DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 4.1, page 206

1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $\boldsymbol{v} = (x, y, z) = (2, 3, 4)$:

$$\begin{array}{l} 1x + 0y + 0z = 2 \\ 0x + 1y + 0z = 3 \\ 0x + 0y + 1z = 4 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Draw the four vectors in the column picture. Two times column 1 plus three times column 2 plus four times column 3 equals the right side $\boldsymbol{b}$.

The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 If the equations in Problem 1 are multiplied by $2, 3, 4$ they become $D\boldsymbol{V} = \boldsymbol{B}$:

$$\begin{array}{l} 2x + 0y + 0z = 4 \\ 0x + 3y + 0z = 9 \\ 0x + 0y + 4z = 16 \end{array} \quad \text{or} \quad D\boldsymbol{V} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} = \boldsymbol{B}$$

Why is the row picture the same? Is the solution $\boldsymbol{V}$ the same as $\boldsymbol{v}$? What is changed in the column picture—the columns or the right combination to give $\boldsymbol{B}$?

The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

3 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.

If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

5 The first of these equations plus the second equals the third:

$$\begin{array}{r} x + y + z = 2 \\ x + 2y + z = 3 \\ 2x + 3y + 2z = 5. \end{array}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line $\mathbf{L}$). Find three solutions on $\mathbf{L}$.

If x, y, z satisfy the first two equations they also satisfy the third equation. The line $\mathbf{L}$ of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$.

6 Move the third plane in Problem 5 to a parallel plane $2x+3y+2z=9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't _____ that line.

Equation $1+$ equation $2-$ equation 3 is now $0=-4$. Line misses plane; *no solution*.

7 In Problem 5 the columns are $(1,1,2)$ and $(1,2,3)$ and $(1,1,2)$. This is a "singular case" because the third column is _____. Find two combinations of the columns that give $\boldsymbol{b}=(2,3,5)$. This is only possible for $\boldsymbol{b}=(4,6,c)$ if $c=$ _____.

Column $3=$ Column 1 makes the matrix singular. Solutions $(x,y,z)=(1,1,0)$ or $(0,1,1)$ and you can add any multiple of $(-1,0,1)$; $\boldsymbol{b}=(4,6,c)$ needs $c=10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns).

8 Normally 4 "planes" in 4-dimensional space meet at a _____. Normally 4 vectors in 4-dimensional space can combine to produce $\boldsymbol{b}$. What combination of $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$, $(1,1,1,1)$ produces $\boldsymbol{b}=(3,3,3,2)$?

Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x}=(3,3,3,2)$ is $\boldsymbol{x}=(0,0,1,2)$ if A has columns $(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)$. The equations are $x+y+z+t=3, y+z+t=3, z+t=3, t=2$.

Problems 9–14 are about multiplying matrices and vectors.

9 Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:

(a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

(a) $A\boldsymbol{x}=(18,5,0)$ and (b) $A\boldsymbol{x}=(3,4,5,5)$.

10 Compute each $A\boldsymbol{x}$ in Problem 9 as a combination of the columns:

$$\text{9(a) becomes} \quad A\boldsymbol{x}=2\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}+2\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}+3\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}=\begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

How many separate multiplications for $A\boldsymbol{x}$, when the matrix is "3 by 3"?

Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x}$. By rows or by columns: **9** separate multiplications for 3 by 3.

11 Find the two components of $A\boldsymbol{x}$ by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

$A\boldsymbol{x}$ equals $(14,22)$ and $(0,0)$ and $(9,7)$.

12 Multiply A times $\boldsymbol{x}$ to find three components of $A\boldsymbol{x}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$A\boldsymbol{x}$ equals (z,y,x) and $(0,0,0)$ and $(3,3,6)$.

13 (a) A matrix with m rows and n columns multiplies a vector with _____ components to produce a vector with _____ components.

(b) The planes from the m equations $A\boldsymbol{x} = \boldsymbol{b}$ are in _____-dimensional space. The combination of the columns of A is in _____-dimensional space.

(a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space, but the columns are in m-dimensional space.

14 Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector $\boldsymbol{x} = (x, y, z, t)$ to produce $\boldsymbol{b}$. The solutions $\boldsymbol{x}$ fill a plane or "hyperplane" in 4-dimensional space. *The plane is* 3-*dimensional with no* 4D *volume*.

$2x+3y+z+5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [\,2 \quad 3 \quad 1 \quad 5\,]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

Problems 15–22 ask for matrices that act in special ways on vectors.

15 (a) What is the 2 by 2 identity matrix? I times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} y \\ x \end{bmatrix}$.

(a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 (a) What 2 by 2 matrix R rotates every vector by $90°$? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} y \\ -x \end{bmatrix}$.

(b) What 2 by 2 matrix R^2 rotates every vector by $180°$?

$90°$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

17 Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z). Q is the inverse of P.

18 What 2 by 2 matrix E subtracts the first component from the second component ? What 3 by 3 matrix does the same ?

$$E \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

19 What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z + x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z - x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1}, the two results are (_____) and (_____).

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = (3, 4, 8)$ and $E^{-1}E\boldsymbol{v}$ recovers $(3, 4, 5)$.

20 What 2 by 2 matrix P_1 projects the vector (x,y) onto the x axis to produce $(x,0)$? What matrix P_2 projects onto the y axis to produce $(0,y)$? If you multiply $(5,7)$ by P_1 and then multiply by P_2, you get (_____) and (_____).

$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis. $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ has $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21 What 2 by 2 matrix R rotates every vector through $45°$? The vector $(1,0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0,1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R.

$R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by 45°. The columns of R are the results from rotating $(1,0)$ and $(0,1)$!

22 Write the dot product of $(1,4,5)$ and (x,y,z) as a matrix multiplication $A\boldsymbol{v}$. The matrix A has one row. The solutions to $A\boldsymbol{v} = \boldsymbol{0}$ lie on a _____ perpendicular to the vector _____. The columns of A are only in _____ -dimensional space.

The dot product $A\boldsymbol{x} = \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = (1 by 3)(3 by 1) is zero for points (x,y,z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 In MATLAB notation, write the commands that define this matrix A and the column vectors $\boldsymbol{v}$ and $\boldsymbol{b}$. What command would test whether or not $A\boldsymbol{v} = \boldsymbol{b}$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$A = [1 \;\; 2 \;\; ; \;\; 3 \;\; 4]$ and $\boldsymbol{x} = [5 \;\; -2]'$ and $\boldsymbol{b} = [1 \;\; 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

24 If you multiply the 4 by 4 all-ones matrix A = ones(4) and the column v = ones(4,1), what is A∗v ? (Computer not needed.) If you multiply B = eye(4) + ones(4) times w = zeros(4,1) + 2∗ones(4,1), what is B∗w ?

ones$(4,4) *$ **ones**$(4,1) = [4 \;\; 4 \;\; 4 \;\; 4]'$; $B * \boldsymbol{w} = [10 \;\; 10 \;\; 10 \;\; 10]'$.

Questions 25–27 review the row and column pictures in 2, 3, and 4 dimensions.

25 Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$.

The row picture has two lines meeting at the solution $(4,2)$. The column picture will have $4(1,1) + 2(-2,1) = 4$(column 1) + 2(column 2) = right side $(0,6)$.

26 For two linear equations in three unknowns x,y,z, the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a _____.

The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a ***line***.

27 For four linear equations in two unknowns x and y, the row picture shows four _____. The column picture is in _____-dimensional space. The equations have no solution unless the vector on the right side is a combination of _____.

