DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 3

1 Draw the graph of $y = e^t$ by hand, for $-1 \le t \le 1$. What is its slope dy/dt at $t = 0$? Add the straight line graph of $y = et$. Where do those two graphs cross?

Solution The derivative of e^t has slope 1 at $t = 0$. The graphs meet at $t = 1$ where their value is e. They don't actually "cross" because the line is tangent to the curve: both have slope $y' = e$ at $t = 1$.

2 Draw the graph of $y_1 = e^{2t}$ on top of $y_2 = 2e^t$. Which function is larger at $t = 0$? Which function is larger at $t = 1$?

Solution From the graphs we see that at $t = 0$, the function $2e^t$ is larger whereas at $t = 1, e^{2t}$ is larger. (e times e is larger than 2 times e).

3 What is the slope of $y = e^{-t}$ at $t = 0$? Find the slope dy/dt at $t = 1$.

Solution The slope of e^{-t} is $-e^{-t}$. At $t = 0$ this is -1. The slope at $t = 1$ is $-e^{-1}$.

4 What "logarithm" do we use for the number t (the exponent) when $e^t = 4$?

Solution We use the natural logarithm to find t from the equation $e^t = 4$. We get that $t = \ln 4 \approx 1.386$.

5 State the chain rule for the derivative dy/dt if $y(t) = f(u(t))$ (chain of f and u).

Solution The chain rule gives:

$$\frac{dy}{dt} = \frac{df(u(t))}{du(t)} \frac{du(t)}{dt}$$

6 The *second* derivative of e^t is again e^t. So $y = e^t$ solves $d^2y/dt^2 = y$. A second order differential equation should have another solution, different from $y = Ce^t$. What is that second solution?

Solution The second solution is $y = e^{-t}$. The second derivative is $-(-e^{-t}) = e^{-t}$.

7 Show that the nonlinear example $dy/dt = y^2$ is solved by $y = C/(1 - Ct)$ for every constant C. The choice $C = 1$ gave $y = 1/(1 - t)$, starting from $y(0) = 1$.

Solution Given that $y = C/(1 - Ct)$, we have:

$$y^2 = C^2/(1 - Ct)^2$$
$$\tfrac{dy}{dt} = C \cdot (-1) \cdot (-C)1/(1 - Ct)^2 = C^2/(1 - Ct)^2$$

8 Why will the solution to $dy/dt = y^2$ grow faster than the solution to $dy/dt = y$ (if we start them both from $y = 1$ at $t = 0$)? The first solution blows up at $t = 1$. The second solution e^t grows exponentially fast but it never blows up.

Solution The solution of the equation $dy/dt = y^2$ for $y(0) = 1$ is $y = 1/(1-t)$, while the solution to $dy/dt = y$ for $y(0) = 1$ is $y = e^t$. Notice that the first solution blows up at $t = 1$ while the second solution e^t grows exponentially fast but never blows up.

9 Find a solution to $dy/dt = -y^2$ starting from $y(0) = 1$. Integrate dy/y^2 and $-dt$. (Or work with $z = 1/y$. Then $\boldsymbol{dz/dt} = (dz/dy)\,(dy/dt) = (-1/y^2)(-y^2) = \mathbf{1}$. From $dz/dt = 1$ you will know $z(t)$ and $y = 1/z$.)

Solution The first method has

$$\begin{aligned} \frac{dy}{y^2} &= -dt \\ \int_{y(0)}^{y} \frac{du}{u^2} &= -\int_0^t dv \quad (u, v \text{ are integration variables}) \\ \frac{-1}{y} + \frac{1}{y(0)} &= -t \\ \frac{-1}{y} &= -t - 1 \\ y &= \frac{1}{1+t} \end{aligned}$$

The approach using $z = 1/y$ leads to $dz/dt = 1$ and $z(0) = 1/1$.
Then $z(t) = 1 + t$ and $y = 1/z = \frac{1}{1+t}$.

10 Which of these differential equations are linear (in y)?

(a) $y' + \sin y = t$ (b) $y' = t^2(y - t)$ (c) $y' + e^t y = t^{10}$.

Solution (a) Since this equation solves a $\sin y$ term, it is not linear in y.

(b) and (c) Since these equations have no nonlinear terms in y, they are linear.

11 The product rule gives what derivative for $e^t e^{-t}$? This function is constant. At $t = 0$ this constant is 1. Then $e^t e^{-t} = 1$ for all t.

Solution $(e^t e^{-t})' = e^t e^{-t} - e^t e^{-t} = 0$ so $e^t e^{-t}$ is a constant (1).

12 $dy/dt = y + 1$ is not solved by $y = e^t + t$. Substitute that y to show it fails. We can't just add the solutions to $y' = y$ and $y' = 1$. What number c makes $y = e^t + c$ into a correct solution?

Solution

$$\frac{dy}{dt} = y + 1 \qquad\qquad \frac{d(e^t + c)}{dt} = e^t + c + 1$$

$$\textbf{Wrong } \frac{d(e^t + t)}{dt} \neq e^t + t + 1 \qquad\qquad \textbf{Correct } \boldsymbol{c = -1}$$

Problem Set 1.3, page 15

1 Set $t = 2$ in the infinite series for e^2. The sum must be e times e, close to 7.39. How many terms in the series to reach a sum of 7? How many terms to pass 7.3?

Solution The series for e^2 has $t = 2$: $e^2 = 1 + 2 + \dfrac{2^2}{2!} + \dfrac{2^3}{3!} + \dfrac{2^4}{4!} + \cdots$

If we include five terms we get: $e^2 \approx 1 + 2 + 2 + \dfrac{8}{6} + \dfrac{16}{24} = 7.0$

If we include seven terms we get: $e^2 \approx 1 + 2 + \dfrac{2^2}{2!} + \dfrac{2^3}{3!} + \dfrac{2^4}{4!} + \dfrac{2^5}{120} + \dfrac{2^6}{720} = 7.35556.$

2 Starting from $y(0)=1$, find the solution to $dy/dt=y$ at time $t=1$. Starting from that $y(1)$, solve $dy/dt=-y$ to time $t=2$. Draw a rough graph of $y(t)$ from $t=0$ to $t=2$. What does this say about e^{-1} times e ?

