DIFFERENTIAL EQUATIONS
AND
LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

Gilbert Strang

Massachusetts Institute of Technology

Book Website	**math.mit.edu/dela**
Email address	**diffeqla@gmail.com**
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Problem Set 7.1, page 393

1 Suppose your pulse is measured at $b_1 = 70$ beats per minute, then $b_2 = 120$, then $b_3 = 80$. The least squares solution to three equations $v = b_1, v = b_2, v = b_3$ with $A^{\mathrm{T}} = [1 \ \ 1 \ \ 1]$ is $\widehat{v} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} =$ _____. Use calculus and projections:

(a) Minimize $E = (v-70)^2 + (v-120)^2 + (v-80)^2$ by solving $dE/dv = 0$.

Solution (a) $\frac{dE}{dv} = 2(v-70) + 2(v-120) + 2(v-80) = 0$ at the minimizing $\widehat{v}$.

Cancel the 2's: $3v = 70 + 120 + 80 = 270$ so $\widehat{v} = v_{\text{average}} = \mathbf{90}$

(b) Project $\boldsymbol{b} = (70, 120, 80)$ onto $\boldsymbol{a} = (1,1,1)$ to find $\widehat{v} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Solution (b) The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is $\boldsymbol{p} = \boldsymbol{a}\widehat{v}$:

$$\boldsymbol{b} = \begin{bmatrix} 70 \\ 120 \\ 80 \end{bmatrix} \qquad \boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \widehat{v} = \frac{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \frac{270}{3} = \mathbf{90}.$$

2 Suppose $Av = \boldsymbol{b}$ has m equations $a_iv = b_i$ in *one unknown* v. For the sum of squares $E = (a_1v - b_1)^2 + \cdots + (a_mv - b_m)^2$, find the minimizing $\widehat{v}$ by calculus. Then form $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ with one column in A, and reach the same $\widehat{v}$.

Solution To minimize E we solve $dE/dv = 0$. For $m = 3$ equations $a_iv = b_i$,

$$\frac{dE}{dv} = 2a_1(a_1v - b_1) + 2a_2(a_2v - b_2) + 2a_3(a_3v - b_3) = 0 \text{ is zero when}$$

$$v = \widehat{v} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_1^2 + a_2^2 + a_3^2} = \frac{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}}.$$

When A has one column, $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ is the same as $(\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a})\widehat{v} = (\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b})$.

3 With $\boldsymbol{b} = (4,1,0,1)$ at the points $x = (0,1,2,3)$ set up and solve the normal equation for the coefficients $\widehat{\boldsymbol{v}} = (C, D)$ in the nearest line $C + Dx$. Start with the four equations $A\boldsymbol{v} = \boldsymbol{b}$ that would be solvable if the points fell on a line.

Solution The unsolvable equation has $m = 4$ points on a line: only $n = 2$ unknowns.

$$A\boldsymbol{v} = \boldsymbol{b} \ \text{ is } \ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \ \text{ leading to } \ A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b} \ :$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \text{ gives } \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \frac{1}{2a}\begin{bmatrix} 60 \\ -20 \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-1} \end{bmatrix}$$

The closest line to the four points is $\boldsymbol{b} = \mathbf{3} - \boldsymbol{x}$.

4 In Problem 3, find the projection $\boldsymbol{p} = A\boldsymbol{v}$. Check that those four values lie on the line $C + Dx$. Compute the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and verify that $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$.

Solution The projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ is

$$\boldsymbol{p} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{with error } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

The best line $C + Dx = 3 - x$ does produce $\boldsymbol{p} = (3, 2, 1, 0)$ at the four points $x = 0, 1, 2, 3$.

Multiply this $\boldsymbol{e}$ by A^{T} to get $A^{\mathrm{T}}\boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as expected.

5 (Problem 3 by calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares : the last one is $(1 - C - 3D)^2$. Find the derivative equations $\partial E/\partial C = \partial E/\partial D = 0$. Divide by 2 to obtain $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution Minimize $E = (4 - C)^2 + (1 - C - D)^2 + (-C - 2D)^2 + (1 - C - 3D)^2$.

The partial derivatives are $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$ at the minimum :

$$-2(4 - C) - 2(1 - C - D) - 2(-C - 2D) - 2(1 - C - 3D) = 0$$

$$-2(1 - C - D) - 4(-C - 2D) - 6(1 - C - 3D) = 0$$

Factoring out -2 and collecting terms this is the same equation $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$!

$$\begin{aligned} 6 - 4C - \ \ 6D &= 0 \\ 4 - 6C - 14D &= 0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

6 For the closest parabola $C + Dt + Et^2$ to the same four points, write down 4 unsolvable equations $A\boldsymbol{v} = \boldsymbol{b}$ for $\boldsymbol{v} = (C, D, E)$. Set up the normal equations for $\widehat{\boldsymbol{v}}$. If you fit the best cubic $C + Dt + Et^2 + Ft^3$ to those four points (thought experiment), what is the error vector $\boldsymbol{e}$?

Solution The parabola $C + Dt + Et^2$ fits the 4 points exactly if $A\boldsymbol{v} = \boldsymbol{b}$:

$$\begin{aligned} t &= 0 & C + 0D + 0E &= 4 \\ t &= 1 & C + 1D + 1E &= 1 \\ t &= 2 & C + 2D + 4E &= 0 \\ t &= 3 & C + 3D + 9E &= 1 \end{aligned} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \qquad .\phi A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 4+1+0+1 \\ 0+1+0+3 \\ 0+1+0+9 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}.$$

The cubic $C + Dt + Et^2 + Ft^3$ can fit 4 points exactly, with **error = zero vector**.

7 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

Solution $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

8 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 7. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$.

Solution $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

9 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 8. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

Solution If $\boldsymbol{b}$ = previous error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection of $\boldsymbol{b}$ is $\boldsymbol{p} = \boldsymbol{0}$.

10 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is _____.

Solution If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \boldsymbol{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

11 Find the best line $C + Dt$ to fit $\boldsymbol{b} = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Solution The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^{\mathrm{T}}A$

12 Find the *plane* that gives the best fit to the 4 values $\boldsymbol{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. At those 4 points, the equations $C + Dx + Ey = b$ are $A\boldsymbol{v} = \boldsymbol{b}$ with 3 unknowns $\boldsymbol{v} = (C, D, E)$.

Solution $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$.

The solution $(C, D, E) = (2, -1, \frac{3}{2})$ gives the best plane $2 - x - \frac{3}{2}y$.

13 With $\boldsymbol{b} = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ set up and solve the normal equations $A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b}$. For the best straight line $C + Dt$, find its four heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

Solution $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives
$E = \|\boldsymbol{e}\|^2 = \mathbf{44}$
$\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

14 (By calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Divide by 2 to obtain the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$.
These normal equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$ are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

15 Which of the four subspaces contains the error vector $\boldsymbol{e}$? Which contains p ? Which contains $\widehat{\boldsymbol{v}}$?

Solution The error $\boldsymbol{e}$ is contained in the nullspace $\boldsymbol{N}(A^{\mathrm{T}})$, since $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$. The projection $\boldsymbol{p}$ is contained in the column space $\boldsymbol{C}(A)$. The vector $\widehat{\boldsymbol{v}}$ of coefficients can be any vector in $\mathbf{R}^n$.

16 Find the height C of the best *horizontal line* to fit $\boldsymbol{b} = (0, 8, 8, 20)$. An exact fit would solve the four unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$ and $A^{\mathrm{T}} = [\,1 \;\; 1 \;\; 1 \;\; 1\,]$.

$A^{\mathrm{T}}A = [\,4\,]$. $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = \mathbf{9} =$ best C. $\boldsymbol{e} = (-9, -1, -1, 11)$.

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

Solution $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$. Why is $P\boldsymbol{e} = \mathbf{0}$?

Solution $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \mathbf{0}$.

19 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 18. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \mathbf{0}$.

Solution If $\boldsymbol{b} =$ error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p} = \mathbf{0}$.

20 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line and $\boldsymbol{e}$. The error is $\boldsymbol{e} = \mathbf{0}$ because this $\boldsymbol{b}$ is _____ ?

Solution If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \mathbf{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

Questions 21–26 ask for projections onto lines. Also errors $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and matrices P.