The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

Challenge Problems

28 Invent a 3 by 3 **magic matrix** M_3 with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be $8, 3, 4$. What is M_3 times $(1, 1, 1)$? What is M_4 times $(1, 1, 1, 1)$ if a 4 by 4 magic matrix has entries $1, \ldots, 16$?

$$M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}; M_3(1,1,1) = (15,15,15);$$

$M_4(1, 1, 1, 1) = (34, 34, 34, 34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

29 Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are the first two columns of a 3 by 3 matrix A. Which third columns $\boldsymbol{w}$ would make this matrix singular ? Describe a typical column picture of $A\boldsymbol{v} = \boldsymbol{b}$ in that singular case, and a typical row picture (for a random $\boldsymbol{b}$).

A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution*.

30 **Multiplying by $\boldsymbol{A}$ is a "linear transformation".** Those important words mean:

If $\boldsymbol{w}$ is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

It is this "***linearity***" $A\boldsymbol{w} = cA\boldsymbol{u} + dA\boldsymbol{v}$ that gives us the name *linear algebra*.

If $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\boldsymbol{u}$ and $A\boldsymbol{v}$ are the columns of A.

Combine $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. **If** $\boldsymbol{w} = \begin{bmatrix} \mathbf{5} \\ \mathbf{7} \end{bmatrix}$ **how is** $A\boldsymbol{w}$ **connected to** $A\boldsymbol{u}$ **and** $A\boldsymbol{v}$?

$\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

31 A 9 by 9 ***Sudoku matrix*** S has the numbers $1, \ldots, 9$ in every row and column, and in every 3 by 3 block. For the all-ones vector $\boldsymbol{v} = (1, \ldots, 1)$, what is $S\boldsymbol{v}$?

A better question is: **Which row exchanges will produce another Sudoku matrix** ? Also, which exchanges of block rows give another Sudoku matrix ?

Section 4.5 will look at all possible permutations (reorderings) of the rows. I see 6 orders for the first 3 rows, all giving Sudoku matrices. Also 6 permutations of the next 3 rows, and of the last 3 rows. And 6 block permutations of the block rows ?

$\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

32 Suppose the second row of A is some number c times the first row:

$$A = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix}.$$

Then if $a \neq 0$, the second column of A is what number d times the first column? **A square matrix with dependent rows will also have dependent columns**. This is a crucial fact coming soon.

The second column is $d = b/a$ times the first column. So the columns are "dependent" when the rows are "dependent".

Problem Set 4.2, page 215

Problems 1–10 are about elimination on 2 by 2 systems.

1 What multiple ℓ_{21} of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this step, solve the triangular system by back substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution?

Multiply by $\ell_{21} = \frac{10}{2} = 5$ and subtract to find $2x + 3y = 14$ and $-6y = 6$. The pivots to circle are 2 and -6. If the right hand side is multiplied by 4, the solution is multiplied by 4.

2 If you find solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ to $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{c}$, what is the solution $\boldsymbol{u}$ to $A\boldsymbol{u} = \boldsymbol{b} + \boldsymbol{c}$? What is the solution $\boldsymbol{U}$ to $A\boldsymbol{U} = 3\boldsymbol{b} + 4\boldsymbol{c}$? (We saw superposition for linear differential equations, it works in the same way for all linear equations.)

If $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{c}$ then $A(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{b} + \boldsymbol{c}$. The solution to $A\boldsymbol{U} = 3\boldsymbol{b} + 4\boldsymbol{c}$ is $U = 3\boldsymbol{v} + 4\boldsymbol{w}$.

3 What multiple of equation 1 should be *subtracted* from equation 2?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution?

Subtract $-\frac{1}{2}$ times equation 1 from equation 2. This leaves $0x + 3y = 3$. Then $y = \mathbf{1}$ and the first equation becomes $2x - 4 = 6$ to give $x = 5$.
If the right side changes from $(6, 0)$ to $(-6, 0)$ the solution changes from $(5, 1)$ to $(-5, -1)$.

4 What multiple ℓ of equation 1 should be subtracted from equation 2 to remove cx?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? The second pivot is missing when $ad = bc$: that is the *singular case*.

Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

Singular system

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

$6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. (The two lines in the row picture are the same line, containing all solutions).

6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 For which a does elimination break down (1) permanently or (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the temporary breakdown by a row exchange.

If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 For which three numbers k does elimination break down? Which is fixed by a row exchange? In these three cases, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture for $\boldsymbol{b} = (1, 2)$ and $(1, 0)$.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).

10 In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ _____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c =$ _____.

The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = c = 16$.

11 (Recommended) A system of linear equations can't have exactly two solutions. If (x, y) and (X, Y) are two solutions to $A\boldsymbol{v} = \boldsymbol{b}$, what is another solution?

If $\boldsymbol{v} = (x, y)$ and also $\boldsymbol{V} = (X, Y)$ solve the system $A\boldsymbol{v} = \boldsymbol{b}$, then another solution is $\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{V}$. (All combinations $\boldsymbol{u} = c\boldsymbol{v} + (1 - c)\boldsymbol{V}$ will be solutions since $A\boldsymbol{u} = cA\boldsymbol{v} + (1 - c)A\boldsymbol{V} = c\boldsymbol{b} + (1 - c)\boldsymbol{b} = \boldsymbol{b}$.)

Problems 12–20 study elimination on 3 by 3 systems (and possible failure).

12 Reduce this system to upper triangular form by two row operations:

$$\begin{array}{lr} & 2x+3y+z\ \ = 8 \\ \text{Eliminate } x \ \rightarrow & 4x+7y+5z = 20 \\ \text{Eliminate } y \ \rightarrow & -2y+2z = 0. \end{array}$$

Circle the pivots. Solve by back substitution for z, y, x.

Elimination leads to an upper triangular system; then comes back substitution.

$$\begin{array}{rcrl} 2x + 3y + \ \ z = 8 & & x = 2 & \\ y + 3z = 4 & \text{gives} & y = 1 & \text{If a zero is at the start of row 2 or 3,} \\ 8z = 8 & & z = 1 & \text{that avoids a row operation.} \end{array}$$

13 Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \qquad &= 3 \\ 4x - 5y + \ \ z &= 7 \\ 2x - \ \ y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract _____ times row _____ from row _____.

$$\begin{array}{rcrcrcr} 2x - 3y \qquad = 3 & & 2x - 3y = 3 & & 2x - 3y = 3 & & x = 3 \\ 4x - 5y + \ \ z = 7 & \text{gives} & y + \ \ z = 1 & \text{and} & y + \ \ z = 1 & \text{and} & y = 1 \\ 2x - \ \ y - 3z = 5 & & 2y + 3z = 2 & & -5z = 0 & & z = 0 \end{array}$$

Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

14 Which number d forces a row exchange? What is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 Which number b leads later to a row exchange? Which b leads to a singular problem that row exchanges cannot fix? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \quad &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. A solution is $(1,1,-1)$.

16 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form.

(b) Construct a 3 by 3 system that needs a row exchange for pivot 2, but breaks down for pivot 3.

(a) **Example of 2 exchanges**
$$\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$
(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then break down**
$$\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$
(rows 1 and 3 are not consistent)

17 If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

Equal rows
$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned}$$

Equal columns
$$\begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

If row $1 =$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column $2 =$ column 1, then column 2 has no pivot.

18 Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\boldsymbol{b} = (1, 10, 100)$ and how many with $\boldsymbol{b} = (0,0,0)$?