Solution $y=e^t$ up to $t=1$, so that $y(1)=e$. Then for $t>1$ the equation $dy/dt=-y$ has $y=Ce^{-t}$. At $t=1$, this becomes $e=Ce^{-1}$ so that $C=e^2$. The solution of $dy/dt=-y$ up to $t=2$ is $y=e^{2-t}$. At $t=2$ we have returned to $y(2)=y(0)=1$. Then $(e^{-1})(e)=1$.

3 Start with $y(0)=\$5000$. If this grows by $dy/dt=.02y$ until $t=5$ and then jumps to $a=.04$ per year until $t=10$, what is the account balance at $t=10$?

Solution

$$t\le 5:\ \frac{dy}{dt}=.02y \qquad 5\le t\le 10:\ \frac{dy}{dt}=.04y \text{ gives } y=Ce^{.04t}$$
$$y=5000e^{.02t} \qquad y(5)=Ce^{-2}=5000e^{.1} \text{ gives } C=5000e^{-.1}$$
$$y(5)=5000e^{.1} \qquad y(t)=5000(e^{.04t-0.1})$$
$$y(10)=5000e^{.3}$$

4 Change Problem 3 to start with \$5000 growing at $dy/dt=.04y$ for the first five years. Then drop to $a=.02$ per year until year $t=10$. What is the account balance at $t=10$?

Solution

$$\begin{aligned}\frac{dy}{dt}&=.04y & \frac{dy}{dt}&=.02y \text{ for } 5\le t\le 10\\ y&=C_1e^{.04t} & y&=C_2e^{.02t}\\ y(0)&=C_1=5000 & y(5)&=C_2e^{.1}=5000e^{.2}\\ y(t)&=5000e^{.04t} \text{ for } t\le 5 & C_2&=5000e^{.1}\\ y(5)&=5000e^{.2} & y(t)&=5000(e^{.02t+0.1})\\ & & y(10)&=5000e^{.3}= \text{ same as in 1.3.3.}\end{aligned}$$

Problems 5–8 are about $y=e^{at}$ and its infinite series.

5 Replace t by at in the exponential series to find e^{at} :
$$e^{at}=1+at+\frac{1}{2}(at)^2+\cdots+\frac{1}{n!}(at)^n+\cdots$$
Take the derivative of every term (keep five terms). Factor out a to show that *the derivative of* e^{at} *equals* ae^{at}. At what time T does e^{at} reach 2 ?

Solution The derivative of this series is obtained by differentiating the terms individually:
$$\begin{aligned}\frac{dy}{dt}&=a+at+\cdots+\frac{1}{(n-1)!}a^nt^{n-1}+\cdots\\ &=a\left(1+at+\frac{1}{2}(at)^2+\cdots+\frac{1}{(n-1)!}a^{n-1}t^{n-1}+\cdots\right)=ae^{at}\end{aligned}$$
If $e^{a\mathrm{T}}=2$ then $a\mathrm{T}=\ln 2$ and $\mathrm{T}=\dfrac{\ln 2}{a}$.

6 Start from $y'=ay$. Take the derivative of that equation. Take the n^{th} derivative. Construct the Taylor series that matches all these derivatives at $t=0$, starting from $1+at+\frac{1}{2}(at)^2$. Confirm that this series for $y(t)$ is exactly the exponential series for e^{at}.

Solution The derivative of $y'=ay$ is $y''=ay'=a^2y$. The next derivative is $y'''=ay''$ which is a^3y. When $y(0)=1$, the derivatives at $t=0$ are $a,a^2,a^3,\ldots$ so the Taylor series is $y(t)=1+at+\dfrac{1}{2}a^2t^2+\cdots=e^{at}$.

7 At what times t do these events happen ?
(a) $e^{at} = e$ (b) $e^{at} = e^2$ (c) $e^{a(t+2)} = e^{at}e^{2a}$.

Solution

(a) $e^{at} = e$ at $t = 1/a$.

(b) $e^{at} = e^2$ at $t = 2/a$.

(c) $e^{a(t+2)} = e^{at}e^{2a}$ at all t.

8 If you multiply the series for e^{at} in Problem 5 by itself you should get the series for e^{2at}. Multiply the first 3 terms by the same 3 terms to see the first 3 terms in e^{2at}.

Solution $(1 + at + \frac{1}{2}a^2t^2)(1 + at + \frac{1}{2}a^2t^2) = 1 + 2at + \left(1 + \frac{1}{2} + \frac{1}{2}\right) a^2t^2 + \cdots$

This agrees with $e^{2at} = 1 + 2at + \frac{1}{2}(2at)^2 + \cdots$

9 (recommended) Find $y(t)$ if $dy/dt = ay$ and $\boldsymbol{y(T) = 1}$ (instead of $y(0) = 1$).

Solution $\frac{dt}{dt} = ay$ gives $y(t) = Ce^{at}$. When $Ce^{a\mathrm{T}} = 1$ at $t = \mathrm{T}$, this gives $C = e^{-a\mathrm{T}}$ and $y(t) = e^{a(t-\mathrm{T})}$.

10 (a) If $dy/dt = (\ln 2)y$, explain why $y(1) = 2y(0)$.

(b) If $dy/dt = -(\ln 2)y$, how is $y(1)$ related to $y(0)$?

Solution

(a) $\frac{dy}{dt} = (\ln 2)y \to y(t) = y(0)e^{t(\ln 2)} \to y(1) = y(0)e^{\ln 2} = 2y(0)$.

(b) $\frac{dy}{dt} = -(\ln 2)y \to y(t) = y(0)e^{-t(\ln 2)} \to y(1) = y(0)e^{-\ln 2} = \frac{1}{2}y(0)$.

11 In a one-year investment of $y(0) = \$100$, suppose the interest rate jumps from 6% to 10% after six months. Does the equivalent rate for a whole year equal 8%, or more than 8%, or less than 8% ?

Solution We solve the equation in two steps, first from $t = 0$ to $t = 6$ months, and then from $t = 6$ months to $t = 12$ months.

$$\begin{aligned} y(t) &= y(0)e^{at} \\ y(0.5) &= \$100e^{0.06\times 0.5} = \$100e^{.03} \\ &= \$103.05 \end{aligned} \qquad \begin{aligned} y(t) &= y(0.5)e^{at} \\ y(1) &= \$103.05e^{0.1\times 0.5} = \$103.05e^{.05} \\ &= \$108.33 \end{aligned}$$

If the money was invested for one year at 8% the amount at $t = 1$ would be:

$y(1) = \$100e^{0.08\times 1} = \$108.33$.