21 Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

Solution (a) The projection $\boldsymbol{p}$ is

$$\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{6}{3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ perpendicular to } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution (b) In this case the projection is

$$\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}\frac{-11}{11} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

22 Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it from $\boldsymbol{p} = \widehat{v}\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$

Solution (b) The projection of $\boldsymbol{b} = (1, 1)$ onto $\boldsymbol{a} = (1, -1)$ is $\boldsymbol{p} = (0, 0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

23 In Problem 22 find the projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto each vector $\boldsymbol{a}$. Verify in both cases that $P^2 = P$. Multiply $P\boldsymbol{b}$ in each case to find the projection $\boldsymbol{p}$.

Solution $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{p} = P_1\boldsymbol{b} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$. $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\boldsymbol{p} = P_2\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

24 Construct the projection matrices P_1 and P_2 onto the lines through the $\boldsymbol{a}$'s in Problem 22. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

Solution The projection matrices P_1 and P_2 (note correction P_2 not $P - 2$) are

$$P_1 = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathbf{T}}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad P_2 = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathbf{T}}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

It is *not true* that $(P_1 + P_2)^2 = P_1 + P_2$. The sum of projection matrices is **not usually** a projection matrix.

25 Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those two matrices P_1P_2 and explain the answer.

Solution $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$.

$P_1P_2 =$ *zero matrix because* $\boldsymbol{a_1}$ is perpendicular to $\boldsymbol{a_2}$.

26 Continuing Problem 25, find the projection matrix P_3 onto $\boldsymbol{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal !

Solution $P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$

We *can* add projections onto *orthogonal vectors*. This is important.

27 Project the vector $\boldsymbol{b} = (1, 1)$ onto the lines through $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (1, 2)$. Draw the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and add $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because the $\boldsymbol{a}$'s are not orthogonal.

Solution The projections of $(1, 1)$ onto the lines through $(1, 0)$ and $(1, 2)$ are $\boldsymbol{p}_1 = (1, 0)$ and $\boldsymbol{p}_2 = (3/5, 6/5) = (0.6, 1.2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.

28 (Quick and recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Solution $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $P =$ square matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{p} = P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix}$.

29 If A is doubled, then $P = 2A(4A^TA)^{-1}2A^T$. This is the same as $A(A^TA)^{-1}A^T$. The column space of $2A$ is the same as ____. Is $\widehat{v}$ the same for A and $2A$?

Solution $2A$ has the same column space as A. Same $\boldsymbol{p}$. But $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

30 What linear combination of $(1,2,-1)$ and $(1,0,1)$ is closest to $\boldsymbol{b} = (2,1,1)$?

Solution $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So $\boldsymbol{b}$ is in the plane: no error $\boldsymbol{e}$. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

31 (*Important*) If $P^2 = P$ show that $(I-P)^2 = I-P$. When P projects onto the column space of A, $I-P$ projects onto which fundamental subspace ?

Solution If $P^2 = P$ then $\boldsymbol{(I-P)^2} = (I-P)(I-P) = I-PI-IP+P^2 = \boldsymbol{I-P}$. When P projects onto the column space, $I-P$ projects onto the *left nullspace*.

32 If P is the 3 by 3 projection matrix onto the line through $(1,1,1)$, then $I-P$ is the projection matrix onto ____.

Solution $I-P$ is the projection onto the plane $x_1+x_2+x_3 = 0$, perpendicular to the direction $(1,1,1)$:

$$I-P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

33 Multiply the matrix $P = A(A^TA)^{-1}A^T$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space so its projection is ____.

Solution $\left(A(A^TA)^{-1}A^T\right)^2 = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T$. So $P^2 = P$. Geometric reason: $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$ for every $\boldsymbol{b}$. So $P^2 = P$.

34 If A is square and invertible, the warning against splitting $(A^TA)^{-1}$ does not apply. Then $AA^{-1}(A^T)^{-1}A^T = I$ is true. *When A is invertible, why is* $P = I$ and $\boldsymbol{e} = \boldsymbol{0}$?

Solution If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $\boldsymbol{e} = \boldsymbol{0}$.

35 An important fact about A^TA is this: ***If*** $\boldsymbol{A^TAx = 0}$ ***then*** $\boldsymbol{Ax = 0}$. *New proof*: The vector $A\boldsymbol{x}$ is in the nullspace of ____. $A\boldsymbol{x}$ is always in the column space of ____. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero.

Solution If $A^TA\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x}$ is in the *nullspace of* A^T. But $A\boldsymbol{x}$ is always in the *column space of* A. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and A^TA have the *same nullspace*.

Notes on mean and variance and test grades

If all grades on a test are 90, the mean is $m = 90$ and the variance is $\sigma^2 = 0$. Suppose the expected grades are $g_1, \ldots, g_N$. Then σ^2 comes from *squaring distances to the mean*:

$$\textbf{Mean}\ \ m = \frac{g_1 + \cdots + g_N}{N} \qquad \textbf{Variance}\ \ \sigma^2 = \frac{(g_1-m)^2 + \cdots + (g_N-m)^2}{N}$$

After every test my class wants to know m and σ. My expectations are usually way off.

36 Show that σ^2 also equals $\frac{1}{N}(g_1^2 + \cdots + g_N^2) - m^2$.

Solution Each term $(g_i - m)^2$ equals $g_i^2 - 2g_i m + m^2$, so

$$\sigma^2 = \frac{(\text{sum of } g_i^2) - 2m(\text{sum of } g_i) + Nm^2}{N} = \frac{(\text{sum of } g_i^2) - 2mNm + Nm^2}{N}$$
$$= \frac{(\text{sum of } g_i^2)}{N} - m^2.$$

37 If you flip a fair coin N times (1 for heads, 0 for tails) what is the expected number m of heads ? What is the variance σ^2 ?

Solution For a fair coin you expect $\boldsymbol{N/2}$ heads in N flips. The variance σ^2 turns out to be $\boldsymbol{N/4}$.

Problem Set 7.2, page 402

1 For a 2 by 2 matrix, suppose the 1 by 1 and 2 by 2 determinants a and $ac - b^2$ are positive. Then $c > b^2/a$ is also positive.

(i) λ_1 and λ_2 have the *same sign* because their product $\lambda_1\lambda_2$ equals ______.

(i) That sign is positive because $\lambda_1 + \lambda_2$ equals ______.

Conclusion: The tests $a > 0, ac - b^2 > 0$ guarantee positive eigenvalues λ_1, λ_2.

Solution Suppose $a > 0$ and $ac > b^2$ so that also $c > b^2/a > 0$.

(i) The eigenvalues have the *same sign* because $\lambda_1\lambda_2 = \det = ac - b^2 > 0$.

(ii) That sign is *positive* because $\lambda_1 + \lambda_2 > 0$ (it equals the trace $a + c > 0$).

2 Which of S_1, S_2, S_3, S_4 has two positive eigenvalues? Use a and $ac - b^2$, don't compute the λ's. Find an $\boldsymbol{x}$ with $\boldsymbol{x}^{\mathrm{T}} S_1 \boldsymbol{x} < 0$, confirming that A_1 fails the test.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

Solution Only $S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues since $101 > 10^2$.

$\boldsymbol{x}^{\mathrm{T}} S_1 \boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms; S_2 has trace c_0; S_3 has det $= 0$.

3 For which numbers b and c are these matrices positive definite?

$$S=\begin{bmatrix}1 & b\\ b & 9\end{bmatrix}\qquad S=\begin{bmatrix}2 & 4\\ 4 & c\end{bmatrix}\qquad S=\begin{bmatrix}c & b\\ b & c\end{bmatrix}.$$

Solution

Positive definite for $-3<b<3$
$$\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}=LDL^{\mathrm{T}}$$

Positive definite for $c>8$
$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}=LDL^{\mathrm{T}}.$$

Positive definite for $c>b$
$$L=\begin{bmatrix}1 & 1\\ -b/c & 0\end{bmatrix}\quad D=\begin{bmatrix}c & 0\\ 0 & c-b/c\end{bmatrix}\quad S=LDL^{\mathrm{T}}.$$

4 What is the energy $q=ax^2+2bxy+cy^2=\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ for each of these matrices? Complete the square to write q as a sum of squares $d_1(\quad)^2+d_2(\quad)^2$.

$$S=\begin{bmatrix}1 & 2\\ 2 & 9\end{bmatrix}\qquad\text{and}\qquad S=\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix}.$$

Solution $f(x,y)=x^2+4xy+9y^2=(x+2y)^2+5y^2;\ x^2+6xy+9y^2=(x+3y)^2.$

5 $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}=2x_1x_2$ certainly has a saddle point and not a minimum at $(0,0)$. What symmetric matrix S produces this energy? What are its eigenvalues?

