NICE FUNCTIONS

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I. Linear Differential Equations

In teaching differential equations, we constantly meet exponentials and polynomials and products like $(t + 1)e^{2t}$. Sines and cosines are included, coming from complex exponentials $e^{i\omega t}$. These functions appear everywhere we look—as solutions $y(t)$, as right hand sides $f(t)$, and as inverse Laplace transforms. The equations themselves are linear with constant coefficients:

$$a_N \frac{d^N y}{dt^N} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t) \tag{1}$$

Then the nice functions appear in all the crucial places:

1. They solve the homogeneous equation (with $f(t) = 0$).

2. They are the successful functions $f(t)$ in the method of undetermined coefficients: Look for $y(t)$ in a finite-dimensional space that contains $f(t)$, and match the left side of equation (1) to $f(t)$ within that space.

    For $f(t) = e^{st}$, the space contains the multiples $y(t) = Ae^{st}$. In this one-dimensional space, the equation gives

$$(a_N s^N + \cdots + a_1 s + a_0)Ae^{st} = e^{st} \tag{2}$$

If s is not a root of that characteristic polynomial $C(s)$, the number A is determined:

$$y(t) = Ae^{st}, \ A = \ \textit{transfer function} \ = \frac{1}{C(s)}.$$

    When s is an m-fold root of $C(s)$, we find $y(t)$ in the space spanned by $e^{st}, te^{st}, \ldots, t^m e^{st}$. *A crucial property is that the function space is invariant under differentiation.* Then the method of undetermined coefficients can succeed. We will identify all such spaces.

3. The Laplace transform of (1) is

$$C(s)Y(s) - (\text{polynomial } Q(s)) = F(s) \tag{3}$$

$Y(s)$ and $F(s)$ are the transforms of $y(t)$ and $f(t)$, and $Q(s)$ has degree less than N. Q comes from the initial conditions on $y, dy/dt, \ldots, d^{N-1}y/dt^{N-1}$.

When $F \equiv 0$, the transform $Y(s)$ is a rational function Q/C with $Y(\infty) = 0$. This is the homogeneous case **1** above.

When $F(s)$ is a rational function with $F(\infty) = 0$, the same is true of $Y(s)$. This will be the inhomogeneous case **2**.

When $f(t)$ is the delta function $\delta(t)$, its transform is $F(s) = 1$. Again $Y(s)$ is rational with $Y(\infty) = 0$. Then the inverse transform—which will be the fundamental solution, the Green's function, the impulse response—is again a nice function. We now describe those functions, and the spaces invariant under differentiation.

II. FINITE-DIMENSIONAL SPACES OF NICE FUNCTIONS

Each complex exponent r and polynomial degree $m - 1 = 0, 1, 2, \ldots$ produce an m-dimensional space of polynomials times e^{rt} :

$$N(r, m) = \left\{ (c_{m-1}t^{m-1} + \cdots + c_1 t + c_0)e^{rt} \right\}. \tag{4}$$

Key point: The derivative of a function in $N(r, m)$ is also in $N(r, m)$. The theorem we want (proved below and probably not original) is that the direct sums of these subspaces $N(r, m)$ produce all the finite-dimensional function spaces invariant under differentiation.

A function is "nice" if it is a combination of polynomials times exponentials. Thus it is a sum $z_1 + \cdots + z_j$ with z_i in $N(r_i, m_i)$.

The homogeneous equation. The N-dimensional subspace of homogeneous solutions is the sum of subspaces

$$H = N(r_1, m_1) \oplus \cdots \oplus N(r_k, m_k)$$

where $r_1, \ldots, r_k$ are the roots of the characteristic polynomial $C(s)$ with multiplicities $m_1, \ldots, m_k$ adding to N. Thus all homogeneous solutions are nice functions. We take this as known.

The inhomogeneous equation. Suppose $f(t)$ is in a space of nice functions: a sum of polynomials times exponentials $e^{s_1 t}, \ldots, e^{s_j t}$:

$$L = N(s_1, n_1) \oplus \cdots \oplus N(s_j, n_j). \tag{5}$$

In most cases we look for a particular solution $y(t)$ in the same space L. The "undetermined coefficients" described in class are the coefficients of $y(t)$ in the natural polynomial-exponential basis for L :

$$y(t) = \sum_{i=0}^{n_1-1} b_{i1} t^i e^{s_1 t} + \cdots + \sum_{i=0}^{n_j-1} b_{ij} t^i e^{s_j t}. \tag{6}$$

Substitute this $y(t)$ into the differential equation (1). With a similar expression for $f(t)$ on the right hand side, match terms to determine the coefficients b.

