\boldsymbol{x}$ The matrix A times the vector $\boldsymbol{x}$ is a **combination of the columns of $\boldsymbol{A}$**. The n columns are multiplied by the n numbers in $\boldsymbol{x}$. Then add those column vectors $x_1\boldsymbol{a}_1, \ldots, x_n\boldsymbol{a}_n$ to find the vector $A\boldsymbol{x}$:

$$\boxed{A\boldsymbol{x} = x_1(\text{column } \boldsymbol{a}_1) + x_2(\text{column } \boldsymbol{a}_2) + x_3(\text{column } \boldsymbol{a}_3) + x_4(\text{column } \boldsymbol{a}_4)} \quad (2)$$

This combination of n columns involves exactly the same multiplications as dot products of $\boldsymbol{x}$ with the m rows. But it is higher level! We have a vector equation instead of three dot products. You see the same $A\boldsymbol{x}$ in equations (1) and (3).

Combination of columns
$$A\boldsymbol{x} = x_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x_2} - \boldsymbol{x_1} \\ \boldsymbol{x_3} - \boldsymbol{x_2} \\ \boldsymbol{x_4} - \boldsymbol{x_3} \end{bmatrix} \quad (3)$$

Let me admit something right away. If I have numbers in A and $\boldsymbol{x}$, and I want to compute $A\boldsymbol{x}$, then I tend to use dot products: the row picture. But if I want to *understand* $A\boldsymbol{x}$, the column picture is better. "The column vector $A\boldsymbol{x}$ is a combination of the columns of A."

We are aiming for a picture of not just one combination $A\boldsymbol{x}$ of the columns (from a particular $\boldsymbol{x}$). What we really want is a picture of **all combinations of the columns** (from multiplying A by all vectors $\boldsymbol{x}$). This figure shows one combination $2\boldsymbol{a}_1 + \boldsymbol{a}_2$ and then it tries to show the plane of all combinations $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ (for every x_1 and x_2).

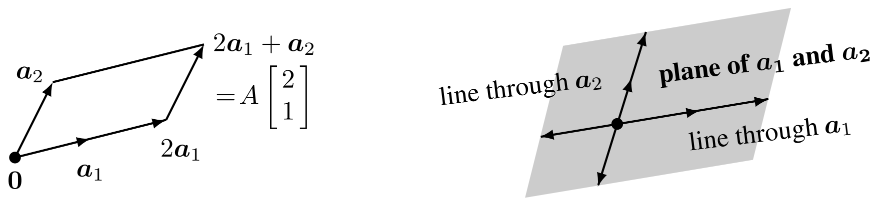

Figure 1.10: A linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. All linear combinations fill a plane.

The next important words are **independence**, **dependence**, and **column space**.

Here is a key point! Columns of A might not contribute anything new. They might be combinations of earlier columns (which we already included). Examples 1 and 2 show columns that give a new direction, and columns that are combinations of previous columns.

Example 1 **Independent columns** $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$ *Each column gives a new direction.* Their combinations fill 3D space $\mathbf{R}^3$.

If we look at all combinations of the columns, we see all vectors (b_1, b_2, b_3): *3D space*. The first column $x_1(1,2,3)$ allows us to match any number b_1. Then $x_2(0,4,5)$ leaves b_1 alone and we can match any number b_2. Finally $x_3(0,0,6)$ doesn't touch b_1 and b_2 and allows us to match any b_3. We have found x_1, x_2, x_3 so that $A_1\boldsymbol{x} = \boldsymbol{b}$.

Independence means: The only combination of columns that produces $A\boldsymbol{x} = (0,0,0)$ is $\boldsymbol{x} = (0,0,0)$. The columns are **independent** when each new column is a vector that we don't already have as a combination of previous columns. That word "independent" will be important.

Example 2 **Dependent columns** $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 6 & 0 & 6 \end{bmatrix}$ **Column 1 $+$ column 2 $=$ column 3** **Their combinations don't fill 3D space** $1+2=3$ $1+4=5$ $6+0=6$

The opposite of independent is "*dependent*". These three columns of A_2 are **dependent**. Column 3 is in the plane of columns 1 and 2. Nothing new from column 3.