Solution $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}=2x_1x_2$ comes from $S=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ which has eigenvalues 1 and -1: S is **indefinite**.

6 Test to see if $A^{\mathrm{T}}A$ is positive definite in each case:

$$A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\quad\text{and}\quad A=\begin{bmatrix}1 & 1\\ 1 & 2\\ 2 & 1\end{bmatrix}\quad\text{and}\quad A=\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}.$$

Solution The first and second matrices have independent columns in A, so $A^{\mathrm{T}}A$ is positive definite. The third matrix has dependent columns so $A^{\mathrm{T}}A$ is only *positive semidefinite*.

7 Which 3 by 3 symmetric matrices S and T produce these quadratic energies?

$\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}=2\left(x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3\right).$ Why is S positive definite?

$\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x}=2\left(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3\right).$ Why is T semidefinite?

Solution

$$S=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$$ is positive *definite*—its determinants are $D_1=2, D_2=3, D_3=4$.

$$T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ is positive } \textit{semidefinite} \text{ with determinants } D_1 = 2, D_2 = 3, D_3 = 0 \text{ .}$$

The energy $\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} = 0$ when $\boldsymbol{x} = (1, 1, 1)$.

8 Compute the three upper left determinants of S to establish positive definiteness. (The first is 2.) Verify that their ratios give the second and third pivots.

$$\textbf{Pivots = ratios of determinants} \qquad S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

Solution The upper left determinants of S are $2, 6, 30$. The pivots are $2, 3, 5$ (ratios of determinants). Notice that the product of pivots is $\mathbf{30}$.

9 For what numbers c and d are S and T positive definite? Test the 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Solution For $c = 1$, the matrix S has eigenvalues $3, 0, 0$. For any c, the eigenvalues all add $c - 1$. So S is positive definite for $c > 1$. (Same answer using determinants.) For T the determinants are $1, d - 4, -4d + 12$. If $d > 4$ then $-4d + 12$ is negative! So T is **never** positive definite for any d.

10 *If S is positive definite then S^{-1} is positive definite.* Best proof: The eigenvalues of S^{-1} are positive because _____. *Second proof* (only for 2 by 2):

$$\text{The entries of } S^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad \text{pass the determinant tests} \quad \_\_\_\_\_ .$$

Solution Positive definite $\Rightarrow$ all eigenvalues $\lambda > 0 \Rightarrow$ all eigenvalues $1/\lambda$ of S^{-1} are positive. Also for 2×2: the determinant tests are passed.

11 *If S and T are positive definite, their sum $S + T$ is positive definite*. Pivots and eigenvalues are not convenient for $S + T$. Better to prove $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} > 0$.

Solution Energy $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0 + 0$

12 A positive definite matrix *cannot have a zero* (or even worse, a negative number) *on its diagonal*. Show that this matrix fails to have $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & \mathbf{0} & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

Solution $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is **zero** when $\boldsymbol{x} = (0, 1, 0)$.

13 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have _____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a _____ on the main diagonal.

Solution If a_{jj} is smaller than all eigenvalues, then $A - a_{jj}I$ would have positive eigenvalues. But this matrix has a zero on the diagonal. But Problem 13, it can't be positive definite. So A_{jj} can't be smaller than all eigenvalues!

14 Show that *if all* $\lambda > 0$ *then* $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$. We must do this for *every* nonzero $\boldsymbol{x}$, not just the eigenvectors. So write $\boldsymbol{x}$ as a combination of the eigenvectors and *explain why all "cross terms" are* $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$. Then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is

$$(c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n)^{\mathrm{T}}(c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n x_n) = c_1^2\lambda_1\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_1 + \cdots + c_n^2\lambda_n\boldsymbol{x}_n^{\mathrm{T}}\boldsymbol{x}_n > 0.$$

Solution The "cross terms" have the form $(c_i x_i)^{\mathrm{T}}(c_j\lambda_j\boldsymbol{x}_j)$. This is zero because symmetric matrices S have orthogonal eigenvectors.

15 Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.
(b) The only positive definite projection matrix is $P = I$.
(c) A diagonal matrix with positive diagonal entries is positive definite.
(d) A symmetric matrix with a positive determinant might not be positive definite!

Solution

(a) All $\lambda_i > 0$ so zero is not an eigenvalue and S is invertible
(b) All projection matrices except $P = I$ are singular
(c) The energy for a positive diagonal matrix is $\boldsymbol{x}^{\mathrm{T}}D\boldsymbol{x} = d_1x_1^2 + \cdots + d_nx_n^2 > 0$ when $\boldsymbol{x} \neq \boldsymbol{0}$
(d) $S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has $\det S = 1$ but S is **negative** definite

16 With positive pivots in D, the factorization $S = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $A = \sqrt{D}L^{\mathrm{T}}$ yields the ***Cholesky factorization*** $S = A^{\mathrm{T}}A$ which is "symmetrized LU":

$$\text{From } A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \text{ find } S. \qquad \text{From } S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \text{ find } A = \textbf{chol}(S).$$

Solution If $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ then $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$ = positive definite S.

$$S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 0 & 9 \end{bmatrix} = LDL^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & \\ & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

so $A = \sqrt{D}\,L^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.

17 Without multiplying $S = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, find

(a) the determinant of S (b) the eigenvalues of S

(c) the eigenvectors of S (d) a reason why S is symmetric positive definite.

Solution $\det S = 10$, $\lambda(S) = 2$ and 5, eigenvectors $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$, S has positive eigenvalues.

18 For $F_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x, y) = x^3 + xy - x$ find the second derivative matrices H_1 and H_2 :

$$\textbf{Test for minimum} \quad H = \begin{bmatrix}\partial^2F/\partial x^2 & \partial^2F/\partial x\partial y\\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2\end{bmatrix} \text{ is positive definite}$$

H_1 is positive definite so F_1 is concave up (= convex). Find the minimum point of F_1 and the saddle point of F_2 (look only where first derivatives are zero).

Solution $F_1 = \frac{1}{4}x^4 + x^2y + y^2$ has $\partial F_1/dx = x^3 + 2xy$ and $\partial F_1/dy = x^2 + 2y$. Then the 2nd derivatives are

$$H_1 = \begin{bmatrix}3x^2 + 2y & 2x\\ 2x & 2\end{bmatrix}. \quad F_2 = x^3 + xy - x \text{ has } H_2 = \begin{bmatrix}6x & 1\\ 1 & 0\end{bmatrix}.$$

19 The graph of $z = x^2 + y^2$ is a bowl opening upward. The graph of $z = x^2 - y^2$ is *a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(0, 0)$?

Solution $ax^2 + 2bxy + cy^2$ has a saddle point $(0, 0)$ if $\partial z/\partial x = \partial z/\partial y = 0$ (which is true) and if $H = 2\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is positive definite.

20 Which values of c give a bowl and which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

Solution The matrix for this problem is $S = \begin{bmatrix}4 & 6\\ 6 & c\end{bmatrix}$ and this has a saddle for $c < 9$. Then $\lambda_1 > 0 > \lambda_2$ because the determinants are $4 > 0$ and $4c - 3b < 0$.

21 When S and T are symmetric positive definite, ST might not even be symmetric. But its eigenvalues are still positive. Start from $ST\boldsymbol{x} = \lambda\boldsymbol{x}$ and take dot products with $T\boldsymbol{x}$. Then prove $\lambda > 0$.

Solution If $ST\boldsymbol{x} = \lambda\boldsymbol{x}$ then $(T\boldsymbol{x})^{\mathrm{T}}ST\boldsymbol{x} = \lambda(T\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x}$. Left side > 0 because S is positive definite, right side has $\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0$ because T is positive definite. Therefore $\lambda > 0$.

22 Suppose C is positive definite (so $\boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \mathbf{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \mathbf{0}$ whenever $\boldsymbol{x} \neq \mathbf{0}$). Apply the energy test to $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}$ to show that $A^{\mathrm{T}}CA$ is positive definite : *the crucial matrix in engineering*.

Solution $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ because $\boldsymbol{y} = A\boldsymbol{x}$ is only zero when $\boldsymbol{x}$ is zero (A has independent columns).