Example $x+2y+3z=0$, $4x+8y+12z=0$, $5x+10y+15z=0$ has 9 different coefficients but rows 2 and 3 become $0=0$: infinitely many solutions.

19 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z=1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z = t-5$. If $q=-4$ the system is singular—no third pivot. Then if $t=5$ the third equation is $0=0$. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Three planes can fail to have an intersection point, *even if no planes are parallel*. The system is singular if row 3 is a combination of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$.

Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2=$row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=1$. No parallel planes but still no solution.

21 Find the pivots and the solution for both systems ($A\boldsymbol{v} = \boldsymbol{b}$ and $S\boldsymbol{w} = \boldsymbol{b}$):

$$\begin{aligned} 2x + y \quad\quad &= 0 \\ x + 2y + z \quad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned} \qquad\qquad \begin{aligned} 2x - y \quad\quad &= 0 \\ -x + 2y - z \quad &= 0 \\ - y + 2z - t &= 0 \\ - z + 2t &= 5. \end{aligned}$$

(a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x+y=0$, $\frac{3}{2}y+z=0$, $\frac{4}{3}z+t=0$, $\frac{5}{4}t=5$ after elimination. Back substitution gives $t=4, z=-3, y=2, x=-1$.

(b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1,2,3,4)$ instead of $(-1,2,-3,4)$.

22 If you extend Problem 21 following the $1,2,1$ pattern or the $-1,2,-1$ pattern, what is the fifth pivot? What is the nth pivot? S is my favorite matrix.

The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If elimination leads to $x+y=1$ and $2y=3$, find three possible original problems.

If ordinary elimination leads to $x+y=1$ and $2y=3$, the original second equation could be $2y+\ell(x+y)=3+\ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y=3$.

24 For which two numbers a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a=2$ or $a=0$.

25 For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of } a.$$

$a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

26 Look for a matrix that has row sums 4 and 8, and column sums 2 and s:

$$\text{Matrix } = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if $s =$ _____. Then find two different matrices that have the correct row and column sums. *Extra credit*: Write down the 4 by 4 system $A\boldsymbol{v} = (4,8,2,s)$ with $\boldsymbol{v} = (a,b,c,d)$ and make A triangular by elimination.

Solvable for $s=10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2+s$ on the right sides). The four equations for a,b,c,d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

27 Elimination in the usual order gives what matrix U and what solution (x, y, z) to this "lower triangular" system? We are really solving by *forward substitution*:

$$\begin{aligned} 3x \qquad\quad &= 3 \\ 6x + 2y \quad &= 8 \\ 9x - 2y + z &= 9. \end{aligned}$$

Elimination leaves the diagonal matrix diag$(3, 2, 1)$ in $3x = 3, 2y = 2, z = 4$. Then $x = 1, y = 1, z = 4$.

28 Create a MATLAB command `A(2, : ) = ...` for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix A is already known.

$A(2, :) = A(2, :) - 3 * A(1, :)$ subtracts 3 times row 1 from row 2.

29 If the last corner entry of A is $A(5, 5) = 11$ and the last pivot of A is $U(5, 5) = 4$, what different entry $A(5, 5)$ would have made A singular?

A change up or down in $A(5, 5)$ produces the same change in $U(5, 5)$. If $A(5, 5) = 11$ gave $U(5, 5) = 4$, then subtract 4: $A(5, 5) = \mathbf{7}$ will give $U(5, 5) = 0$ and a singular matrix—zero in the last pivot position $U(5, 5)$.

Challenge Problems

30 Suppose elimination takes A to U without row exchanges. Then row i of U is a combination of which rows of A? If $A\boldsymbol{v} = \mathbf{0}$, is $U\boldsymbol{v} = \mathbf{0}$? If $A\boldsymbol{v} = \boldsymbol{b}$, is $U\boldsymbol{v} = \boldsymbol{b}$?

Row j of U is a combination of rows $1, \ldots, j$ of A. If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular.*

31 Start with 100 equations $A\boldsymbol{v} = \mathbf{0}$ for 100 unknowns $\boldsymbol{v} = (v_1, \ldots, v_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".

(a) Elimination takes linear combinations of the rows. So this singular system has the singular property: Some linear combination of the 100 **rows** is _____ .

(b) Singular systems $A\boldsymbol{v} = \mathbf{0}$ have infinitely many solutions. This means that some linear combination of the 100 ***columns*** is _____ .

(c) Invent a 100 by 100 singular matrix with no zero entries.

(d) For your matrix, describe in words the row picture and the column picture of $A\boldsymbol{v} = \mathbf{0}$. Not necessary to draw 100-dimensional space.

The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A =$ **eye**(100). A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 4.3, page 223

Problems 1–16 are about the laws of matrix multiplication .

1 A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results ?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B+C).$$

If all entries of A, B, C, D are 1, then $BA = 3$ **ones**(5) is 5 by 5; $AB = 5$ **ones**(3) is 3 by 3; $ABD = 15$ **ones**$(3,1)$ is 3 by 1. DBA and $A(B+C)$ are not defined.

2 What rows or columns or matrices do you multiply to find

(a) the third column of AB ?

(b) the first row of AB ?

(c) the entry in row 3, column 4 of AB ?

(d) the entry in row 1, column 1 of CDE ?

(a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

3 Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

$AB + AC$ is the same as $A(B+C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

4 In Problem 3, multiply A times BC. Then multiply AB times C.

$A(BC) = (AB)C$ by the *associative law*. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ from column 1 of AB and row 2 of C (multiply columns times rows).

5 Compute A^2 and A^3. Make a prediction for A^5 and A^n :

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

(a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \underline{\qquad} + B^2$.

$(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

7 True or false. Give a specific example when false:

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

(d) $(AB)^2 = A^2B^2$.

(a) True (b) False (c) True (d) False: usually $(AB)^2 \neq A^2B^2$.

8 How is each row of DA and EA related to the rows of A, when

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}?$$

How is each column of AD and AE related to the columns of A?

The rows of DA are 3 (row 1 of A) and 5 (row 2 of A). Both rows of EA are row 2 of A. The columns of AD are 3 (column 1 of A) and 5 (column 2 of A). The first column of AE is zero, the second is column 1 of A + column 2 of A.

9 Row 1 of A is added to row 2. This gives EA below. Then column 1 of EA is added to column 2 to produce $(EA)F$. Notice E and F in boldface.

$$EA = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

$$(EA)F = (EA) \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+c & a+c+b+d \end{bmatrix}.$$

Do those steps in the opposite order, first multiply AF and then $E(AF)$. Compare with $(EA)F$. What law is obeyed by matrix multiplication?

$AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 Row 1 of A is added to row 2 to produce EA. Then F adds row 2 of EA to row 1. Now F is on the left, for row operations. The result is $F(EA)$:

$$F(EA) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix}.$$

Do those steps in the opposite order: first add row 2 to row 1 by FA, then add row 1 of FA to row 2. What law is or is not obeyed by matrix multiplication?

$FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not the same as $F(EA)$ because multiplication is not commutative.

11 (3 by 3 matrices) Choose the only B so that for every matrix A

(a) $BA = 4A$

(b) $BA = 4B$ (tricky)

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged

(d) All rows of BA are the same as row 1 of A.

(a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

12 Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I. The only matrices that commute with B and C and all other 2 by 2 matrices are $A =$ multiple of I.

$AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $\boldsymbol{b} = \boldsymbol{c} = \mathbf{0}$. Then $AC = CA$ gives $\boldsymbol{a} = \boldsymbol{d}$. The only matrices that commute with B and C (and all other matrices) are multiples of I: $A = aI$.