I usually test independence going from left to right. The column $(1,1,6)$ is no problem. Column 2 is *not* a multiple of column 1 and $(2,4,0)$ gives a new direction. But column 3 is the sum of columns 1 and 2. The third column vector $(3,5,6)$ is not independent of $(1,1,6)$ and $(2,4,0)$. We only have two independent columns.

If I went from right to left, I would start with independent columns 3 and 2. Then column 1 is a combination (column 3 minus column 2). Either way we find that the three columns are in the **same plane**: two independent columns produce a plane in 3D.

That plane is the column space of this matrix: Plane $=$ all combinations of the columns.

Dependent columns in Example 2 $\boxed{\text{column } 1 + \text{column } 2 - \text{column } 3 \text{ is } (\mathbf{0},\mathbf{0},\mathbf{0}).}$

Example 3 $A_3 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 15 & 20 \end{bmatrix}$ Now $\boldsymbol{a}_2$ is 3 times $\boldsymbol{a}_1$. And $\boldsymbol{a}_3$ is 4 times $\boldsymbol{a}_1$. **Every pair of columns is dependent.**

This example is important. You could call it an extreme case. All three columns of A_3 lie on the **same line** in 3-dimensional space. That line consists of all column vectors $(c, 2c, 5c)$—all the multiples of $(1,2,5)$. Notice that $c = 0$ gives the point $(0,0,0)$.

That line in 3D is the column space for this matrix $\boldsymbol{A_3}$. The line contains all vectors $A_3\boldsymbol{x}$. By allowing every vector $\boldsymbol{x}$, we fill in the column space of A_3—and here we only filled one line. That is almost the smallest possible column space.

$\boxed{\text{The } \textbf{column space of } \boldsymbol{A} \text{ is the set of all vectors } A\boldsymbol{x} : \textbf{All combinations of the columns}.}$

Thinking About the Column Space of A

"Vector spaces" are a central topic. Examples are coming unusually early. They give you a chance to see what linear algebra is about. The combinations of all columns produce the column space, but you only need r independent columns. So we start with column 1, and go **from left to right** in identifying independent columns. Here are two examples A_4 and A_5.

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_5 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} \end{bmatrix}$$

A_4 has four independent columns. For example, column 4 is not a combination of columns $1, 2, 3$. There are no dependent columns in A_4. Triangular matrices like A_4 are easy *provided the main diagonal has no zeros*. Here the diagonal is $1, 1, 1, 1$.

A_5 is not so easy. Columns 1 and 2 and 3 are independent. The big question is whether column 4 is independent—or is it a combination of columns $1, 2, 3$? To match the final 1 in column 4, that combination will have to start with column 1.

To cancel the 1 in the top left corner of A_5, we need *minus* the second column. Then we need *plus* column 3 so that -1 and $+1$ in row 2 will also cancel. Now we see what is true about this matrix A_5 :

$$\textbf{Column 4 of } \boldsymbol{A_5} = \textbf{Column 1} - \textbf{Column 2} + \textbf{Column 3.} \tag{4}$$

So column 4 of A_5 is *a combination of* columns $1, 2, 3$. A_5 has only 3 independent columns.

The next step is to "visualize" the column space—all combinations of the four columns. That word is in quotes because the task may be impossible. I don't think that drawing a 4-dimensional figure would help (possibly this is wrong). The first matrix A_4 is a good place to start, because its column space is the full 4-dimensional space $\mathbf{R}^4$.

Do you see why $\mathbf{C}(A_4) = \mathbf{R}^4$? If we look to algebra, we see that every vector $\boldsymbol{v}$ in $\mathbf{R}^4$ is a combination of the columns. By writing $\boldsymbol{v}$ as (v_1, v_2, v_3, v_4), we can literally show the exact combination that produces every vector $\boldsymbol{v}$ from A_4 :

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = (v_1 - v_2)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (v_2 - v_3)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (v_3 - v_4)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (v_4)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{5}$$

This says that $\boldsymbol{v}$ is a combination of the columns. More than that, equation (5) shows what the combination is. **We have solved the four equations $\boldsymbol{A_4 x} = \boldsymbol{v}$!** The four unknowns in $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ are now known in the four parentheses of equation (5).