23 Find the eigenvalues and unit eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$. Then find $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \text{ and } AA^{\mathrm{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that $\boldsymbol{u}_1$ is a unit eigenvector of AA^{T}. Complete the matrices U, Σ, V.

$$\textbf{SVD} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathbf{T}}.$$

Solution $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has eigenvalues 50 and 0. Its eigenvectors are $\boldsymbol{v}_1 = (1,2)/\sqrt{5}$ and $\boldsymbol{v}_2 = (-2,1)/\sqrt{5}$. Then $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sqrt{50} = (50,100)/\sqrt{250}$. The SVD is $\dfrac{\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

24 Write down orthonormal bases for the four fundamental subspaces of this A.

Solution $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has bases $\begin{bmatrix} 1 \\ 3 \end{bmatrix}/\sqrt{10}$ for $\mathbf{C}(A)$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}/\sqrt{5}$ for row space $\mathbf{C}(A^{\mathrm{T}})$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}/\sqrt{5}$ for $\mathbf{N}(A)$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}/\sqrt{10}$ for $\mathbf{N}(A^{\mathrm{T}})$.

25 (a) Why is the trace of $A^{\mathrm{T}}A$ equal to the sum of all a_{ij}^2 ?

(b) For every rank-one matrix, why is $\sigma_1^2 =$ sum of all a_{ij}^2 ?

Solution The diagonal entries of $A^{\mathrm{T}}A$ are $||$column $1||^2$ to $||$column $n||^2$. The sum of those is the sum of all a_{ij}^2. The trace of $A^{\mathrm{T}}A$ is always the sum of all σ_i^2 and for a rank one matrix that sum is only σ_1^2.

26 Find the eigenvalues and unit eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. Keep each $A\boldsymbol{v} = \sigma\boldsymbol{u}$:

$$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^{\mathrm{T}}$.

Solution A is symmetric with $A^{\mathrm{T}}A = A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ with eigenvalues x from $x^2 - 3x + 1 = 0$ and $x = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$. Then $\sigma = \sqrt{x} = \frac{1}{2}\left(\sqrt{5} \pm 1\right)$.

27 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide $\pm$ signs in U). Σ *has the same shape as* A.

Solution $A^{\mathrm{T}}A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues 3 and 1, so A has singular values $\sqrt{3}$ and 1. The unit eigenvectors are $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$. $AA^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has eigenvalues 3 and 1 and 0 and eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ divided by $\sqrt{6}, \sqrt{2}, \sqrt{3}$.

28 Construct the matrix with rank one that has $A\boldsymbol{v} = 12\boldsymbol{u}$ for $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ and $\boldsymbol{u} = \frac{1}{3}(2,2,1)$. Its only singular value is $\sigma_1 =$ _____.

Solution $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ has $A\boldsymbol{v} = 12\boldsymbol{u}$ for that unit vector $\boldsymbol{v}$. The only singular value is $\sigma_1 = 12$. (Other A are also possible.)

29 Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by ***as small a matrix as possible*** to produce a singular matrix A_0. Hint: U and V do not change.

$$\text{From} \qquad A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} \quad \text{find the nearest } A_0.$$

Solution The nearest singular matrix is $A_0 = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V^{\mathrm{T}}$. Since U and V are orthogonal matrices, the size of $A - A_0$ is only σ_2. In other words, $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$ is the closest rank 1 matrix to A.

30 The SVD for $A + I$ doesn't use $\Sigma + I$. *Why is $\sigma(A+I)$ not just $\sigma(A) + I$?*

Solution The SVD of $A + I$ uses the eigenvectors of $(A+I)^{\mathrm{T}}(A+I)$. Those are not the eigenvectors of $A^{\mathrm{T}}A$ (or $A^{\mathrm{T}}A + I$).

31 Multiply $A^{\mathrm{T}}A\boldsymbol{v} = \sigma^2\boldsymbol{v}$ by A. Put in parentheses to show that $A\boldsymbol{v}$ is an eigenvector of AA^{T}. We divide by its length $||A\boldsymbol{v}|| = \sigma$ to get the unit eigenvector $\boldsymbol{u}$.

Solution A times $A^{\mathrm{T}}A\boldsymbol{v} = \sigma^2\boldsymbol{v}$ is $(AA^{\mathrm{T}})A\boldsymbol{v} = \sigma^2(A\boldsymbol{v})$. So $A\boldsymbol{v}$ is an eigenvector of AA^{T}.

32 My favorite example of the SVD is when $Av(x) = dv/dx$, with the endpoint conditions $v(0) = 0$ and $v(1) = 0$. We are looking for orthogonal functions $v(x)$ so that their derivatives $Av = dv/dx$ are also orthogonal. The perfect choice is $v_1 = \sin \pi x$ and $v_2 = \sin 2\pi x$ and $v_k = \sin k\pi x$. Then each u_k is a cosine.

The derivative of v_1 is $Av_1 = \pi \cos \pi x = \pi u_1$. The singular values are $\sigma_1 = \pi$ and $\sigma_k = k\pi$. Orthogonality of the sines (and orthogonality of the cosines) is the foundation for Fourier series.

You may object to $AV = U\Sigma$. The derivative $A = d/dx$ is not a matrix ! The orthogonal factor V has functions $\sin k\pi x$ in its columns, not vectors. The matrix U has cosine functions $\cos k\pi x$. Since when is this allowed ? One answer is to refer you to the **chebfun** package on the web. This extends linear algebra to matrices whose columns are functions—not vectors.

Another answer is to replace d/dx by a first difference matrix A. Its shape will be $N+1$ by N. A has 1's down the diagonal and -1's on the diagonal below. Then $AV = U\Sigma$ has discrete sines in V and discrete cosines in U. For $N = 2$ those will be sines and cosines of $30°$ and $60°$ in $\boldsymbol{v}_1$ and $\boldsymbol{u}_1$.

$**$ Can you construct the difference matrix A (3 by 2) and $A^{\mathrm{T}}A$ (2 by 2)? The discrete sines are $\boldsymbol{v}_1 = (\sqrt{3}/2, \sqrt{3}/2)$ and $\boldsymbol{v}_2 = (\sqrt{3}/2, -\sqrt{3}/2)$. Test that $A\boldsymbol{v}_1$ is orthogonal to $A\boldsymbol{v}_2$. What are the singular values σ_1 and σ_2 in Σ ?

Solution The sines and cosines are perfect examples of the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s for the operator (infinite-dimensional matrix) $A =$ derivative d/dx. The sines $\boldsymbol{v}_k = \sin \pi kx$ are orthogonal, the cosines $\boldsymbol{u}_k = \cos \pi kx$ are orthogonal, and $A\boldsymbol{v}_k = \sigma_k \boldsymbol{u}_k$. (The derivative of a sine is a cosine with $\sigma_k = \pi k$.) For differences instead of derivatives, we can try the matrix $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

Problem Set 7.3, page 413

1 *Transpose the derivative with integration by parts*: $(dy/dx, g) = -(y, dg/dx)$. Ay is dy/dx with boundary conditions $y(0) = 0$ and $y(1) = 0$. Why is $\int y'g dx$ equal to $-\int yg'dx$? Then A^{T} (which is normally written as A^*) is $A^{\mathrm{T}}g = -dg/dx$ with **no** boundary conditions on g. $A^{\mathrm{T}}Ay$ is $-y''$ with $y(0) = 0$ and $y(1) = 0$.

Solution Integration by parts for $0 \le x \le 1$ produces boundary terms at $x = 0$ and 1:

$$\int_0^1 \frac{dy}{dx}\, g(x)\, dx = -\int_0^1 y(x)\, \frac{dg}{dx}\, dx + y(x)\, g(x)\Big|_{x=0}^{x=1}$$

The boundary terms are zero if $y(0) = y(1) = 0$. Then the adjoint (or transpose) of d/dx is $-d/dx$, with no boundary condition on g when there are 2 boundary conditions on y (fixed-fixed).

Problems 2-6 have boundary conditions at $x = 0$ and $x = 1$: no initial conditions.

2 Solve this boundary value problem in two steps. Find the complete solution $y_p + y_n$ with two constants in y_n, and find those constants from the boundary conditions:

Solve $-y'' = 12x^2$ with $y(0) = 0$ and $y(1) = 0$ and $y_p = -x^4$.