The equivalent rate for the whole year is indeed exactly 8%.

12 If you invest $y(0) = \$100$ at 4% interest compounded continuously, then $dy/dt = .04y$. Why do you have more than $\$104$ at the end of the year ?

Solution The quantitative reason for why this is happening is obtained from solving the equation:

$$\begin{aligned} \frac{dy}{dt} &= 0.04y \to y(t) = y(0)e^{.04t} \\ y(1) &= 100e^{0.04} \approx \$104.08. \end{aligned}$$

The intuitive reason is that **the interest accumulates interest**.

13 What linear differential equation $dy/dt = a(t)y$ is satisfied by $y(t) = e^{\cos t}$?

Solution The chain rule for $f(u(t))$ has $y(t) = f(u) = e^u$ and $u(t) = \sin t$:

$$\frac{dy}{dt} = \frac{df(u(t))}{dt} = \frac{df}{dt}\frac{du}{dt} = e^u \cos t = y \cos t. \text{ Then } \boldsymbol{a(t) = \cos(t)}.$$

14 If the interest rate is $a = 0.1$ per year in $y' = ay$, how many years does it take for your investment to be multiplied by e ? How many years to be multiplied by e^2 ?

Solution If the interest rate is $a = 0.1$, then $y(t) = y(0)e^{0.1t}$. For $t = 10$, the value is $y(t) = y(0)\, e$. For $t = 20$, the value is $y(t) = y(0)\, e^2$.

15 Write the first four terms in the series for $y = e^{t^2}$. Check that $dy/dt = 2ty$.

Solution

$$y = e^{t^2} = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots$$

$$\frac{dy}{dt} = 2t + 2t^3 + t^5 + \cdots = 2t\left(1 + t^2 + \frac{1}{2}t^4 + \cdots\right) = 2te^{t^2}.$$

16 Find the derivative of $Y(t) = \left(1 + \frac{t}{n}\right)^n$. If n is large, this dY/dt is close to Y !

Solution The derivative of $Y(t) = \left(1 + \frac{t}{n}\right)^n$ with respect to t is $n(\frac{1}{n})\left(1 + \frac{t}{n}\right)^{n-1} = \left(1 + \frac{t}{n}\right)^{n-1}$. For large n the extra factor $1 + \frac{t}{n}$ is nearly 1, and dY/dt is near Y.

17 (Key to future sections). Suppose the exponent in $y = e^{u(t)}$ is $u(t) =$ integral of $a(t)$. What equation $dy/dt =$ _____ y does this solve ? If $u(0) = 0$ what is the starting value $y(0)$?

Solution Differentiating $y = e^{\int a(t)\, dt}$ with respect to t by the chain rule yields $y' = a(t)e^{\int a(t)\, dt}$. Therefore $\boldsymbol{dy/dt = a(t)y}$. If $u(0) = 0$ we have $y(0) = e^{u(0)} = 1$.

18 The Taylor series comes from $e^{d/dx}f(x)$, when you write out $e^{d/dx} = 1 + d/dx + \frac{1}{2}(d/dx)^2 + \cdots$ as a sum of higher and higher derivatives. Applying the series to $f(x)$ at $x = 0$ would give the value $f + f' + \frac{1}{2}f'' + \cdots$ at $\boldsymbol{x = 0}$.
The Taylor series says: This is equal to $f(x)$ at $x =$ _____ .

Solution

$$\boldsymbol{f(1)} = f(0) + 1f'(0) + \frac{1}{2}1^2 f''(0) + \cdots \quad \text{This is exactly}$$

$$f(1) = \left(1 + \frac{d}{dx} + \frac{1}{2}\left(\frac{d}{dx}\right)^2 + \cdots\right) f(x) \text{ at } x = 0.$$

19 (Computer or calculator, 2.xx is close enough) Find the time t when $e^t = 10$. The initial $y(0)$ has increased by an order of magnitude—a factor of 10. The exact statement of the answer is $t =$ _____ . At what time t does e^t reach 100 ?

Solution The exact time when $e^t = 10$ is $t = \ln 10$. This is $t \approx 2.30$ or 2.3026.

Then the time when $e^{\mathrm{T}} = 100$ is $\mathrm{T} = \ln 100 = \ln 10^2 = 2 \ln 10 \approx 4.605$.

Note that the time when $e^t = \frac{1}{10}$ is $t = -\ln 10$ and not $t = \frac{1}{\ln 10}$.

20 The most important curve in probability is the bell-shaped graph of $e^{-t^2/2}$. With a calculator or computer find this function at $t = -2, -1, 0, 1, 2$. Sketch the graph of $e^{-t^2/2}$ from $t = -\infty$ to $t = \infty$. *It never goes below zero.*

Solution At $t = 1$ and $t = -1$, we have $e^{-t^2/2} = e^{-1/2} = 1/\sqrt{e} \approx \mathbf{.606}$

At $t = 2$ and $t = -2$, we have $e^{-t^2/2} = e^{-2} \approx \mathbf{.13}$.

21 Explain why $y_1 = e^{(a+b+c)t}$ is the same as $y_2 = e^{at}e^{bt}e^{ct}$. They both start at $y(0) = 1$. They both solve what differential equation?

Solution The exponent rule is used twice to find $e^{(a+b+c)\,t} = e^{at+bt+ct} = e^{at+bt}e^{ct} = e^{at}e^{bt}e^{ct}$.

This function must solve $\boldsymbol{\frac{dy}{dt} = (a+b+c)y}$. The product rule confirms this.

22 For $y' = y$ with $a = 1$, Euler's first step chooses $Y_1 = (1 + \Delta t)Y_0$. Backward Euler chooses $Y_1 = Y_0/(1 - \Delta t)$. Explain why $1 + \Delta t$ is smaller than the exact $e^{\Delta t}$ and $1/(1 - \Delta t)$ is larger than $e^{\Delta t}$. (Compare the series for $1/(1 - x)$ with e^x.)