13 Which of the following matrices are guaranteed to equal $(A - B)^2$: $A^2 - B^2$, $(B - A)^2$, $A^2 - 2AB + B^2$, $A(A - B) - B(A - B)$, $A^2 - AB - BA + B^2$?

$(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$. In a typical case (when $AB \neq BA$) the matrix $A^2 - 2AB + B^2$ is different from $(A - B)^2$.

14 True or false:

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

(a) True (A^2 is only defined when A is square) (b) False (if A is m by n and B is n by m, then AB is m by m and BA is n by n). (c) True (d) False (take $B = 0$).

15 If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector $\boldsymbol{x}$ with n components?

(b) A multiplies an n by p matrix B?

(c) A multiplies itself to produce A^2? Here $m = n$ and A is square.

(a) mn (use every entry of A) (b) $mnp = p \times$part (a) (c) n^3 (n^2 dot products).

16 For $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, compute these answers *and nothing more*:

(a) column 2 of AB (b) row 2 of AB (c) row 2 of A^2

(d) row 2 of A^3.

(a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

Problems 17–19 use a_{ij} for the entry in row i, column j of A.

17 Write down the 3 by 3 matrix A whose entries are

(a) $a_{ij} =$ minimum of i and j (b) $a_{ij} = (-1)^{i+j}$ (c) $a_{ij} = i/j$.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ has $a_{ij} = \min(i,j)$. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has $a_{ij} = (-1)^{i+j} =$ "alternating sign matrix". $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$ has $a_{ij} = i/j$ (this will be an example of a *rank one matrix*).

18 What words would you use to describe each of these classes of matrices? Give a 3 by 3 example in each class. Which matrix belongs to all four classes?

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i < j$ (c) $a_{ij} = a_{ji}$
(d) $a_{ij} = a_{1j}$.

Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 The entries of A are a_{ij}. Assuming that zeros don't appear, what is

(a) the first pivot?
(b) the multiplier ℓ_{31} of row 1 to be subtracted from row 3?
(c) the new entry that replaces a_{32} after that subtraction?
(d) the second pivot?

(a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

Problems 20–24 involve powers of A.

20 Compute A^2, A^3, A^4 and also $A\boldsymbol{v}, A^2\boldsymbol{v}, A^3\boldsymbol{v}, A^4\boldsymbol{v}$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

$A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^4 =$ zero matrix for *strictly triangular* A.

Then $A\boldsymbol{v} = A\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$, $A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$, $A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A^4\boldsymbol{v} = 0$.

21 Find all the powers $A^2, A^3, \ldots$ and $AB, (AB)^2, \ldots$ for

$$A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$A = A^2 = A^3 = \cdots = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ but $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$ and $(AB)^2 =$ zero matrix!

22 By trial and error find real nonzero 2 by 2 matrices such that

$$A^2 = -I \qquad BC = 0 \qquad DE = -ED \text{ (not allowing } DE = 0).$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;
$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

23 (a) Find a nonzero matrix A for which $A^2 = 0$.
(b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. Note: Any matrix $A =$ column times row $= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ will have $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = 0$ if $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ but $A^3 = 0$; strictly triangular as in Problem 20.

24 By experiment with $n = 2$ and $n = 3$ predict A^n for these matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

$$(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix},\ (A_2)^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ (A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}.$$

Problems 25–31 use column-row multiplication and block multiplication.

25 Multiply A times I using columns of A (3 by 3) times rows of I.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} d \\ e \\ h \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

26 Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\qquad} = \underline{\qquad}.$$

Columns of A times rows of B $\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}3 & 3 & 0\end{bmatrix}+\begin{bmatrix}0\\4\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\end{bmatrix}=\begin{bmatrix}3 & 3 & 0\\6 & 6 & 0\\6 & 6 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\4 & 8 & 4\\1 & 2 & 1\end{bmatrix}=$
$\begin{bmatrix}3 & 3 & 0\\10 & 14 & 4\\7 & 8 & 1\end{bmatrix}=AB.$

27 Show that the product of two upper triangular matrices is always upper triangular:

$$AB=\begin{bmatrix}x & x & x\\0 & x & x\\0 & 0 & x\end{bmatrix}\begin{bmatrix}x & x & x\\0 & x & x\\0 & 0 & x\end{bmatrix}=\begin{bmatrix}x & & \\0 & & \\0 & 0 & x\end{bmatrix}.$$

Proof using dot products (*Row-times-column*) (Row 2 of A) $\cdot$ (column 1 of B)$=0$. Which other dot products give zeros?

Proof using full matrices (*Column-times-row*) Draw x's and 0's in (column 2 of A) times (row 2 of B). Also show (column 3 of A) times (row 3 of B).

(a) (row 3 of A) $\cdot$ (column 1 of B) and (row 3 of A) $\cdot$ (column 2 of B) are both zero.

(b) $\begin{bmatrix}x\\x\\0\end{bmatrix}\begin{bmatrix}0 & x & x\end{bmatrix}=\begin{bmatrix}0 & x & x\\0 & x & x\\0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}x\\x\\x\end{bmatrix}\begin{bmatrix}0 & 0 & x\end{bmatrix}=\begin{bmatrix}0 & 0 & x\\0 & 0 & x\\0 & 0 & x\end{bmatrix}$: **both upper**.

28 If A is 2 by 3 with rows 1, 1, 1 and 2, 2, 2, and B is 3 by 4 with columns 1, 1, 1 and 2, 2, 2 and 3, 3, 3 and 4, 4, 4, use each of the four multiplication rules to find AB:

(1) Rows of A times columns of B. **Inner products** (each entry in AB)

(2) Matrix A times columns of B. **Columns of AB**

(3) Rows of A times the matrix B. **Rows of AB**

(4) Columns of A times rows of B. **Outer products** (3 matrices add to AB)

$$AB=\begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{bmatrix}=\begin{bmatrix}3 & 6 & 9 & 12\\6 & 12 & 18 & 24\end{bmatrix}.$$

(1) Two rows of A times four columns of $B=$ **eight** numbers

(2) A times the first column of B gives $\begin{bmatrix}3\\6\end{bmatrix}$. The later columns are multiplied by 2, 3, and 4.

(3) The first row of A is multiplied by B to give 3, 6, 9, 12. The second row of A is doubled so the second row of AB is doubled.

(4) Column times row multiplication gives three matrices (in this case they are all the same!)
$\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}=\begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 6 & 8\end{bmatrix}$ times 3 gives AB.

29 Which matrices E_{21} and E_{31} produce zeros in the $(2,1)$ and $(3,1)$ positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA.

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ produce zeros in the $2,1$ and $3,1$ entries.

Multiply E's to get $E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. Then $EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is the result of both E's since $(E_{31}E_{21})A = E_{31}(E_{21}A)$.

30 **Block multiplication** produces zeros below the pivot in one big step:

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & D - \boldsymbol{cb}/a \end{bmatrix} \text{ with vectors } \mathbf{0}, \boldsymbol{b}, \boldsymbol{c}.$$

In Problem 29, what are $\boldsymbol{c}$ and D and what is the block $D - \boldsymbol{cb}/a$?

In **29**, $\boldsymbol{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - \boldsymbol{cb}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

31 With $i^2 = -1$, the product of $(A + iB)$ and $(\boldsymbol{x} + i\boldsymbol{y})$ is $A\boldsymbol{x} + iB\boldsymbol{x} + iA\boldsymbol{y} - B\boldsymbol{y}$. Use blocks to separate the real part without i from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ B\boldsymbol{x} + A\boldsymbol{y} \end{bmatrix}$ real part / imaginary part. Complex matrix times complex vector needs **4** real times real multiplications.