Solution $y_p = -x^4$ solves $-y_p'' = 12x^2$. It has $y_p(0) = 0$ and $y_p = -1$. We need to add the solution to $-Y'' = 0$ with $Y(0) = 0$ and $Y(1) = 1$. Then $Y = A + Bx$ has $A = 0$ and $B = 1$. The complete solution is $y = -x^4 + x$.

3 Solve the same equation $-y'' = 12x^2$ with $y(0) = 0$ and $y'(1) = 0$ (zero slope).

Solution Changing $y(1) = 0$ to $y'(1) = 0$ will change the solution to $y = -x^4 + Bx$ with $y' = -4x^3 + B$. For $y'(1) = 0$ we need $B = 4$.

4 Solve the same equation $-y'' = 12x^2$ with $y'(0) = 0$ and $y(1) = 0$. Then try for both slopes $y'(0) = 0$ and $y'(1) = 0$: *this has no solution* $y = -x^4 + Ax + B$.

Solution With $y'(0) = 0$ the solution we want is $y = -x^4 + A$. The constant A is determined by $y(1) = -1 + A = 0$. We cannot have $y'(1) = 0$ because $y' = -4x^3$.

5 Solve $-y'' = 6x$ with $y(0) = 2$ and $y(1) = 4$. Boundary values need not be zero.

Solution $-y'' = 6x$ leads to $y = -x^3 + A + Bx$. The boundary conditions are $y(0) = A = 2$ and $y(1) = -1 + 2 + B = 4$. Then $B = 3$ and $y = -x^3 + 2 + 3x$.

6 Solve $-y'' = e^x$ with $y(0) = 5$ and $y(1) = 0$, starting from $y = y_p + y_n$.

Solution $-y'' = e^x$ leads to $y = -e^x + A + Bx$. The first boundary condition is $y(0) = -1 + A = 5$ so that $A = \mathbf{6}$. Then $y(1) = -e + 6 + B = 0$ and $B = e - 6$.

Problems 7-11 are about the LU factors and the inverses of second difference matrices.

7 The matrix T with $T_{11} = 1$ factors perfectly into $LU = A^{\mathrm{T}}A$ (all its pivots are 1).

$$\boldsymbol{T} = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = LU.$$

Each elimination step adds the pivot row to the next row (and L subtracts to recover T from U). The inverses of those difference matrices L and U are **sum matrices**. Then the inverse of $T = LU$ is $U^{-1}L^{-1}$:

$$T^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} = U^{-1}L^{-1}.$$

Compute T^{-1} for $N = 4$ (as shown) and for any N.

Solution $T^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ T is fixed-free second difference matrix. For any N, T^{-1} has the same pattern with first row $N \;\; N-1 \;\; \cdots \;\; 2 \;\; 1$

8 The matrix equation $TY = (0, 1, 0, 0) =$ *delta vector* is like the differential equation $-y'' = \delta(x - a)$ with $a = 2\Delta\, x = \frac{2}{5}$. The boundary conditions are $y'(0) = 0$ and $y(1) = 0$. Solve for $y(x)$ and graph it from 0 to 1. Also graph $Y =$ second column of T^{-1} at the points $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. The two graphs are ramp functions.

Solution Two integrations of the delta function $\delta(x)$ will produce the unit ramp $R(x) = 0$ for $x \leq 0, R(x) = x$ for $x \geq 0$. Shifting $\delta(x)$ to $\delta\left(x - \frac{2}{5}\right)$ will shift the solution to $y = -R\left(x - \frac{2}{5}\right) + A + Bx$. Then $y'(0) = -1 + B$ gives $B = 1$, and $y(1) = 0$ gives $-\frac{3}{5} + A + 1 = 0$ and $A = -\frac{2}{5}$.

9 The matrix B has $B_{11} = 1$ (like $T_{11} = 1$) and also $B_{NN} = 1$ (where $T_{NN} = 2$). Why does B have the same pivots 1, 1, ... as T, except for zero in the last pivot position? The early pivots don't know $B_{NN} = 1$.

Then B is not invertible: $-y'' = \delta(x - a)$ has no solution with $y'(0) = y'(1) = 0$.

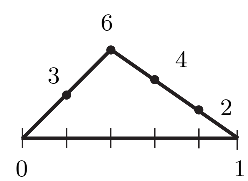
Solution B starts with the pivots $1, 1, 1, \ldots$ (as T did) but reducing the N, N entry by 1 will reduce the last pivot by 1. So we have last pivot = zero and B is not invertible. The analog for differential equations is $y' = 0$ at both endpoints: No ramp function except $y = 0$ can meet those boundary conditions.

10 When you compute K^{-1}, multiply by $\det K = N + 1$ to get nice numbers:

Column 2 of $5K^{-1}$ solves the equation $K\boldsymbol{v} = \mathbf{5}\boldsymbol{\delta}$ when the delta vector is $\boldsymbol{\delta} =$ _____

We know from $KK^{-1} = I$ that K times each column of K^{-1} is a delta vector.

$$\mathbf{5}K^{-1} = \begin{bmatrix} 4 & \mathbf{3} & 2 & 1 \\ 3 & \mathbf{6} & 4 & 2 \\ 2 & \mathbf{4} & 6 & 3 \\ 1 & \mathbf{2} & 3 & 4 \end{bmatrix}$$



graph of column 2

Solution Column 2 of $5K^{-1}$ is like the solution to $-y'' = 5\delta\left(x - \frac{2}{5}\right)$. The column of $5K^{-1}$ has a max in row 2 and the solution $y(x)$ has a max at $x = \frac{2}{5}$.

11 K comes with two boundary conditions. T only has $y(1) = 0$. B has no boundary conditions on y. Verify that $K = A^{\mathrm{T}}A$. Then remove the first row of A to get $T = A_1^{\mathrm{T}}A_1$. Then remove the last row to get dependent rows: $B = A_0^{\mathrm{T}}A_0$.

The backward first difference $A = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix}$ gives $K = A^{\mathrm{T}}A$.

Solution A is the matrix in Problem 7 with 1's on the main diagonal and -1's on the diagonal above. $A^{\mathrm{T}}A$ is the symmetric second difference matrix with three nonzero diagonals. Those diagonals contain -1's and 2's and -1's. Then removing the top row of A gives a rectangular A_1 with $A_1^{\mathrm{T}}A_1 = T$ as in Problem 7 ($T_{11} = 1$ not 2). Removing the last row gives A_2 with $A_2^{\mathrm{T}}A_2 = B$ and $B_{NN} = 1$ not 2.

12 Multiply K_3 by its eigenvector $\boldsymbol{y}_n = (\sin n\pi h, \sin 2n\pi h, \sin 3n\pi h)$ to verify that the eigenvalues λ_1, λ_2, λ_3 are $\lambda_n = 2 - 2\cos\frac{n\pi}{4}$ in $K\boldsymbol{y}_n = \lambda_n\boldsymbol{y}_n$. This uses the trigonometric identity $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.

Solution The eigenvectors of K are "sine vectors" just as the eigenfunctions of $-y'' = \lambda y$ with $y(0) = 0 = y(1)$ are sine functions.

13 Those eigenvalues of K_3 are $2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$. Those add to 6, which is the trace of K_3. Multiply those eigenvalues to get the determinant of K_3.

Solution Multiplying $2 - \sqrt{2}$ times $2 + \sqrt{2}$ gives $4 - 2 = 2$. Then multiplying by 2 gives 4. This is the determinant (and $2 - \sqrt{2}, 2 + \sqrt{2}, 2$ are the eigenvalues) of 3 by 3 matrix K_3.

14 The slope of a ramp function is a step function. The slope of a step function is a delta function. Suppose the ramp function is $r(x) = -x$ for $x \le 0$ and $r(x) = x$ for $x \ge 0$ (so $r(x) = |x|$). *Find dr/dx and d^2r/dx^2.*

Solution For the down-up ramp function $r(\boldsymbol{x}) = |x| =$ absolute value of x, the derivatives are $dr/dx = -1$ then $+1$ and $d^2r/dx^2 = 2\delta(x)$ because dr/dx jumps by 2 at $x = 0$.