In case an exponent s in $f(t)$ coincides with a root r of the characteristic polynomial, this is "resonance." The system of linear equations for the coefficients b will be singular. To reach a nonsingular system, increase the polynomial degrees for that exponent s by its multiplicity m as a root of $C(s)$. An extreme example of resonance will make the point:

$$\frac{d^N y}{dt^N} = 1 + t \tag{7}$$

The characteristic polynomial is $C(s) = s^N$ with root $r = 0$ and multiplicity N. The right side $1 + t$ has the same exponent 0. So we add N to the degrees:

Look for $y = b_0 t^N + b_1 t^{N+1}$.   Find $y = \dfrac{1}{N!} t^N + \dfrac{1}{(N+1)!} t^{N+1}$.

This method of undetermined coefficients gives students a direct approach to the solution $y(t)$.

In applying the method to nice functions in L, are we overlooking anyone? Is there another finite dimensional space invariant under differentiation? If so, the same method would solve the inhomogeneous equation (1) by a function $y(t)$ in that space.

Our small result is that no spaces other than L can succeed. The method applies only to nice functions $f(t)$. That limitation (found in all textbooks) is correct. A broad hint by Mike Artin suggested the proof we give here.

Theorem. *The only finite-dimensional spaces invariant under differentiation are spaces of nice functions.*

Proof. If $y(t)$ lies in such a space of dimension d, then its derivatives $y'(t), y''(t), \ldots, y^{(d)}(t)$ also lie in the space. These $d+1$ functions must be linearly dependent. So $y(t)$ solves a homogeneous linear equation of order d with constant coefficients. Therefore $y(t)$ is a nice function. $\square$

For the operation of differentiation on the space L in (5), the eigenvectors are the exponentials e^{st}. If L also includes te^{st}, this is a "generalized eigenvector" with eigenvalue $\lambda = s$. Then $e^{st}, te^{st}, \ldots, t^{n-1}e^{st}$ is a chain corresponding to a block of size n in the Jordan form for the operation of differentiation on L. The entire Jordan form is a direct sum of blocks with eigenvalues $s_1, \ldots, s_j$ and block sizes $n_1, \ldots, n_j$.

III. The Laplace transforms of nice functions

The transform of an exponential e^{at} is $1/(s-a)$. The transform of $t^n e^{at}$ is

$$\int_0^\infty t^n e^{at} e^{-st} dt = \frac{n!}{(s-a)^{n+1}} \tag{8}$$

Then the transform of the nice function $y(t)$ in (6) is

$$Y(s) = \sum_{i=0}^{n_1-1} b_{i1} \frac{i!}{(s-s_1)^{i+1}} + \cdots + \sum_{i=0}^{n_j-1} b_{ij} \frac{i!}{(s-s_j)^{i+1}}. \tag{9}$$

This is the partial fraction representation of $Y(s)$. It is a rational function of s and it approaches $Y(\infty) = 0$ as $|s| \to \infty$.

Our observation (in no way new) is that the Laplace transform maps the space L into the space of rational functions with poles $s_1, \ldots, s_j$ of orders $n_1, \ldots, n_j$:

$$Y(s) = \frac{\text{polynomial of degree } n_1 + \cdots + n_j - 1}{(s-s_1)^{n_1} \ldots (s-s_j)^{n_j}}. \tag{10}$$

Thus the Laplace transforms of nice functions are rational functions with $Y(\infty) = 0$.

Our small point is that the fundamental solution $g(t)$ of a constant coefficient linear equation is *"piecewise nice"*. It is zero for $t < 0$ and it solves

$$a_N \frac{d^N g}{dt^N} + \cdots + a_1 \frac{dg}{dt} + a_0 g = \delta(t). \tag{11}$$

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The Laplace transform is $(a_N s^N + \cdots + a_1 s + a_0)G = 1$. So $G(s) = 1/C(s)$ is a rational function and the Green's function $g(t)$ is nice.

For second order equations this is $G(s) = 1/a_2(s - r_1)(s - r_2)$. The inverse transform is

$$g(t) = \frac{1}{a_2(r_1 - r_2)}(e^{r_1 t} - e^{r_2 t}).$$

This same Green's function $g(t)$ also solves the *homogeneous* equation with $g(0) = 0$ and $g'(0) = 1/a_2$.

Our new textbook [1] emphasizes that $g(t)$ leads to a particular solution of equation (1) for any right side $f(t)$:

$$y(t) = \int_0^t g(t - T) f(T) \, dT. \tag{12}$$

In engineering, $\delta(t)$ is an impulse and $g(t)$ is the *impulse response*. Its importance could hardly be overestimated.

REFERENCE

[1] Gilbert Strang, *Differential Equations and Linear Algebra,* Wellesley–Cambridge Press, 2014.