32 (*Very important*) Suppose you solve $A\boldsymbol{v} = \boldsymbol{b}$ for three special right sides $\boldsymbol{b}$:

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are the columns of a matrix X, what is A times X?

A times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

33 If the three solutions in Question 32 are $\boldsymbol{v}_1 = (1,1,1)$ and $\boldsymbol{v}_2 = (0,1,1)$ and $\boldsymbol{v}_3 = (0,0,1)$, solve $A\boldsymbol{v} = \boldsymbol{b}$ when $\boldsymbol{b} = (3,5,8)$. Challenge problem: What is A?

$\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $\boldsymbol{x}_1 = (1,1,1), \boldsymbol{x}_2 = (0,1,1), \boldsymbol{x}_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

34 Practical question Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The same answer comes from A times BC, now with $mnq + npq$ separate multiplications. Notice npq for BC.

(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?

(b) With N-component vectors, would you choose $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ or $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$?

(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

Multiplying $AB = (m$ by $n)(n$ by $p)$ needs mnp multiplications. Then $(AB)C$ needs mpq more. Multiply $BC = (n$ by $p)(p$ by $q)$ needs npq and then $A(BC)$ needs mnq.

(a) If m, n, p, q are $2, 4, 7, 10$ we compare $(2)(4)(7) + (2)(7)(10) = \mathbf{196}$ with the larger number $(2)(4)(10) + (4)(7)(10) = \mathbf{360}$. So AB first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are N by 1, then $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ needs $2N$ multiplications but $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$ needs N^2 to find $\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}$ and N^2 more to multiply by the row vector $\boldsymbol{u}^{\mathrm{T}}$. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1} + n^{-1}$ with $p^{-1} + m^{-1}$. This yields a simple important rule. If matrices A and B are multiplying $\boldsymbol{v}$ for $AB\boldsymbol{v}$, **don't multiply the matrices first**.

35 Unexpected fact A friend in England looked at powers of a 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \quad A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \quad A^4 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

He noticed that the ratios $2/3$ and $10/15$ and $54/81$ are all the same. This is true for all powers. It doesn't work for an $n \times n$ matrix, unless A is tridiagonal. One neat proof is to look at the equal $(1,1)$ entries of A^nA and AA^n. Can you use that idea to show that $B/C = 2/3$ in this example?

The off-diagonal ratio $\frac{2}{3}$ in $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ stays the same for all powers of A^n. Peter Larcombe gave a proof by induction. Ira Gessel compared the $(1,1)$ entries on the left and right sides of the true equation $A^nA = AA^n$:

$$A^nA = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The $(1,1)$ entries give $A + 3B = A + 2C$ and therefore $B/C = 2/3$. This ratio stays the same for A^{-1}.

The same idea applies when the matrix A is N by N, provided it is tridiagonal (three nonzero diagonals):

$$\text{The } (1,1) \text{ entry of } A^nA = \begin{bmatrix} A & B & E \\ C & D & F \\ G & H & I \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ 3 & 4 & 5 \\ & 6 & 7 \end{bmatrix} \text{ is still } A + 3B.$$

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1 Find the inverses of A, B, C (directly or from the 2 by 2 formula):

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

A simple row exchange has $P^2 = I$ so $P^{-1} = P$. Here $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always P^{-1} = "transpose" of P, coming in Section 2.7.

3 Solve for the first column (x, y) and second column (t, z) of A^{-1}:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination.

4 Show that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ is not invertible by trying to solve $AA^{-1} = I$ for column 1 of A^{-1}:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(*For a different* A, could column 1 of A^{-1} be possible to find but not column 2?)

The equations are $x + 2y = 1$ and $3x + 6y = 0$. No solution because 3 times equation 1 gives $3x + 6y = 3$.

5 Find an upper triangular U (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$.

An upper triangular U with $U^2 = I$ is $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ for any a. And also $-U$.

6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.
(b) If $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, find two different matrices such that $AB = AC$.

(a) Multiply $AB = AC$ by A^{-1} to find $B = C$ (since A is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\boldsymbol{v} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{v} = \boldsymbol{b}$?

(c) What happens to row 3 in elimination?

(a) In $A\boldsymbol{x} = (1, 0, 0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Why is no third pivot ?

(a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

9 Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible and how would you find B^{-1} from A^{-1}?

If you exchange rows 1 and 2 of A to reach B, you exchange **columns** 1 and 2 of A^{-1} to reach B^{-1}. In matrix notation, $B = PA$ has $B^{-1} = A^{-1}P^{-1} = A^{-1}P$ for this P.

10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block of B).

11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

(a) If $B = -A$ then certainly $A + B$ = zero matrix is not invertible. (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular but $A + B = I$ is invertible.

12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

Multiply $C = AB$ on the right by C^{-1} and on the left by A^{-1} to get $A^{-1} = BC^{-1}$.

13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

$M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply on the left by C and the right by A : $B^{-1} = CM^{-1}A$.

14 If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

Notice the order. The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A$ is ______.

$B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

15 Prove that a matrix with a column of zeros cannot have an inverse.

If A has a column of zeros, so does BA. Then $BA = I$ is impossible. There is no A^{-1}.

16 Multiply $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ times $\begin{bmatrix} \mathbf{d} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{a} \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

17 (a) What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

$E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E$. Reverse the order and change -1 to $+1$ to get inverses $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} =$ $L = E^{-1}$. Notice the 1's unchanged by multiplying in this order.

18 If B is the inverse of A^2, show that AB is the inverse of A.

$A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

19 (Recommended) A is a 4 by 4 matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

$$A^{-1} = \begin{bmatrix} 1 & -a & 0 & 0 \\ & 1 & -b & 0 \\ & & 1 & -c \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ab & abc \\ & 1 & b & bc \\ & & 1 & c \\ & & & 1 \end{bmatrix}.$$

20 Find the numbers a and b that give the inverse of 5 ∗ eye(4) − ones(4,4) :

$$[\,5I-\text{ones}\,]^{-1} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 ∗ eye(5) − ones(5,5) ? In MATLAB, $I =$ eye.

The $(1,1)$ entry requires $4a - 3b = 1$; the $(1,2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b = a$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

21 Sixteen 2 by 2 matrices contain only 1's and 0's. How many of them are invertible?

Six of the sixteen $0-1$ matrices are invertible, including all four with three 1's.

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1}.

22 Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix};$$
$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

23 Follow the 3 by 3 text example of Gauss-Jordan but with all plus signs in A. Eliminate above and below the pivots to reduce $[\,A \;\; I\,]$ to $[\,I \;\; A^{-1}\,]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

$$[\boldsymbol{A}\;\; \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] =$$
$[\boldsymbol{I}\;\boldsymbol{A}^{-1}]$.

24 Use Gauss-Jordan elimination on $[\,U \;\; I\,]$ to find the upper triangular U^{-1}:

$$\boldsymbol{U}\boldsymbol{U}^{-1} = \boldsymbol{I} \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

25 Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A \;\; I\,]$ and $[\,B \;\; I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{\mathbf{-1}} = \dfrac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$; $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so B^{-1} does not exist.

26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{6} \end{bmatrix}$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1}.

$E_{21}A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. $E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to reach $DE_{12}E_{21}A = I$. Then $A^{-1} = DE_{12}E_{21} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$.