15 Find the second differences $y_{n+1} - 2y_n + y_{n-1}$ of these infinitely long vectors $\boldsymbol{y}$:

Constant	$(\ldots, 1, 1, 1, 1, 1, \ldots)$
Linear	$(\ldots, -1, 0, 1, 2, 3, \ldots)$
Quadratic	$(\ldots, 1, 0, 1, 4, 9, \ldots)$
Cubic	$(\ldots, -1, 0, 1, 8, 27, \ldots)$
Ramp	$(\ldots, 0, 0, 0, 1, 2, \ldots)$
Exponential	$(\ldots, e^{-i\omega}, e^0, e^{i\omega}, e^{2i\omega}, \ldots)$.

It is amazing how closely those second differences follow second derivatives for $y(x) = 1, x, x^2, x^3, \max(x, 0)$, and $e^{i\omega x}$. From $e^{i\omega x}$ we also get $\cos\omega x$ and $\sin\omega x$.

Solution The six second differences are : zero vector, zero vector, constant vector of 2's, 6 times the linear vector, (for ramp : delta vector with $\delta_0 = 1$), $e^{i\omega} - 2 + e^{-i\omega} = 2\cos\omega - 2$ times the exponential vector. **Like 2nd derivatives** of $1, x, x^2, x^3$, ramp, $e^{i\omega x}$.

Problem Set 7.4, page 422

1 What solution to Laplace's equation completes "degree 3" in the table of pairs of solutions ? We have one solution $u = x^3 - 3xy^2$, and we need another solution.

Solution Start with $s = -y^3$. Then $s_{yy} = -6y$, and therefore we need $s_{xx} = 6y$. Integrating twice with respect to x gives $3y^2x$. Therefore the second function is $s(x, y) = -y^3 + 3x^2y$.

2 What are the two solutions of degree 4, the real and imaginary parts of $(x + iy)^4$? Check $u_{xx} + u_{yy} = 0$ for both solutions.

Solution Expanding $(x + iy)^4$ gives

$$(x + iy)^4 = x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i$$

Therefore the two solutions would be :

$$u(x, y) = \boldsymbol{x^4 - 6x^2y^2 + y^4} \quad \text{and} \quad s(x, y) = \boldsymbol{4x^3y - 4xy^3}$$

Checking the first solution :

$$\frac{\partial^2(x^4 - 6x^2y^2 + y^4)}{\partial x^2} + \frac{\partial^2(x^4 - 6x^2y^2 + y^4)}{\partial y^2} = (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0$$

Checking the second solution :

$$\frac{\partial^2(4x^3y - 4xy^3)}{\partial x^2} + \frac{\partial^2(4x^3y - 4xy^3)}{\partial y^2} = (24xy - 0) + (0 - 24xy) = 0$$

3 What is the second x-derivative of $(x+iy)^n$? What is the second y-derivative ? Those cancel in $u_{xx}+u_{yy}$ because $i^2=-1$.

Solution The second x-derivative of $(x+iy)^n$ is :

$$\frac{\partial^2(x+iy)^n}{\partial x^2}=n(n-1)(x+iy)^{n-2}$$

The second y-derivative of $(x+iy)^n$ cancels that because

$$\frac{\partial^2(x+iy)^n}{\partial y^2}=i\cdot i\cdot n(n-1)(x+iy)^{n-2}=-n(n-1)(x+iy)^{n-2}$$

4 For the solved 2×2 example inside a 4×4 square grid, write the four equations (9) at the four interior nodes. Move the known boundary values 0 and 4 to the right hand sides of the equations. You should see $K2$D on the left side multiplying the correct solution $\boldsymbol{U}=(U_{11},U_{12},U_{21},U_{22})=(1,2,2,3)$.

Solution The equations at the interior node would be :

$$\begin{aligned}4U_{1,1}-U_{2,1}-U_{0,1}-U_{1,2}-U_{1,0}&=0\\4U_{1,2}-U_{2,2}-U_{0,2}-U_{1,3}-U_{1,1}&=0\\4U_{2,1}-U_{3,1}-U_{1,1}-U_{2,2}-U_{2,0}&=0\\4U_{2,2}-U_{3,2}-U_{1,2}-U_{2,3}-U_{2,1}&=0\end{aligned}$$

Substituting the known boundary values leaves :

$$\begin{aligned}4U_{1,1}-U_{2,1}-U_{1,2}&=4\\4U_{1,2}-U_{2,2}-U_{1,1}&=8\\4U_{2,1}-U_{1,1}-U_{2,2}&=0\\4U_{2,2}-U_{1,2}-U_{2,1}&=4\end{aligned}$$

Writing this in matrix form gives :

$$\begin{bmatrix}4&-1&0&-1\\-1&4&-1&0\\0&-1&4&-1\\-1&0&-1&4\end{bmatrix}\begin{bmatrix}U_{1,1}\\U_{1,2}\\U_{2,1}\\U_{2,2}\end{bmatrix}=\begin{bmatrix}4\\8\\0\\4\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}U_{1,1}\\U_{1,2}\\U_{2,1}\\U_{2,2}\end{bmatrix}=\begin{bmatrix}2\\3\\1\\2\end{bmatrix}$$

5 Suppose the boundary values on the 4×4 grid change to $U=0$ on three sides and $U=8$ on the fourth side. Find the four inside values so that each one is the average of its neighbors.

Solution The values at the 16 nodes will be

$$\begin{matrix}0&0&0&0\\0&\frac12&\frac12&0\\0&\frac32&\frac32&0\\0/4&4&4&0/4\end{matrix}$$

Notice that the corner boundary values **do not enter** the 5-point equations around interior points. Every interior value must be the average of its four neighbors. By symmetry the two middle columns must be the same.

6 (MATLAB) Find the inverse $(K2\text{D})^{-1}$ of the 4 by 4 matrix displayed for the square grid.

Solution The circulant matrix $K2$D on page 422 has a circulant inverse:

$$(K2\text{D})^{-1} = \frac{1}{24}\begin{bmatrix} 7 & 2 & 1 & 2 \\ 2 & 7 & 2 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 1 & 2 & 7 \end{bmatrix}.$$

7 Solve this Poisson finite difference equation (right side $\neq 0$) for the inside values $U_{11}, U_{12}, U_{21}, U_{22}$. All boundary values like U_{10} and U_{13} are zero. The boundary has i or j equal to 0 or 3, the interior has i *and* j equal to 1 or 2:

$$4U_{ij} - U_{i-1,j} - U_{i+1,j} - U_{i,j-1} - U_{i,j+1} = \mathbf{1} \text{ at four inside points.}$$

Solution The interior solution to the Poisson equation (on this small grid) is

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

On a larger grid U_{ij} will not be constant in the interior.

8 A 5×5 grid has a 3 by 3 interior grid: 9 unknown values U_{11} to U_{33}. Create the 9×9 difference matrix $K2$D.

Solution Order the points by rows to get $U_{11}, U_{12}, U_{13}, U_{21}, U_{22}, U_{23}, U_{31}, U_{32}, U_{33}$. Then $K2D$ is symmetric with 3 by 3 blocks:

$$K2D = \begin{bmatrix} A & -I & 0 \\ -I & A & -I \\ 0 & -I & A \end{bmatrix} \qquad A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

9 Use $\text{eig}(K2\text{D})$ to find the nine eigenvalues of $K2$D in Problem 8. Those eigenvalues will be positive! The matrix $K2$D is symmetric positive definite.

Solution $\text{eig}(K2\text{D})$ in Problem 8 produces 9 eigenvalues between 0 and 4:

The eigenvalues come from $\text{eig}(K2\text{D})$ and explicitly from equation (11). Notice that pairs of eigenvalues add to 8. The eigenvalue distribution is symmetric around $\lambda = 4$:

$$1.1716 \quad 2.5828 \quad 2.5828 \quad 4.0 \quad 4.0 \quad 4.0 \quad 5.4142 \quad 5.4142 \quad 6.8284$$

10 If $u(x)$ solves $u_{xx} = 0$ and $v(y)$ solves $v_{yy} = 0$, verify that $u(x)v(y)$ solves Laplace's equation. Why is this only a 4-dimensional space of solutions? Separation of variables does not give all solutions—only the solutions with separable boundary conditions.

Solution If $\dfrac{\partial^2 u}{\partial x^2} = 0$ and $\dfrac{\partial^2 v}{\partial y^2} = 0$ then

$$\frac{\partial^2 u(x)v(y)}{\partial x^2} + \frac{\partial^2 u(x)v(y)}{\partial y^2} = v(y)\frac{\partial^2 u(x)}{\partial x^2} + u(x)\frac{\partial^2 v(y)}{\partial y^2}$$
$$= v \cdot 0 + u \cdot 0 = 0$$

Therefore $u(x)v(y)$ solves Laplace's equation. But the only solutions found this way are $u(x)v(y) = (A + Bx)(C + Dy)$.