Geometrically, every vector $\boldsymbol{v}$ is a combination of the 4 columns of A_4. Here is one way to look at A_4. The first column $(1, 0, 0, 0)$ is responsible for a line in 4-dimensional space. That line contains every vector $(c_1, 0, 0, 0)$. The second column is responsible for another line, containing every vector $(c_2, c_2, 0, 0)$. *If you add every vector* $(c_1, 0, 0, 0)$ *to every vector* $(c_2, c_2, 0, 0)$, you get a 2-dimensional plane inside 4-dimensional space.

That was the first two columns. The main rule of linear algebra is *keep going*. The last two columns give two more directions in $\mathbf{R}^4$, and they are *independent of the first two*. At the end, equation (5) shows how **every 4-dimensional vector is a combination of the four columns of A_4**. The column space of A_4 is all of $\mathbf{R}^4$.

If we attempt the same plan for the matrix A_5, the first 3 columns cooperate. But column 4 of A_5 is a combination of columns $1, 2, 3$. Those three columns combine to give a *three-dimensional subspace* inside $\mathbf{R}^4$. Column 4 happens to be *in that subspace*.

That three-dimensional subspace is the whole column space $\mathbf{C}(A_5)$. We can only solve $A_2\boldsymbol{x} = \boldsymbol{v}$ when $\boldsymbol{v}$ is in $\mathbf{C}(A_5)$. The matrix A_5 only has three independent columns.

I always write $\mathbf{C}(A)$ for the column space of A. When A has m rows, the columns are vectors in m-dimensional space $\mathbf{R}^m$. The column space might fill all of $\mathbf{R}^m$ or it might not. For $m = 3$, here are all four possibilities for column spaces in 3-dimensional space:

1.	The whole space $\mathbf{R}^3$	3 independent columns
2.	A plane in $\mathbf{R}^3$ going through $(0,0,0)$	2 independent columns
3.	A line in $\mathbf{R}^3$ going through $(0,0,0)$	1 independent column
4.	The single point $(0,0,0)$ in $\mathbf{R}^3$ (when A is a matrix of zeros !)	

Here are simple matrices to show those four possibilities for the column space $\mathbf{C}(A)$:

$$\begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{C}(A) = \mathbf{R}^3 = xyz$ space $\quad \mathbf{C}(A) = xy$ plane $\quad \mathbf{C}(A) = x$ axis $\quad \mathbf{C}(A) =$ one point $(0,0,0)$

Author's note The words "column space" have not appeared in Chapter 1 of my previous books. I thought the idea of a *space* was too important to come so soon. Now I think that the best way to understand such an important idea is to see it early and often. It is examples more than definitions that make ideas clear—in mathematics as in life.

Here is a succession of questions. With practice in the next section 1.4, you will find the keys to the answers. They give a real understanding of any matrix A.

1. How many columns of A are independent? That number r is the **"rank"** of A.

2. Which are the first r independent columns? They are a **"basis"** for the column space.

3. What combinations of those r basic columns produce the remaining $n - r$ columns?

4. Write A as an m by r column matrix C times an r by n matrix R : $\boldsymbol{A = CR}$.

5. (Amazing) The r rows of R are a basis for the **row space** of A: combinations of rows.

Section I.4 will explain how to multiply those matrices C and R. The result is $A = CR$. C contains columns from A. Please notice that the rows of R *do not* come directly from A.

Matrices of Rank One

Now we come to the building blocks for all matrices. *Every matrix of rank r is the sum of r matrices of rank one*. For a rank one matrix, all column vectors lie along the same line. That line through $(0, 0, 0)$ is the whole column space of the rank one matrix.

Example $A_6 = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 12 & -8 \\ 2 & 6 & -4 \end{bmatrix}$ has rank $\boldsymbol{r = 1}$. All columns : same direction !