27 Invert these matrices A by the Gauss-Jordan method starting with $[\,A \quad I\,]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

28 Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}$.
This is $\begin{bmatrix} I & A^{-1} \end{bmatrix}$: row exchanges are certainly allowed in Gauss-Jordan.

29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1's down the main diagonal is invertible.
(c) If A is invertible then A^{-1} and A^2 are invertible.

(a) True (If A has a row of zeros, then every AB has too, and $AB = I$ is impossible) (b) False (the matrix of all ones is singular even with diagonal 1's: *ones* (3) has 3 equal rows) (c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

This A is not invertible for $c = 7$ (equal columns), $c = 2$ (equal rows), $c = 0$ (zero column).

31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)} \begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A \quad I\,]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. When the triangular A alternates 1 and -1 on its diagonal, A^{-1} is *bidiagonal* with 1's on the diagonal and first superdiagonal.

33 **(Puzzle)** Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

A can be invertible with diagonal zeros. B is singular because each row adds to zero.

34 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

$\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

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Questions 1–9 are about transposes A^{T} and symmetric matrices $S = S^{\mathrm{T}}$.

1 Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ has $A^{\mathrm{T}} = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$;
$A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ has $A^{\mathrm{T}} = A$ and $A^{-1} = \dfrac{1}{c^2} \begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^{\mathrm{T}}$.

2 (a) Find 2 by 2 symmetric matrices A and B so that AB is not symmetric.

(b) With $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$, show that $AB = BA$ ensures that AB will now be symmetric. The product is symmetric only when A commutes with B.

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ give $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad$ and $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) If $AB = BA$ and $A^{\mathrm{T}} = A, B^{\mathrm{T}} = B$ then $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} = BA = AB$. Thus AB is symmetric when A and B commute.

3 (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order*?

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is _____ triangular.

(a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$.
(b) If U is upper triangular, so is U^{-1}: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

4 Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matrix).

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. The diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

5 Every square matrix A has a symmetric part and an antisymmetric part:

$$A = \text{symmetric} + \text{antisymmetric} = \left(\frac{A + A^{\mathrm{T}}}{2}\right) + \left(\frac{A - A^{\mathrm{T}}}{2}\right).$$

Transpose the antisymmetric part to get *minus* that part. Split these in two parts:

$$A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Transposing $\frac{1}{2}(A - A^{\mathrm{T}})$ gives $\frac{1}{2}(A^{\mathrm{T}} - A)$: this part is antisymmetric.

$$\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 3 \\ -4 & -3 & 0 \end{bmatrix}.$$

6 The transpose of a block matrix $M = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is $M^{\mathrm{T}} =$ _____. Test an example to be sure. Under what conditions on A, B, C, D is the block matrix symmetric?

$M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = C$ and $D^{\mathrm{T}} = D$.

7 True or false:

(a) The block matrix $\left[\begin{smallmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{smallmatrix}\right]$ is automatically symmetric.

(b) If A and B are symmetric then their product AB is symmetric.

(c) If A is not symmetric then A^{-1} is not symmetric.

(d) When A, B, C are symmetric, the transpose of ABC is CBA.

(a) False: $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is symmetric only if $A = A^{\mathrm{T}}$. (b) False: The transpose of AB is $B^{\mathrm{T}}A^{\mathrm{T}} = BA$ when A and B are symmetric $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ transposes to $\begin{bmatrix} 0 & A^{\mathrm{T}} \\ A^{\mathrm{T}} & 0 \end{bmatrix}$. So $(AB)^{\mathrm{T}} = AB$ needs $BA = AB$. (c) True: Invertible symmetric matrices have symmetric in verses! Easiest proof is to transpose $AA^{-1} = I$. (d) True: $(ABC)^{\mathrm{T}}$ is $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}(= CBA$ for symmetric matrices A, B, and C).

8 (a) How many entries of S can be chosen independently, if $S = S^{\mathrm{T}}$ is 5 by 5?

(b) How many entries can be chosen if A is *skew-symmetric*? ($A^{\mathrm{T}} = -A$).

Answers: **15** and **10**. If $S = S^{\mathrm{T}}$ is 5 by 5, its 5 diagonal entries and 10 entries above the diagonal are free to choose. If $A^{\mathrm{T}} = -A$, the 5 diagonal entries of A must be zero.

9 Transpose the equation $A^{-1}A = I$. The result shows that the inverse of A^{T} is _____. If S is symmetric, **how does this show that S^{-1} is also symmetric**?

$A^{-1}A = I$ transposes to $A^{\mathrm{T}}(A^{-1})^{\mathrm{T}} = I$. This shows that the inverse of A^{T} is $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$. If S is symmetric ($S^{\mathrm{T}} = S$) then this statement becomes $S^{-1} = (S^{-1})^{\mathrm{T}}$. Therefore **$S^{-1}$ is symmetric**.

Questions 10–14 are about permutation matrices.

10 Why are there $n!$ permutation matrices of size n? They give $n!$ orders of $1, \ldots, n$.

The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices . . . ($n!$ overall).

11 If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

$$P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ but } P_2P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If P_3 and P_4 exchange *different* pairs of rows, $P_3P_4 = P_4P_3$ does both exchanges.

12 There are 12 "*even*" permutations of $(1, 2, 3, 4)$, with an *even number of exchanges*. Two of them are $(1, 2, 3, 4)$ with no exchanges and $(4, 3, 2, 1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, just order the numbers.

$(3, 1, 2, 4)$ and $(2, 3, 1, 4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2, 1, 4, 3)$ and $(3, 4, 1, 2)$ exchange 2 pairs. $(1, 2, 3, 4)$, $(4, 3, 2, 1)$ make 12 even P's.

13 If P has 1's on the antidiagonal from $(1, n)$ to $(n, 1)$, describe PAP. Is P even?

The "reverse identity" P takes $(1, \ldots, n)$ into $(n, \ldots, 1)$. When rows and also columns are reversed, $(PAP)_{ij}$ is $(A)_{n-i+1,n-j+1}$. In particular $(PAP)_{11}$ is A_{nn}.

14 (a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

(b) Find a 4 by 4 permutation with $P^4 \neq I$.

A cyclic $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose will have $P^3 = I : (1, 2, 3) \to (2, 3, 1) \to (3, 1, 2) \to (1, 2, 3)$. $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same P has $\widehat{P}^4 = \widehat{P} \neq I$.

Questions 15–18 are about first differences A and second differences $A^{\mathrm{T}}A$ and AA^{T}.

15 Write down the 5 by 4 backward difference matrix A.

(a) Compute the symmetric second difference matrices $S = A^{\mathrm{T}}A$ and $L = AA^{\mathrm{T}}$.

(b) Show that S is invertible by finding S^{-1}. Show that L is singular.

$$A = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad S = A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$L = AA^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}.$$

L (5 by 5) is singular: $L\boldsymbol{x} = \mathbf{0}$ for $\boldsymbol{x} = (1,1,1,1,1)$.

S (4 by 4) is invertible: $S^{-1} = \dfrac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

16 In Problem 15, find the pivots of S and L (4 by 4 and 5 by 5). The pivots of S in equation (8) are $2, 3/2, 4/3$. The pivots of L in equation (10) are $1, 1, 1, 0$ (fail).

The pivots of S are $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$. Multiply those pivots to find determinant $= 5$. This explains 1/5 in S^{-1}.

The pivots of L are $1, 1, 1, 1, 0$ (no pivot).