Problem Set 7.5, page 428

Problems 1 — 5 are about complete graphs. Every pair of nodes has an edge.

1 With $n = 5$ nodes and all edges, find the diagonal entries of $A^{\mathrm{T}}A$ (the degrees of the nodes). All the off-diagonal entries of $A^{\mathrm{T}}A$ are -1. Show the reduced matrix R without row 5 and column 5. Node 5 is "grounded" and $v_5 = 0$.

Solution The complete graph (all edges included) has no zeros in $A^{\mathrm{T}}A$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \qquad \textbf{Singular}!$$

The grounded matrix would be

$$(A^{\mathrm{T}}A)_{\text{reduced}} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \qquad \textbf{Invetible}!$$

2 Show that the *trace* of $A^{\mathrm{T}}A$ (sum down the diagonal = sum of eigenvalues) is $n^2 - n$. What is the trace of the reduced (and invertible) matrix R of size $n-1$?

Solution $A^{\mathrm{T}}A$ is n by n and each diagonal entry is $n-1$. Therefore the trace is $n(n-1) = \boldsymbol{n^2 - n}$. The reduced matrix R has $n-1$ diagonal entries, each still equal to $n-1$. Therefore the trace is $(n-1)(n-1) = n^2 - 2n + 1$.

3 For $n = 4$, write the 3 by 3 matrix $R = (A_{\text{reduced}})^{\mathrm{T}}(A_{\text{reduced}})$. Show that $RR^{-1} = I$ when R^{-1} has all entries $\frac{1}{4}$ off the diagonal and $\frac{2}{4}$ on the diagonal.

Solution

$$\textbf{Reduced matrix } R = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

R by its proposed inverse gives

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

4 For every n, the reduced matrix R of size $n-1$ is *invertible*. Show that $RR^{-1} = I$ when R^{-1} has all entries $1/n$ off the diagonal and $2/n$ on the diagonal.

Solution

$$\frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-1-1 & 3-2-1 & 3-1-2 \\ -2+3-1 & -1+6-1 & -1+3-2 \\ -2-1+3 & -1-2+3 & -1-1+6 \end{bmatrix} = I.$$

5 Write the 6 by 3 matrix $M = A_{\text{reduced}}$ when $n = 4$. The equation $M\boldsymbol{v} = \boldsymbol{b}$ is to be solved by least squares. The vector $\boldsymbol{b}$ is like scores in 6 games between 4 teams (team 4 always scores zero; it is grounded). Knowing the inverse of $R = M^{\mathrm{T}}M$, what is the least squares ranking $\widehat{v}_1$ for team 1 from solving $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$?

Solution Remove column 4 of A when node 4 is grounded ($x_4 = 0$).

$$M = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ has independent columns}$$

The least squares solution $\widehat{v}$ to $M\boldsymbol{v} = \boldsymbol{b}$ comes from $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$. This $\widehat{\boldsymbol{v}}$ gives the predicted point spreads when all teams play all other teams. The first component $\widehat{v}_1$ would come from the first row of $(M^{\mathrm{T}}M)^{-1}$ multiplying by $M^{\mathrm{T}}\boldsymbol{b}$. Note that

$$M^{\mathrm{T}}M = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \text{ and } (M^{\mathrm{T}}M)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

6 For the tree graph with 4 nodes, $A^{\mathrm{T}}A$ is in equation (1). What is the 3 by 3 matrix $R = (A^{\mathrm{T}}A)_{\text{reduced}}$? How do we know it is positive definite?

Solution The reduced form of $A^{\mathrm{T}}A$ removes row 4 and column 4 :

$$\text{Singular } A^{\mathrm{T}}A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ reduces to invertible } \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The first is positive semidefinite (A has dependent columns). the second is positive definite (the reduced A has 3 independent columns).

7 (a) If you are given the matrix A, how could you reconstruct the graph?

Solution Each row of A tells you an edge in the graph.

(b) If you are given $L = A^{\mathrm{T}}A$, how could you reconstruct the graph (no arrows) ?

Solution Each nonzero off the main diagonal of $A^{\mathrm{T}}A$ tells you an edge.

(c) If you are given $K = A^{\mathrm{T}}CA$, how could you reconstruct the weighted graph?

Solution Each nonzero off the main diagonal tells you the weight of that edge.

8 Find $K = A^{\mathrm{T}}CA$ for a line of 3 resistors with conductances $c_1 = 1$, $c_2 = 4$, $c_3 = 9$. Write K_{reduced} and show that this matrix is positive definite.

Solution A **circle** of three resistors has 3 edges and 3 nodes :

$$A^{\mathrm{T}}CA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 & -1 \\ -4 & 13 & -9 \\ -1 & -9 & 10 \end{bmatrix} \text{ is only } \textbf{semidefinite} \qquad .$$

$$(A^{\mathrm{T}}CA)_{\text{reduced}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{13} \end{bmatrix}$$

The determinant tests $5 > 0$ and $(5)(13) > 4^2$ are passed.

9 A 3 by 3 square grid has $n=9$ nodes and $m=12$ edges. Number nodes by rows.

(a) How many nonzeros among the 81 entries of $L=A^{\mathrm{T}}A$?

Solution The 9 nodes ordered by rows have $2,3,2,3,4,3,2,3,2$ neighbors around them. Those add to 24 nonzeros off the diagonal. The 9 diagonal entries make 33 nonzeros out of $9^2=81$ entries in $L=A^{\mathrm{T}}A$.

(b) Write down the 9 diagonal entries in the degree matrix D: they are not all 4.

Solution Those 9 numbers are the degrees of the 9 nodes (= diagonal entries in $A^{\mathrm{T}}A$).

(c) Why does the middle row of $L=D-W$ have four -1's? Notice $L=K2\mathrm{D}$!

Solution The middle node in the grid has 4 **neighbors**.

10 Suppose all conductances in equation (5) are equal to c. Solve equation (6) for the voltages v_2 and v_3 and find the current I flowing out of node 1 (and into the ground at node 4). What is the "system conductance" I/V from node 1 to node 4?

This overall conductance I/V should be larger than the individual conductances c.

Solution The reduced equation (6) with conductances $=c$ is

$$\begin{bmatrix} 3c & -c \\ -c & 2c \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} cV \\ cV \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0.6V \\ 0.8V \end{bmatrix}.$$

Then the flows on the five edges in Figure 7.6 use A in equation (2). Remember the minus sign:

$$-cA\boldsymbol{v}=-c\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} V \\ 0.6V \\ 0.8V \\ 0 \end{bmatrix}=cV\begin{bmatrix} 0.4 \\ 0.2 \\ -0.2 \\ 1.0 \\ 0.6 \end{bmatrix}$$

The total flow (on edges $1+2+4$ out of node 1, or on edges $3+4$ into the grounded node 4, is $I=1.6cV$. The overall system conductance is $1.6c$, greater than the individual conductance c on each edge.

11 The multiplication $A^{\mathrm{T}}A$ can be columns of A^{T} times rows of A. For the tree with $m=3$ edges and $n=4$ nodes, each (column times row) is $(4\times 1)(1\times 4)=4\times 4$. Write down those three column-times-row matrices and add to get $L=A^{\mathrm{T}}A$.

Solution Suppose the 3 tree edges go out of node 1 to nodes $2,3,4$. (The problem allows to choose other trees, including a line of 4 nodes.) Then

$$A=\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad A^{\mathrm{T}}A=\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}= \text{ sum of (columns of} A^{\mathrm{T}})(\text{ rows of } A)$$

$$=\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}+\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}+\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}.$$

12 A graph with two separate 3-node trees is *not connected*. Write its 6 by 4 incidence matrix A. Find *two* solutions to $A\boldsymbol{v} = \mathbf{0}$, not just one solution $\boldsymbol{v} = (1,1,1,1,1,1)$. To reduce $A^{\mathrm{T}}A$ we must ground *two* nodes and remove two rows and columns.