Solution $1 + \Delta t$ is certainly smaller than $e^{\Delta t} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \cdots$

$\frac{1}{1-\Delta t} = 1 + \Delta t + (\Delta t)^2 + (\Delta t)^3 + \cdots$ is larger than $e^{\Delta t}$, because the coefficients drop below 1 in $e^{\Delta t}$.

Problem Set 1.4, page 27

1 All solutions to $dy/dt = -y + 2$ approach the steady state where dy/dt is zero and $y = y_\infty =$ ___. That constant $y = y_\infty$ is a particular solution y_p.

Which $y_n = Ce^{-t}$ combines with this steady state y_p to start from $y(0) = 4$? This question chose $y_p + y_n$ to be y_∞+ *transient* (decaying to zero).

Solution $y_\infty = 2 = y_p$ at the steady state when $\frac{dy}{dt} = 0$. Then $y_n = 2e^{-t}$ gives $y = y_n + y_p = 2 + 2e^{-t} = 4$ at $t = 0$.

2 For the same equation $dy/dt = -y + 2$, choose the null solution y_n that starts from $y(0) = 4$. Find the particular solution y_p that starts from $y(0) = 0$.

This splitting chooses y_n and y_p as $e^{at}y(0)$ + integral of $e^{a(t-T)}q$ in equation (4).

Solution For the same equation as 11.4.1, $y_n = 4e^{-t}$ has the correct $y(0) = 4$. Now y_p must be $2 - 2e^{-t}$ to start at $y_p(0) = 0$. Of course $y_n + y_p$ is still $2 + 2e^{-t}$.

3 The equation $dy/dt = -2y + 8$ also has two natural splittings $\boldsymbol{y_S + y_T = y_N + y_P}$:

1. Steady $(\boldsymbol{y_S = y_\infty})$ + Transient $(\boldsymbol{y_T \to 0})$. What are those parts if $y(0) = 6$?

2. $(y_N' = -2y_N$ from $\boldsymbol{y_N(0) = 6}) + (y_P' = -2y_P + 8$ starting from $\boldsymbol{y_P(0) = 0})$.

Solution **1.** $y_S = 4$ (when $\frac{dy}{dt} = 0$: steady state) and $y_{\mathrm{T}} = 2e^{-2t}$.

2. $y_N = 6e^{-2t}$ and $y_P = 4 - 4e^{-2t}$ starts at $y_P(0) = 0$.

Again $y_S + y_{\mathrm{T}} = y_N + y_P$: two splittings of y.

4 All null solutions to $u - 2v = 0$ have the form $(u, v) = (c,$ _____$)$.

One particular solution to $u - 2v = 3$ has the form $(u, v) = (7,$ _____$)$.

Every solution to $u - 2v = 3$ has the form $(7,$ _____$) + c(1,$ _____$)$.

But also every solution has the form $(3,$ _____$) + C(1,$ _____$)$ for $C = c + 4$.

Solution All null solutions to $u - 2v = 0$ have the form $(u, v) = \boldsymbol{(c, \frac{1}{2}c)}$.

One particular solution to $u - 2v = 3$ has the form $(u, v) = \boldsymbol{(7, 2)}$.

Every solution to $u - 2v = 3$ has the form $\boldsymbol{(7, 2) + c(1, \frac{1}{2})}$.

But also every solution has the form $\boldsymbol{(3, 0) + C(1, \frac{1}{2})}$. Here $C = c + 4$.

5 The equation $dy/dt = 5$ with $y(0) = 2$ is solved by $y =$ _____. A natural splitting $y_n(t) =$ ___ and $y_p(t) =$ ___ comes from $y_n = e^{at}y(0)$ and $y_p = \int e^{a(t-T)} 5\,dT$. This small example has $\boldsymbol{a = 0}$ (so ay is absent) and $\boldsymbol{c = 0}$ (the source is $q = 5e^{0t}$). When $a = c$ we have "resonance." A factor t will appear in the solution y.

Solution $dy/dt = 5$ with $y(0) = 2$ is solved by $y = 2+5t$. A natural splitting $y_n(t) = 2$ and $y_p(t) = 5t$ comes from $y_n(0) = y(0)$ and $y_p = \int e^{a(t-s)} 5ds = 5t$ (since $a = 0$).

Starting with Problem 6, choose the very particular $\boldsymbol{y_p}$ that starts from $\boldsymbol{y_p(0) = 0}$.

6 For these equations starting at $y(0) = 1$, find $y_n(t)$ and $y_p(t)$ and $y(t) = y_n + y_p$.
(a) $y' - 9y = 90$ (b) $y' + 9y = 90$

Solution (a) Since the forcing function is a we use equation 6:

$$\begin{aligned} y_n(t) &= e^{9t} \\ y_p(t) &= \tfrac{90}{9}(e^{9t} - 1) = 10(e^{9t} - 1) \\ y(t) &= y_n(t) + y_p(t) = e^{9t} + 10(e^{9t} - 1) = 11e^{9t} - 10. \end{aligned}$$

(b) We again use equation 6, noting that $a = -9$. The steady state will be $y_\infty = 10$.

$$\begin{aligned} y_n(t) &= e^{-9t} \\ y_p(t) &= \tfrac{90}{-9}(e^{-9t} - 1) \\ y(t) &= y_n(t) + y_p(t) = e^{-9t} - 10(e^{-9t} - 1) = 10 - 9e^{-9t}. \end{aligned}$$

7 Find a linear differential equation that produces $y_n(t) = e^{2t}$ and $y_p(t) = 5(e^{8t} - 1)$.

Solution $y_n = e^{2t}$ needs $a = 2$. Then $y_p = 5(e^{8t} - 1)$ starts from $y_p(0) = 0$, telling us that $y(0) = y_n(0) = 1$. This y_p is a response to the forcing term $(e^{8t} + 1)$. So the equation for $y = e^{2t} + 5e^{8t} - 5$ must be $\frac{dy}{dt} = 2y + (e^{8t} + 1)$. Substitute y :

$$2e^{2t} + 40e^{8t} = 2e^{2t} + 10e^{8t} - 10 + (e^{8t} + 1).$$

Comparing the two sides, $C = 30$ and $D = 10$. Harder than expected.