Columns 2 and 3 are multiples of the first column $\boldsymbol{a}_1 = (1, 4, 2)$. Column 2 is $3\boldsymbol{a}_1$ and column 3 is $-2\boldsymbol{a}_1$. The column space $\mathbf{C}(A_6)$ is only the line of all vectors $c\boldsymbol{a}_1 = (c, 4c, 2c)$.

Here is a wonderful fact about any rank one matrix. You may have noticed the rows of A_6. **All the rows are multiples of one row**. When the column space is a single line in m-dimensional space, the row space is a single line in n-dimensional space. All rows of this matrix A_6 are multiples of _____ .

An example like A_6 raises a basic question. **If all columns are in the same direction, why does it happen that all rows are in the same direction ?** To find an answer, look first at this 2 by 2 matrix. Column 2 is m times column 1 so the column rank is 1.

$$A = \begin{bmatrix} a & ma \\ b & mb \end{bmatrix} \qquad \text{Is row 2 a multiple of row 1 ?}$$

Yes ! The second row (b, mb) is $\frac{b}{a}$ times the first row (a, ma). If the column rank is 1, then the row rank is 1. To cover every possibility we have to check the case when $a = 0$. Then the first row $\begin{bmatrix} 0 & 0 \end{bmatrix}$ is 0 times row 2. So the row space is the line through row 2.

Our 2 by 2 proof is complete. Let me look next at this 3 by 3 matrix of rank 1 :

$$A = \begin{bmatrix} a & ma & pa \\ b & mb & pb \\ c & mc & pc \end{bmatrix} \qquad \begin{array}{l} \text{Column 2 is } m \text{ times column 1} \\ \text{Column 3 is } p \text{ times column 1} \\ \text{Rows 2 and 3 are } b/a \text{ and } c/a \text{ times row 1} \end{array}$$

This matrix does not have two independent columns. Is it the same for the rows of A ? Is row 2 in the same direction as row 1 ? *Yes*. Is row 3 in the same direction as row 1 ? *Yes*. *The rule still holds*. The row rank of this A is also 1 (equal to the column rank).

Let me jump from rank one matrices to all matrices. At this point we could make a guess : It looks possible that **row rank equals column rank for every matrix**. If A has r independent columns, then A has r independent rows. A wonderful fact !

I believe that this is the first great theorem in linear algebra. So far we have only seen the case of rank one matrices. The next section 1.4 will explain matrix multiplication AB and lead us toward an understanding of "row rank = column rank" for all matrices.

Problem Set 1.3

This chapter introduces column spaces. But we don't yet have a computational system to decide independence or dependence of column vectors. So these problems stay with whole numbers and small matrices.

1 Describe the column space of these matrices: a point, a line, a plane, all of 3D.

$$A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 5 & 6 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 5 \\ 2 & 10 \\ 1 & 5 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2 Find a combination of the columns that produces $(0,0,0)$: column space = *plane*.

$$\text{Dependent columns} \qquad A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

3 Describe the column spaces in $\mathbf{R}^3$ of B and C :

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} B & -B \end{bmatrix} \qquad \text{(3 by 4 block matrix)}$$

4 Multiply $A\boldsymbol{x}$ and $B\boldsymbol{y}$ and $I\boldsymbol{z}$ using dot products as in (rows of A) $\cdot\, \boldsymbol{x}$:

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \qquad B\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \\ 10 \end{bmatrix} \qquad I\boldsymbol{z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

5 Multiply the same A times $\boldsymbol{x}$ and B times $\boldsymbol{y}$ and I times $\boldsymbol{z}$ using combinations of the columns of A and B and I, as in $A\boldsymbol{x} = 1(\text{column } 1) + 2(\text{column } 2) + 5(\text{column } 3)$.

6 In Problem 4, how many independent columns does A have? How many independent columns in B? How many independent columns in $A+B$?