17 (Computer problem) Create the 9 by 10 backward difference matrix A. Multiply to find $S = A^{\mathrm{T}}A$ and $L = AA^{\mathrm{T}}$. If you have linear algebra software, ask for the determinants det(S) and det(L).

Challenge : By experiment find det(S) when $S = A^{\mathrm{T}}A$ is n by n.

Correction The backward difference matrix A will be **10 by 9**. Then $S = A^{\mathrm{T}}A$ is 9 by 9 (the $-1, 2, -1$ matrix) with $\det S = 10$. In general $\det S = n$ when A is n by $n-1$.
$L = AA^{\mathrm{T}}$ is 10 by 10 (the $-1, 2 - -1$ matrix except that $L_{11} = 1$ and $L_{nn} = 1$). Then L is singular and $\det L = 0$.

18 (Infinite computer problem) Imagine that the second difference matrix S is infinitely large. The diagonals of 2's and -1's go from minus infinity to plus infinity:

Infinite tridiagonal matrix $\quad S = \begin{bmatrix} \cdot & \cdot & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & \cdot & \cdot \end{bmatrix}$

(a) Multiply S times the infinite *all-ones* vector $\boldsymbol{v} = (\ldots, 1, 1, 1, 1, \ldots)$

(b) Multiply S times the infinite *linear* vector $\boldsymbol{w} = (\ldots, 0, 1, 2, 3, \ldots)$

(c) Multiply S times the infinite *squares* vector $\boldsymbol{u} = (\ldots, 0, 1, 4, 9, \ldots)$.

(d) Multiply S times the infinite *cubes* vector $\boldsymbol{c} = (\ldots, 0, 1, 8, 27, \ldots)$.

The answers correspond to second derivatives (with minus sign) of 1 and x^2 and x^3.

S times **all-ones**	gives the zero vector
S times **linear** $\boldsymbol{w}$	gives the zero vector
S times **squares** $\boldsymbol{u}$	gives -2 times **all-ones**
S times **cubes** $\boldsymbol{c}$	gives -6 times **linear** $\boldsymbol{w}$

Those correspond to $0, 0, -2, -6x =$ **minus** the second derivatives of $1, x, x^2, x^3$.

Questions 19–28 are about matrices with $\boldsymbol{Q^{\mathrm{T}}Q = I}$. If Q is square, then it is an orthogonal matrix and $\boldsymbol{Q^{\mathrm{T}} = Q^{-1}}$ and $\boldsymbol{QQ^{\mathrm{T}} = I}$.

19 Complete these matrices to be orthogonal matrices:

(a) $Q = \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix}$ (b) $Q = \frac{1}{3}\begin{bmatrix} -1 & & \\ 2 & & \\ 2 & & \end{bmatrix}$ (c) $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ 1 & -1 & & \\ 1 & -1 & & \end{bmatrix}$.

$$Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \quad Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Note: You could complete to Q with different columns than these.

20 (a) Suppose Q is an orthogonal matrix. Why is $Q^{-1} = Q^{\mathrm{T}}$ also an orthogonal matrix?

(b) From $Q^{\mathrm{T}}Q = I$, the columns of Q are orthogonal unit vectors (orthonormal vectors). Why are the rows of Q (square matrix) also orthonormal vectors?

(a) Q^{-1} is also orthogonal because $(Q^{-1})^{\mathrm{T}}(Q^{-1}) = (Q^{\mathrm{T}})^{\mathrm{T}}Q^{\mathrm{T}} = QQ^{\mathrm{T}} = I$.

(b) The rows of Q are orthonormal vectors because $QQ^{\mathrm{T}} = I$. For square matrices, Q^{T} is a right-inverse of Q whenever it is a left-inverse of Q. So rows are orthonormal when columns are orthonormal.

21 (a) Which vectors can be the first column of an orthogonal matrix?

(b) If $Q_1^{\mathrm{T}}Q_1 = I$ and $Q_2^{\mathrm{T}}Q_2 = I$, is it true that $(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = I$? Assume that the matrix shapes allow the multiplication Q_1Q_2.

(a) Any unit vector (length 1) can be the first column of Q.

(b) YES, $(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = Q_2^{\mathrm{T}}(Q_1^{\mathrm{T}}Q_1)Q_2 = Q_2^{\mathrm{T}}Q_2 = I$.

22 If $\boldsymbol{u}$ is a unit column vector (length 1, $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$), show why $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is

(a) a symmetric matrix: $H = H^{\mathrm{T}}$ (b) an orthogonal matrix: $H^{\mathrm{T}}H = I$.

The Householder matrix $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is symmetric (because $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is symmetric) and also orthogonal (because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$):

$$H^{\mathrm{T}}H = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^2 = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = I.$$

23 If $\boldsymbol{u} = (\cos\theta, \sin\theta)$, what are the four entries in $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$? Show that $H\boldsymbol{u} = -\boldsymbol{u}$ and $H\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} = (-\sin\theta, \cos\theta)$. This H is a **reflection matrix**: the $\boldsymbol{v}$-line is a mirror and the $\boldsymbol{u}$-line is reflected across that mirror.

$$H = I - 2\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\begin{bmatrix}\cos\theta & sin\theta\end{bmatrix} = \begin{bmatrix}1-2\cos^2\theta & -2\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & 1-2\sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ -\sin 2\theta & -\cos^2\theta\end{bmatrix}.$$

$$H\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathbf{T}}\boldsymbol{u} = -\boldsymbol{u} \qquad H\boldsymbol{v} = \boldsymbol{v} - 2\boldsymbol{u}\boldsymbol{u}^{\mathbf{T}}\boldsymbol{v} = \boldsymbol{v} \quad \text{since } \boldsymbol{u}^{\mathbf{T}}\boldsymbol{v} = 0.$$

24 Suppose the matrix Q is orthogonal and also upper triangular. What can Q look like? Must it be diagonal?

If Q is orthogonal and upper triangular, its first column must be $\boldsymbol{q}_1 = (\pm 1, 0, \ldots, 0)$. Then its second column $\boldsymbol{q}_2$ must start with 0 to have the orthogonality $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$. Then $\boldsymbol{q}_2 = (0, \pm 1, 0, \ldots, 0)$. Then $\boldsymbol{q}_3$ must start with 0, 0 to have $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_3 = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_3 = 0$ (and so onward). Thus Q is diagonal: $Q = \text{diag}\,(\pm 1, \ldots, \pm 1)$.

25 (a) To construct a 3 by 3 orthogonal matrix Q whose first column is in the direction $\boldsymbol{w}$, what first column $\boldsymbol{q}_1 = c\boldsymbol{w}$ would you choose?

(b) The next column $\boldsymbol{q}_2$ can be any unit vector perpendicular to $\boldsymbol{q}_1$. To find $\boldsymbol{q}_3$, choose a solution $\boldsymbol{v} = (v_1, v_2, v_3)$ to the two equations $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{v} = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{v} = 0$. *Why is there always a nonzero solution* $\boldsymbol{v}$?

(a) The first column of Q will be $\boldsymbol{q}_1 = \boldsymbol{w}/||w||$ to have length 1.

(b) The next column $\boldsymbol{q}_2$ has $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ and $||\boldsymbol{q}_2|| = 1$. Then there will be a vector $\boldsymbol{v}$ orthogonal to $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ because $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{v} = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{v} = 0$ give 2 linear equations in 3 unknowns v_1, v_2, v_3.