8 Find a resonant equation $(a = c)$ that produces $y_n(t) = e^{2t}$ and $y_p(t) = 3te^{2t}$.

Solution Clearly $a = c = 2$. The equation must be $dy/dt = 2y + Be^{2t}$. Substituting $y = e^{2t} + 3te^{2t}$ gives $2e^{2t} + 3e^{2t} + 6te^{2t} = 2(e^{2t} + 3te^{2t}) + Be^{2t}$ and then $B = \mathbf{3}$.

9 $y' = 3y + e^{3t}$ has $y_n = e^{3t}y(0)$. Find the resonant y_p with $y_p(0) = 0$.

Solution The resonant y_p has the form Cte^{3t} starting from $y_p(0) = 0$. Substitute in the equation:

$$\frac{dy}{dt} = 3y + e^{3t} \text{ is } Ce^{3t} + 3Cte^{3t} = 3Cte^{3t} + e^{3t} \text{ and then } C = \mathbf{1}.$$

Problems 10–13 are about $\boldsymbol{y' - ay =}$ constant source $\boldsymbol{q}$.

10 Solve these linear equations in the form $y = y_n + y_p$ with $y_n = y(0)e^{at}$.

(a) $y' - 4y = -8$ (b) $y' + 4y = 8$ Which one has a steady state?

Solution (a) $y' - 4y = -8$ has $a = 4$ and $y_p = 2$. But 2 is not a steady state at $t = \infty$ because the solution $y_n = y(0)e^{4t}$ is exploding.

(b) $y' + 4y = 8$ has $a = -4$ and again $y_p = 2$. This 2 is a steady state because $a < 0$ and $y_n \to 0$.

11 Find a formula for $y(t)$ with $y(0) = 1$ and draw its graph. What is y_∞ ?

(a) $y' + 2y = 6$ (b) $y' + 2y = -6$

Solution (a) $y' + 2y = 6$ has $a = -2$ and $y_\infty = 3$ and $y = y(0)e^{-2t} + 3$.

(b) $y' + 2y = -6$ has $a = -2$ and $y_\infty = -3$ and $y = y(0)e^{-2t} - 3$.

12 Write the equations in Problem 11 as $Y' = -2Y$ with $Y = y - y_\infty$. What is $Y(0)$?

Solution With $Y = y - y_\infty$ and $Y(0) = y(0) - y_\infty$, the equations in 1.4.11 are $Y' = -2Y$. (The solutions are $Y(t) = Y(0)e^{-2t}$ which is $y(t) - y_\infty = (y(0) - y_\infty)e^{-2t}$ or $y(t) = y(0)e^{-2t} + y_\infty(1 - e^{-2t})$.

13 If a drip feeds $q = 0.3$ grams per minute into your arm, and your body eliminates the drug at the rate $6y$ grams per minute, what is the steady state concentration y_∞ ? Then *in* = *out* and y_∞ is constant. Write a differential equation for $Y = y - y_\infty$.

Solution The steady state has $y_{\text{in}} = y_{\text{out}}$ or $0.3 = 6y_\infty$ or $y_\infty = 0.05$. The equation for $Y = y - y_\infty$ is $Y' = aY = -6Y$. The solution is $Y(t) = Y(0)e^{-6t}$ or $y(t) = y_\infty + (y(0) - y_\infty)e^{-6t}$.

Problems 14–18 are about $\boldsymbol{y' - ay =}$ step function $\boldsymbol{H(t - T)}$:

14 Why is y_∞ the same for $y' + y = H(t - 2)$ and $y' + y = H(t - 10)$?

Solution Notice $a = -1$. The steady states are the same because the step functions $H(t - 2)$ and $H(t - 10)$ are the same after time $t = 10$.

15 Draw the ramp function that solves $y' = H(t - T)$ with $y(0) = 2$.

Solution The solution is a ramp with $y(t) = y(0) = 2$ up to time T and then $y(t) = 2 + t - T$ beyond time T.

16 Find $y_n(t)$ and $y_p(t)$ as in equation (10), with step function inputs starting at $T = 4$.

(a) $y' - 5y = 3H(t - 4)$ (b) $y' + y = 7H(t - 4)$ (*What is* y_∞ ?)

Solution (a) $y_p(t) = \frac{3}{5}(e^{5(t-4)} - 1)$ for $t \geq 4$ with no steady state.

(b) $y_p(t) = \frac{7}{-1}(e^{-(t-4)} - 1)$ for $t \geq 4$ with $a = -1$ and $y_\infty = 7$.

17 Suppose the step function turns on at $T = 4$ and off at $T = 6$. Then $q(t) = H(t - 4) - H(t - 6)$. Starting from $y(0) = 0$, solve $y' + 2y = q(t)$. What is y_∞ ?

Solution The solution has 3 parts. First $y(t) = y(0) = 0$ up to $t = 4$. Then $H(t - 4)$ turns on and $y(t) = \frac{1}{-2}(e^{-2(t-4)} - 1)$. This reaches $y(6) = -\frac{1}{2}(e^{-4} - 1)$ at time $t = 6$. After $t = 6$, the source is turned off and the solution decays to zero: $y(t) = y(6)e^{-2(t-6)}$.

Method 2: We use the same steps as in equations (8) - (10), noting that $y(0) = 0$.

$$\begin{aligned}
(e^{2t}y)' &= e^{2t}H(t-4) - e^{2t}H(t-6) \\
e^{2t}y(t) - e^{2t}y(0) &= \int_4^t e^{2x}dx - \int_6^t e^{2x}dx \\
e^{2t}y(t) &= -\tfrac{1}{2}(e^{2\cdot 4} - e^{2t})H(t-4) + \tfrac{1}{2}(e^{2\cdot 6 - e^{2t}})H(t-6) \\
y(t) &= -\tfrac{1}{2}(e^{8-2t} - 1)H(t-4) + \tfrac{1}{2}(e^{12-2t} - 1)H(t-6)
\end{aligned}$$

.