7 Can you find A and B (both with two independent columns) so that $A+B$ has
(a) 1 independent column (b) No independent columns (c) 4 independent columns

8 The "column space" of a matrix contains all combinations of the columns. Describe the column spaces in $\mathbf{R}^3$ of A and B and C :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 4 \end{bmatrix}$$

9 Find a 3 by 3 matrix A with 3 independent columns and all nine entries $= 1$ or 2. (What is the maximum possible number of 1's?)

10 Complete A and B so that they are rank one matrices. What are the column spaces of A and B ? What are the row spaces of A and B ?

$$A = \begin{bmatrix} 3 & \\ 5 & 15 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -5 \\ 4 & & \end{bmatrix}$$

11 Suppose A is a 5 by 2 matrix with columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. We include one more column to produce B $(5$ by $3)$. Do A and B have the same column space if
(a) the new column is the zero vector ? (b) the new column is $(1, 1, 1)$?
(c) the new column is the difference $\boldsymbol{a}_2 - \boldsymbol{a}_1$?

12 Explain this important sentence. It connects column spaces to linear equations.

$$\boxed{A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution vector } \boldsymbol{x} \text{ if the vector } \boldsymbol{b} \text{ is in the column space of } A.}$$

The equation $A\boldsymbol{x} = \boldsymbol{b}$ looks for a combination of columns of A that produces $\boldsymbol{b}$. What vector will solve $A\boldsymbol{x} = \boldsymbol{b}$ for these right hand sides $\boldsymbol{b}$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -2 \\ -2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

13 Find two 3 by 3 matrices A and B with the same column space $=$ the plane of all vectors perpendicular to $(1, 1, 1)$. What is the column space of $A + B$?

14 Which numbers q would leave A with two independent columns ?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & q \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & q \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & q \end{bmatrix}$$

15 Suppose A times $\boldsymbol{x}$ equals $\boldsymbol{b}$. If you add $\boldsymbol{b}$ as an extra column of A, explain why the rank r (number of independent columns) stays the same.

16 *True or false*

(a) If the 5 by 2 matrices A and B have independent columns, so does $A + B$.

(b) If the m by n matrix A has independent columns, then $m \geq n$.

(c) A random 3 by 3 matrix almost surely has independent columns.

17 If A and B have rank 1, what are the possible ranks of $A + B$? Give an example of each possibility.

18 Find the linear combination $3\boldsymbol{s}_1 + 4\boldsymbol{s}_2 + 5\boldsymbol{s}_3 = \boldsymbol{b}$. Then write $\boldsymbol{b}$ as a matrix-vector multiplication $S\boldsymbol{x}$, with $3, 4, 5$ in $\boldsymbol{x}$. Compute the three dot products (row of S) $\cdot\, \boldsymbol{x}$:

$$\boldsymbol{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ go into the columns of } S\,.$$

19 Solve these equations $S\boldsymbol{y} = \boldsymbol{b}$ with $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ in the columns of the sum matrix S:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The sum of the first 3 odd numbers is _____. The sum of the first 10 is _____.

20 Solve these three equations for y_1, y_2, y_3 in terms of c_1, c_2, c_3:

$$S\boldsymbol{y} = \boldsymbol{c} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Write the solution $\boldsymbol{y}$ as a matrix A times the vector $\boldsymbol{c}$. A is the "*inverse matrix*" S^{-1}. Are the columns of S independent or dependent?

21 The three rows of this square matrix A are dependent. Then linear algebra says that the three columns must also be dependent. Find $\boldsymbol{x}$ in $A\boldsymbol{x} = \mathbf{0}$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 4 & 7 & 9 \end{bmatrix} \qquad \begin{array}{l} \text{Row 1} + \text{ row 2} = \text{ row 3} \\ \text{Two independent rows} \\ \text{Then only two independent columns} \end{array}$$

22 Which numbers c give dependent columns? Then a combination of columns is zero.

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} c & 1 \\ 4 & c \end{bmatrix}$$

23 If the columns combine into $A\boldsymbol{x} = \mathbf{0}$ then each row of A has $\textbf{row} \cdot \boldsymbol{x} = 0$:

$$\text{If} \quad \begin{bmatrix} & & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{then by rows} \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?