26 Why is every solution $\boldsymbol{v}$ to $A\boldsymbol{v} = \mathbf{0}$ orthogonal to every row of A?

Writing out $A\boldsymbol{v} = \mathbf{0}$ shows that every row is orthogonal to $\boldsymbol{v}$:

$$\begin{bmatrix} row\ 1 \\ \cdots \\ row\ n \end{bmatrix}\begin{bmatrix} \\ \boldsymbol{v} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}.$$

27 Suppose $Q^{\mathrm{T}}Q = I$ but Q is not square. The matrix $P = QQ^{\mathrm{T}}$ is not I. But show that P is symmetric and $P^2 = P$. This is a **projection matrix**.

If Q has n orthogonal columns and $n < m$, then the m by m matrix $P = QQ^{\mathrm{T}}$ is not I. (Some vector $\boldsymbol{v}$ in R^m will solve the n equations $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$. Then $QQ^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$ and $QQ^{\mathrm{T}} \neq I$.) But P is symmetric and $P^2 = QQ^{\mathrm{T}}QQ^{\mathrm{T}} = QIQ^{\mathrm{T}} = P$. Thus P is a **projection matrix**.

28 A 5 by 4 matrix Q can have $Q^{\mathrm{T}}Q = I$ but *it cannot possibly have* $QQ^{\mathrm{T}} = I$. Explain in words why the four equations $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$ must have a nonzero solution $\boldsymbol{v}$. Then $\boldsymbol{v}$ is not the same as $QQ^{\mathrm{T}}\boldsymbol{v}$ and I is not the same as QQ^{T}.

The four equations $Q^{\mathrm{T}}\boldsymbol{v} = 0$ have 5 unknowns v_1, v_2, v_3, v_4, v_5. With only 4 rows, Q^{T} cannot have more than 4 pivots. There must be a free column in Q^{T} and a *nonzero* special solution to $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$.

Challenge Problems

29 Can you find a rotation matrix Q so that QDQ^{T} is a permutation ?

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \quad \text{equals} \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

With $\theta = 45^\circ$, $\dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix} \qquad \dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$

30 Split an orthogonal matrix $(Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I)$ into two rectangular submatrices:

$$Q = [\,Q_1 \mid Q_2\,] \quad \text{and} \quad Q^{\mathrm{T}}Q = \begin{bmatrix}Q_1^{\mathrm{T}}Q_1 & Q_1^{\mathrm{T}}Q_2\\ Q_2^{\mathrm{T}}Q_1 & Q_2^{\mathrm{T}}Q_2\end{bmatrix}$$

(a) What are those four blocks in $Q^{\mathrm{T}}Q = I$?

(b) $QQ^{\mathrm{T}} = Q_1Q_1^{\mathrm{T}} + Q_2Q_2^{\mathrm{T}} = I$ is column times row multiplication. Insert the diagonal matrix $D = \begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$ and do the same multiplication for QDQ^{T}.

Note: The description of all symmetric orthogonal matrices S in (**??**) becomes $S = QDQ^{\mathrm{T}} = Q_1Q_1^{\mathrm{T}} - Q_2Q_2^{\mathrm{T}}$. This is exactly the reflection matrix $I - 2Q_2Q_2^{\mathrm{T}}$.

(a) The four blocks in $Q^{\mathrm{T}}Q$ are $I, 0, 0, I$ because all the columns of Q_1 are orthogonal to all the columns of Q_2. (All together they are the columns of the orthogonal matrix Q.)

(b) Column times row multiplication gives

$$\begin{bmatrix}Q_1 & Q_2\end{bmatrix}\begin{bmatrix}Q_1^{\mathrm{T}}\\ Q_2^{\mathrm{T}}\end{bmatrix} = Q_1Q_1^{\mathrm{T}} + Q_2Q_2^{\mathrm{T}} = I.$$

$$QDQ^{\mathrm{T}} = \begin{bmatrix}Q_1 & Q_2\end{bmatrix} D \begin{bmatrix}Q_1^{\mathrm{T}}\\ Q_2^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}Q_1 & Q_2\end{bmatrix}\begin{bmatrix}Q_1^{\mathrm{T}}\\ -Q_2^{\mathrm{T}}\end{bmatrix} = Q_1Q_1^{\mathrm{T}} - Q_2Q_2^{\mathrm{T}}$$

$= I - 2Q_2Q_2^{\mathrm{T}}$.

Then QDQ^{T} is both symmetric and orthogonal.

31 The real reason that the transpose "flips A across its main diagonal" is to obey this dot product law: $(A\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot (A^{\mathrm{T}}\boldsymbol{w})$. That rule $(A\boldsymbol{v})^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{w})$ **becomes integration by parts in calculus**, where $A = d/dx$ and $A^{\mathrm{T}} = -d/dx$.

(a) For 2 by 2 matrices, write out both sides (4 terms) and compare:

$$\left(\begin{bmatrix}a & \boldsymbol{b}\\ \boldsymbol{c} & d\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right) \cdot \begin{bmatrix}w_1\\ w_2\end{bmatrix} \quad \text{is equal to} \quad \begin{bmatrix}v_1\\ v_2\end{bmatrix} \cdot \left(\begin{bmatrix}a & \boldsymbol{c}\\ \boldsymbol{b} & d\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}\right).$$

(b) The rule $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ comes slowly but directly from part (a):

$$(AB)\,\boldsymbol{v} \cdot \boldsymbol{w} = A(B\boldsymbol{v}) \cdot \boldsymbol{w} = B\boldsymbol{v} \cdot A^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v} \cdot B^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{w}) = \boldsymbol{v} \cdot (B^{\mathrm{T}}A^{\mathrm{T}})\boldsymbol{w}$$

Steps 1 and 4 are the _____ law. Steps 2 and 3 are the dot product law.

The connection between $(Ax)^{\mathrm{T}} y = x\left(A^{\mathrm{T}} y\right)$ and integration by parts is developed in the Chapter 7 Notes. The idea is that A becomes the derivative d/dx and the dot product becomes an integral:

$$(Af)^{\mathrm{T}} g = \int \frac{df}{dx}\, g(x)\, dx = -\int f(x) \frac{dg}{dx}\, dx = f^{\mathrm{T}}\left(A^{\mathrm{T}} g\right).$$

That last step identifies $A^{\mathrm{T}} g$ as $-dg/dx$. So the first derivative $A = d/dx$ is like an *anti*symmetric matrix. Our functions f and g are zero at the ends of the integration interval, so the "by parts formula" above has zero from the other usual term $[fg]_0^1$.

In $31(b)$, steps 1 and 4 are the **associative law** $(AB)\boldsymbol{v} = A(B\boldsymbol{v})$.

32 How is a matrix $S = S^{\mathrm{T}}$ decided by its entries on and above the diagonal? How is Q with orthonormal columns decided by its entries *below* the diagonal? Together this matches the number of entries in an n by n matrix. So it is reasonable that every matrix can be factored into $A = SQ$ (like $re^{i\theta}$).

If S is symmetric, then the entries on and above the diagonal tell you the entries below the diagonal. If Q is orthogonal, here is how the entries *below the diagonal* decide the matrix. In column 1, the top entry Q_{11} has to complete a unit vector (no choice except a $\pm$ sign). In column 2, the two top entries are decided by (1) orthogonality to column 1 and (2) unit vector. Every column, in order, has no free numbers available on and above the diagonal.

So there are a total of n^2 choices available: on and above the diagonal of S and below the diagonal of Q. This n^2 matches the number of equations in $A = SQ$ (linear equations in $S = AQ^{\mathrm{T}}$). "polar factorization" of a matrix is possible.