For $t \to \infty$, we have:

$$y_\infty = \frac{1}{2}(e^{8-2\cdot\infty} - 1)H(t-4) + \frac{1}{2}(e^{12-2\cdot\infty} - 1)H(t-6) = \mathbf{0}.$$

18 Suppose $y' = H(t-1) + H(t-2) + H(t-3)$, starting at $y(0) = 0$. Find $y(t)$.

Solution We integrate both sides of the equation.

$$\begin{aligned}\int_0^t y'(t)dt &= \int_0^t (H(t-1) + H(t-2) + H(t-3))dt \\ y(t) - y(0) &= R(t-1) + R(t-2) + R(t-3) \\ y(t) &= R(t-1) + R(t-2) + R(t-3)\end{aligned}$$

$R(t)$ is the unit ramp function $= \max(0, t)$.

Problems 19–25 are about delta functions and solutions to $y' - ay = q\,\delta(t-T)$.

19 For all $t > 0$ find these integrals $a(t), b(t), c(t)$ of point sources and graph $b(t)$:

(a) $\displaystyle\int_0^t \delta(T-2)\,dT$ (b) $\displaystyle\int_0^t (\delta(T-2) - \delta(T-3))\,dT$ (c) $\displaystyle\int_0^t \delta(T-2)\delta(T-3)dT$

Solution For $t < 2$, the spike in $\delta(t-2)$ does not appear in the integral from 0 to t :

$$\text{(a)} \quad \int_0^t \delta(T-2)dT = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

The integral (b) equals **1 for $\mathbf{2 \leq t < 3}$**. This is the difference $H(t-2) - H(t-3)$. The integral (c) is **zero** because $\delta(T-2)\delta(T-3)$ is everywhere zero.

20 Why are these answers reasonable? (They are all correct.)

(a) $\displaystyle\int_{-\infty}^{\infty} e^t\delta(t)dt = 1$ (b) $\displaystyle\int_{-\infty}^{\infty} (\delta(t))^2 dt = \infty$ (c) $\displaystyle\int_{-\infty}^{\infty} e^T\delta(t-T)dT = e^t$

Solution (a) The difference $e^t\delta(t) - \delta(t)$ is everywhere zero (notice it is zero at $t = 0$). So $e^t\delta(t)$ and $\delta(t)$ have the same integral (from $-\infty$ to ∞ that integral is 1). This reasoning can be made more precise.

(b) This is the difference between the step functions $H(t-2)$ and $H(t-3)$. So it equals 1 for $2 \leq t \leq 3$ and otherwise zero.

(c) As in part (a), the difference between $e^T\delta(t-T)$ and $e^t\delta(t-T)$ is zero at $t = T$ (and also zero at every other t). So

$$\int_{-\infty}^{\infty} e^T\delta(t-T)dT = e^t \int_{-\infty}^{\infty} \delta(t-T)dT = e^t.$$

21 *The solution to* $y' = 2y + \delta(t-3)$ *jumps up by* 1 *at* $t = 3$. Before and after $t = 3$, the delta function is zero and y grows like e^{2t}. Draw the graph of $y(t)$ when (a) $y(0) = 0$ and (b) $y(0) = 1$. Write formulas for $y(t)$ before and after $t = 3$.

Solution (a) $y(0) = 0$ gives $y(t) = 0$ until $t = 3$. Then $y(3) = 1$ from the jump. After the jump we are solving $y' = 2y$ and y grows exponentially from $y(3) = 1$. So $y(t) = \boldsymbol{e^{2(t-3)}}$.

(b) $y(0) = 1$ gives $y(t) = e^{2t}$ until $t = 3$. The jump produces $y(3) = e^6 + 1$. Then exponential growth gives $y(t) = e^{2(t-3)}(e^6 + 1) = \boldsymbol{e^{2t} + e^{2(t-3)}}$. One part grows from $t = 0$, one part grows from $t = 3$ as before.

22 Solve these differential equations starting at $y(0) = 2$:

(a) $y' - y = \delta(t-2)$ (b) $y' + y = \delta(t-2)$. (*What is* y_∞ ?)

Solution (a) $y' - y = \delta(t-2)$ starts with $y(t) = y(0)e^t = 2e^t$ up to the jump at $t = 2$. The jump brings another term into $y(t) = \boldsymbol{2e^t + e^{t-2}}$ for $t \geq 2$. Note the jump of $e^{t-2} = 1$ at $t = 2$.

(b) $y' + y = \delta(t-2)$ starts with $y(t) = y(0)e^{-t} = 2e^{-t}$ up to $t = 2$. The jump of 1 at $t = 2$ starts another exponential $e^{-(t-2)}$ (decaying because $a = -1$). Then $y(t) = \boldsymbol{2e^{-t} + e^{-(t-2)}}$.

23 Solve $dy/dt = H(t-1) + \delta(t-1)$ starting from $y(0) = 0$: jump and ramp.

Solution Nothing happens and $y(t) = 0$ until $t = 1$. Then $H(t-1)$ starts a ramp in $y(t)$ and there is a jump from $\delta(t-1)$. So $y(t) = \textbf{ramp} + \textbf{constant} = \max(0, t-1) + 1$.

24 (My small favorite) What is the steady state y_∞ for $y' = -y + \delta(t-1) + H(t-3)$?

Solution $dy/dt = 0$ at the steady state y_{ss}. Then $-y + \delta(t-1) + H(t-3)$ is $-y_\infty + 0 + 1$ and $\boldsymbol{y_\infty = 1}$.

25 Which q and $y(0)$ in $y' - 3y = q(t)$ produce the step solution $y(t) = H(t-1)$?

Solution We simply substitute the particular solution $y(t) = H(t-1)$ into the original differential equation with $y(0) = 0)$:

$$\delta(t-1) - 3H(t-1) = q(t)$$

Notice how $\delta(t-1)$ in $q(t)$ produces the jump $H(t-1)$ in y, and then $-3H(t-1)$ in $q(t)$ cancels the $-3y$ and keeps $dy/dt = 0$ after $t = 1$.

Problems 26–31 are about exponential sources $\boldsymbol{q(t) = Qe^{ct}}$ and resonance.

26 Solve these equations $y' - ay = Qe^{ct}$ as in (19), starting from y(0) = 2:
(a) $y' - y = 8e^{3t}$ (b) $y' + y = 8e^{-3t}$ (*What is* y_∞ ?)