Solution The incidence matrix for two 3-node trees is

$$A = \begin{bmatrix} A_{\text{tree}} & 0 \\ 0 & A_{\text{tree}} \end{bmatrix} \text{ with } A_{\text{tree}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ (for example)}$$

The columns of A_{tree} add to zero so we have 2 independent solutions to $A\boldsymbol{v} = \mathbf{0}$:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ come from } A_{\text{tree}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

13 "Element matrices" from column times row appear in the **finite element method**. Include the numbers c_1, c_2, c_3 in the element matrices K_1, K_1, K_3.

$$K_i = (\text{row } i \text{ of } A)^{\mathrm{T}} \ (\boldsymbol{c_i}) \ (\text{row } i \text{ of } A) \qquad K = A^{\mathrm{T}}CA = \boldsymbol{K_1} + \boldsymbol{K_2} + \boldsymbol{K_3}.$$

Write the element matrices that add to $A^{\mathrm{T}}A$ in (1) for the 4-node line graph.

$$A^{\mathrm{T}}A = \begin{bmatrix} [K_1] & & \\ & [K_2] & \\ & & [K_3] \end{bmatrix} = \text{assembly of the nonzero entries of } K_1 + K_2 + K_3 \text{ from edges } 1, 2, \text{ and } 3$$

Solution The three "element matrices" for the three edges come from multiplying the three columns of A^{T} by the three rows of A. Then $A^{\mathrm{T}}A$ equals

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} [-1 \quad 1 \quad 0 \quad 0] + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} [0 \quad -1 \quad 1 \quad 0] + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} [0 \quad 0 \quad -1 \quad 1].$$

When the diagonal matrix C is included, those are multiplied by c_1, c_2, and c_3. Those products produce 2 by 2 blocks of nonzeros in 4×4 matrices:

$$K_1 = c_1 \begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & & \\ & & & \end{bmatrix} \quad K_2 = c_2 \begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix} \quad K_3 = c_3 \begin{bmatrix} & & & \\ & & & \\ & & 1 & -1 \\ & & -1 & 1 \end{bmatrix}$$

Then $\boldsymbol{A^{\mathrm{T}}CA = K_1 + K_2 + K_3}$. This 'assembly" of the element stiffness matrices just requires placing the nonzeros correctly into the final matrix $A^{\mathrm{T}}CA$.

14 An n by n grid has n^2 nodes. How many edges in this graph? How many interior nodes? How many nonzeros in A and in $L = A^{\mathrm{T}}A$? *There are no zeros in* L^{-1} !

Solution An n by n grid has n horizontal rows ($n-1$ edges on each row) and n vertical columns ($n-1$ edges down each column). Altogether $\boldsymbol{2n(n-1)}$ edges. There are

$(\boldsymbol{n}-\mathbf{2})^{\mathbf{2}}$ interior nodes—a square grid with the boundary nodes removed to reduce n to $n-2$.

Every edge produces 2 nonzeros (-1 and $+1$) in A. Then A has $\mathbf{4}\boldsymbol{n}(\boldsymbol{n}-\mathbf{1})$ nonzeros. The matrix $A^{\mathrm{T}}A$ has size n^2 with n^2 diagonal nonzeros—and off the diagonal of $A^{\mathrm{T}}A$ there are two -1's for each edge: altogether $n^2+4n(n-1)=\mathbf{5}\boldsymbol{n}^{\mathbf{2}}-\mathbf{4}\boldsymbol{n}$ nonzeros out of n^4 entries. For $n=2$, this means 12 nonzeros in a 4 by 4 matrix.

15 When only $\boldsymbol{e}=C^{-1}\boldsymbol{w}$ is eliminated from the 3-step framework, equation (**??**) shows

$$\textbf{Saddle-point matrix}\atop\textbf{Not positive definite}\qquad\begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first block row by $A^{\mathrm{T}}C$ and subtract from the second block row:

$$\textbf{After block elimination}\qquad\begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f}-A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

After m positive pivots from C^{-1}, why does this matrix have negative pivots? The two-field problem for $\boldsymbol{w}$ and $\boldsymbol{v}$ is finding a saddle point, not a minimum.

Solution The three equations $e=b-Av$ and $w=Ce$ and $A^{\mathrm{T}}w=f$ reduce to two equations when e is replaced by $C^{-1}w$:

$$\begin{aligned} C^{-1}\boldsymbol{w}&=\boldsymbol{b}-Av \\ A^{\mathrm{T}}\boldsymbol{w}&=\boldsymbol{f} \end{aligned}\qquad\text{become}\qquad\begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first equation by $A^{\mathrm{T}}C$ to get $A^{\mathrm{T}}\boldsymbol{w}=A^{\mathrm{T}}C\boldsymbol{b}-A^{\mathrm{T}}CA\boldsymbol{v}$. Subtract from the second equation $A^{\mathrm{T}}\boldsymbol{w}=\boldsymbol{f}$, to eliminate w:

$$A^{\mathrm{T}}C\boldsymbol{b}-A^{\mathrm{T}}CA\boldsymbol{v}=\boldsymbol{f}.$$

This gives the second row of the block matrix after elimination:

$$\begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f}-A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

The pivots of that matrix on the left side start with $1/c_1, 1/c_2, \ldots, 1/c_m$. Then we get the n pivots of $-A^{\mathrm{T}}CA$ which are **negative**, because this matrix is negative definite.

Altogether we are finding a saddle point $(\boldsymbol{v},\boldsymbol{w})$ of the energy (quadratic function). The derivative of that quadratic gives our linear equations. The block matrix in those equations has m positive eigenvalues and n negative eigenvalues.

16 The least squares equation $A^{\mathrm{T}}A\boldsymbol{v}=A^{\mathrm{T}}\boldsymbol{b}$ comes from the projection equation $A^{\mathrm{T}}\boldsymbol{e}=\mathbf{0}$ for the error $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{v}$. Write those two equations in the symmetric saddle point form of Problem 7 (with $\boldsymbol{f}=\mathbf{0}$).

In this case $\boldsymbol{w}=\boldsymbol{e}$ because the weighting matrix is $C=I$.

Solution Ordinary least squares for $A\boldsymbol{v}=\boldsymbol{b}$ separates the data vector $\boldsymbol{b}$ in two perpendicular parts:

$$\boldsymbol{b}=(A\widehat{\boldsymbol{v}})+(\boldsymbol{b}-A\widehat{\boldsymbol{v}})=(\text{projection of }\boldsymbol{b})+(\text{error in }\boldsymbol{b}).$$

The error $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{v}$ satisfies $A^{\mathrm{T}}\boldsymbol{e}=A^{\mathrm{T}}\boldsymbol{b}-A^{\mathrm{T}}A\boldsymbol{v}=\mathbf{0}$ (which means that $A^{\mathrm{T}}A\boldsymbol{v}=A^{\mathrm{T}}\boldsymbol{b}$, the key equation). That equation $d^{\mathrm{T}}\boldsymbol{e}=\mathbf{0}$ is Kirchhoff's Current Law for flows in a network. It is a candidate for the "most important equation in applied mathematics"—the conservation equation or continuity equation "flow in = flow out."

In the form of Problem 15 (with $C = I$) the equations are

$$\begin{bmatrix} I & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \boldsymbol{e} + A\boldsymbol{v} = \boldsymbol{b} \\ A^{\mathrm{T}}\boldsymbol{e} \quad = \mathbf{0}\,. \end{matrix}$$

17 Find the three eigenvalues and three pivots and the determinant of this saddle point matrix with $C = I$. One eigenvalue is negative because A has one column:

$$m = 2, n = 1 \qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution The eigenvalues come from $\det(M - \lambda I) = 0$:

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 1 \\ -1 & 1 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)^2 - 2(1-\lambda) = 0\,.$$

Then $(1-\lambda)(\lambda^2 - \lambda - 2) = 0$ and $(1-\lambda)(\lambda-2)(\lambda+1) = 0$ and the eigenvalues are $\lambda = \mathbf{1}, \mathbf{2}, \mathbf{-1}$. Check the sum $1 + 2 - 1 = \mathbf{2}$ equal to the trace (sum down the main diagonal $1 + 1 + 0 = \mathbf{2}$).

The determinant is the product $\lambda_1\lambda_2\lambda_3 = (1)(2)(-1) = -2$. Notice $m = 2$ positive λ's and $n = 1$ negative eigenvalue.

Elimination finds the three pivots (which also multiply to give $\det M = -2$):

$$\begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{-2} \end{bmatrix}.$$