Solution

(a) $a = \mathbf{1}, c = \mathbf{3}$ and $y(0) = \mathbf{2}$

$$y(t) = y(0)e^{at} + 8\frac{e^{ct} - e^{at}}{c - a}$$
$$y(t) = 2e^t + 8\frac{e^{3t} - e^t}{3 - 1}$$
$$y(t) = 2e^t + 4(e^{3t} - e^t)$$
$$y(t) = 4e^{3t} - 2e^t$$

y goes to ∞ as $t \to \infty$

(b) $a = \mathbf{-1}, c = \mathbf{-3}$ and $y(0) = \mathbf{2}$

$$y(t) = y(0)e^{at} + 8\frac{e^{-3t} - e^{-t}}{c - a}$$
$$y(t) = 2e^{-t} + 8\frac{e^{-3t} - e^{-t}}{-3 - (-1)}$$
$$y(t) = 2e^{-t} - 4(e^{-3t} - e^{-t})$$
$$y(t) = -4e^{-3t} + 2e^{-t}$$

y goes to 0 as $t \to \infty$

.

27 When $c = 2.01$ is very close to $a = 2$, solve $y' - 2y = e^{ct}$ starting from $y(0) = 1$. By hand or by computer, draw the graph of $y(t)$: near resonance.

Solution We substitute the values $a = 2, c = 2.01$ and $y(0) = 1$ into equation (18):

$$y(t) = y(0)e^{at} + \frac{e^{ct} - e^{at}}{c - a}$$

$$y(t) = 2e^{at} + \frac{e^{2t} - e^{2.01t}}{2.01 - 2}$$

$$y(t) = 2e^{2t} + 100(e^{2t} - e^{2.01t})$$

$$y(t) = 101e^{2t} - 100e^{2.01t}$$

The graph of this function shows the "near resonance" when $c \approx a$.

28 When $c = 2$ is exactly equal to $a = 2$, solve $y' - 2y = e^{2t}$ starting from $y(0) = 1$. This is resonance as in equation (20). By hand or computer, draw the graph of $y(t)$.

Solution We substitute $a = 2, c = 2$ (resonance) and $y(0) = 1$ into equation (19):

$$y(t) = y(0)e^{at} + te^{at} = e^{2t} + te^{2t}.$$

29 Solve $y' + 4y = 8e^{-4t} + 20$ starting from $y(0) = 0$. What is y_∞ ?

Solution We have $a = -4, c = -4$ and $y(0) = 0$. Equation (19) with resonance leads to $8te^{-4t}$. The constant source 20 leads to $20(e^{-4t} - 1)$. By linearity $y(t) = 8te^{-4t} + 20(e^{-4t} - 1)$. The steady state is $y_\infty = -20$.

30 The solution to $y' - ay = e^{ct}$ didn't come from the main formula (4), but it could. Integrate $e^{-as}e^{cs}$ in (4) to reach the very particular solution $(e^{ct} - e^{at})/(c - a)$.

Solution

$$
\begin{aligned}
y(t) &= e^{at}y(0) + e^{at}\int_0^t e^{-aT}q(T)dT \\
&= e^{at}y(0) + e^{at}\int_0^t e^{-aT}e^{cT}dT \\
&= e^{at}y(0) + e^{at}\int_0^t e^{(c-a)T}dT \\
&= e^{at}y(0) + e^{at}\left(\frac{e^{(c-a)t}-e^{0}}{c-a}\right) \\
&= e^{at}y(0) + \frac{e^{ct}-e^{at}}{c-a} = y_n + y_{vp}
\end{aligned}
$$

31 *The easiest possible equation* $y' = 1$ *has resonance*! The solution $y = t$ shows the factor t. What number is the growth rate a and also the exponent c in the source?

Solution The growth rate in $y' = 1$ or $dy/dt = e^{0t}$ is $\boldsymbol{a = 0}$. The source is e^{ct} with $\boldsymbol{c = 0}$. **Resonance** $\boldsymbol{a = c}$. The resonant solution $y(t) = te^{at}$ is $\boldsymbol{y = t}$, certainly correct for the equation $dy/dt = 1$.

32 Suppose you know two solutions y_1 and y_2 to the equation $y' - a(t)y = q(t)$.

(a) Find a null solution to $y' - a(t)y = 0$.
(b) Find all null solutions y_n. Find all particular solutions y_p.

Solution (a) $y = y_1 - y_2$ will be a null solution by linearity.

(b) $y = C(y_1 - y_2)$ will give all null solutions. Then $y = C(y_1 - y_2) + y_1$ will give all particular solutions. (Also $y = c(y_1 - y_2) + y_2$ will also give all particular solutions.)

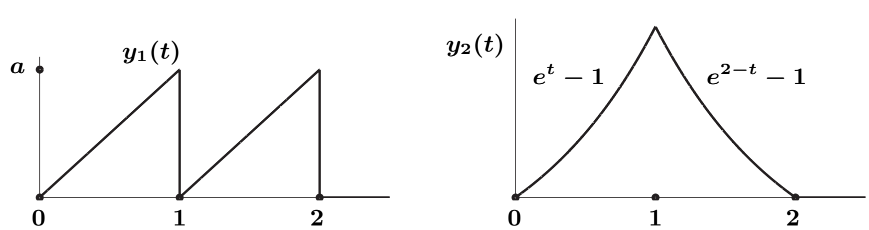
33 Turn back to the first page of this Section 1.4. Without looking, can you write down a solution to $y' - ay = q(t)$ for all four source functions $\boldsymbol{q}, \boldsymbol{H(t)}, \boldsymbol{\delta(t)}, \boldsymbol{e^{ct}}$?

Solution Equations (5), (7), (14), (19).

34 Three of those sources in Problem 33 are actually the same, if you choose the right values for q and c and $y(0)$. What are those values?

Solution The sources $q = 1$ and $q = H(t)$ and $q = e^{0t}$ are all the same for $t \geq 0$.

35 What differential equations $y' = ay + q(t)$ would be solved by $y_1(t)$ and $y_2(t)$? Jumps, ramps, corners—maybe harder than expected (`math.mit.edu/dela/Pset1.4`).



Solution (a) $\dfrac{dy_1}{dt} = 1 - \delta(t-1) - \delta(t-2)$ with $a = 0$.

(b) $\dfrac{dy_2}{dt} = y_2 + 1$ up to $t = 1$. Add in $-2e\,\delta(t-1)$ to drop the slope from e to $-e$ at $t = 1$. After $t = 1$ we need $dy_2/dt = -y_2 - 1$ to keep $y_2 = e^{2-t} - 1$.

Problem Set 1.5, page 37

Problems 1-6 are about the sinusoidal identity (9). It is stated again in Problem 1.

1 These steps lead again to the sinusoidal identity. This approach doesn't start with the usual formula $\cos(\omega t - \phi) = \cos\omega t\ \cos\phi + \sin\omega t\ \sin\phi$ from trigonometry. The identity says:

$$\textbf{If } \boldsymbol{A + iB = R\,e^{i\phi}} \textbf{ then } \boldsymbol{A \cos\omega t + B \sin\omega t = R\cos(\omega t - \phi)}.$$

Here are the four steps to find that real part of $Re^{i(\omega t-\phi)}$. Explain Step 3 where $R\,e^{-i\phi}$ equals $A - iB$:

$$\boldsymbol{R\ \cos(\omega t - \phi)} = \text{Re}\left[R\,e^{i(\omega t-\phi)}\right] = \text{Re}\left[e^{i\omega t}(R\,e^{-i\phi})\right] = \ (\textit{what is } R\,e^{-i\phi}\,?)$$
$$= \text{Re}\left[(\cos\omega t + i\ \sin\omega t)\ (A - iB)\right] = \boldsymbol{A \cos\omega t + B \sin\omega t}.$$

Solution The key point is that if $A + iB = Re^{i\phi}$ then $A - iB = Re^{-i\phi}$ (the complex conjugate).

2 To express $\sin 5t + \cos 5t$ as $R\ \cos(\omega t - \phi)$, what are R and ϕ?

Solution The sinusoidal identity has $A = 1, B = 1,$ and $\omega = 5$. Therefore:
$R^2 = A^2 + B^2 = 2 \to R = \sqrt{2}$ and $\tan\phi = \dfrac{1}{1} \to \phi = \dfrac{\pi}{4}$. Answer $\sqrt{2}\cos\left(5t - \dfrac{\pi}{4}\right)$.

3 To express $6\ \cos 2t + 8\ \sin 2t$ as $R\ \cos(2t - \phi)$, what are R and $\tan\phi$ and ϕ?

Solution Use the Sinusoidal Identity with $A = 6, B = 8$ and $\omega = 2$.

$$R^2 = A^2 + B^2 = 6^2 + 8^2 = 100 \text{ and } R = 10$$

$$\tan\phi = \tfrac{B}{A} = \tfrac{8}{6} = \tfrac{4}{3} \text{ and } \phi \text{ is in the positive quadrant } 0 \textit{ to } \tfrac{\pi}{2}\ (\textit{ not } \pi \textit{ to } \tfrac{3\pi}{2})$$

$$6\cos(2t) + 8\sin(2t) = 10\cos\left(2t - \arctan\left(\frac{4}{3}\right)\right)$$

4 Integrate $\cos\omega t$ to find $(\sin\omega t)/\omega$ in this complex way.

(i) $dy_{\text{real}}/dt = \cos\omega t$ is the real part of $dy_{\text{complex}}/dt = e^{i\omega t}$.

(ii) Take the real part of the complex solution.

Solution (i) The complex equation $y' = e^{i\omega t}$ leads to $y = \dfrac{e^{i\omega t}}{i\omega}$.

(ii) Take the real part of that solution (since the real part of the right side is $\cos\omega t$).

$$\text{Re}\frac{e^{i\omega t}}{i\omega} = \text{Re}\left[\frac{\cos\omega t}{i\omega} + \frac{\sin\omega t}{\omega}\right] = \frac{\sin\omega t}{\omega}.$$

5 The sinusoidal identity for $A = 0$ and $B = -1$ says that $-\sin \omega t = R\cos(\omega t - \phi)$. Find R and ϕ.

Solution $R^2 = A^2 + B^2 = 0^2 + 1^2 = 1 \to R = 1$

$\tan\phi = \frac{1}{0} = \infty \to \phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$: Here it is $\frac{3\pi}{2}$, since $A + iB = -i$

Therefore we have

SOLUTION: $-\sin\omega t = \cos(\omega t - \frac{3\pi}{2})$

CHECK: $t = 0$ gives $0 = 0, \omega t = \frac{\pi}{2}$ gives $-1 = -1$.

6 Why is the sinusoidal identity useless for the source $q(t) = \cos\, t + \sin\, 2t$?

Solution The sinusoidal identity needs the same ω in all terms. But the first term has $\omega = 1$ while the second term has $\omega = 2$.

7 Write $2+3i$ as $re^{i\phi}$, so that $\frac{1}{2+3i} = \frac{1}{r}e^{-i\phi}$. Then write $y = e^{i\omega t}/(2+3i)$ in polar form. Then find the real and imaginary parts of y. And also find those real and imaginary parts directly from $(2-3i)e^{i\omega t}/(2-3i)(2+3i)$.

Solution
$$r = \sqrt{2^2 + 3^2} = \sqrt{13} \text{ and } \phi = \arctan(3/2)$$
$$2 + 3i = \sqrt{13}\, e^{i \arctan(3/2)}$$
$$y = e^{i\omega t}/(2+3i) = \sqrt{13}\, e^{i\arctan(3/2) + i\omega t}$$

Writing this in cartesian (rectangular) form gives
$$\text{real part} = \sqrt{13}\cos(\arctan(3/2) + \omega t) = 2\cos(\omega t) - 3\sin(\omega t)$$
$$\text{imag part} = \sqrt{13}\sin(\arctan(3/2) + \omega t) = 3\cos(\omega t) + 2\sin(\omega t)$$

We can also find the real and imaginary parts from:
$$\frac{(2-3i)e^{i\omega t}}{(2-3i)(2+3i)} = \frac{2-3i}{13}e^{i\omega t} = \frac{2-3i}{13}(\cos(\omega t) + i\sin(\omega t)).$$

8 Write these functions $A\cos\omega t + B\sin\omega t$ in the form $R\cos(\omega t - \phi)$: Right triangle with sides A, B, R and angle ϕ.

(1) $\cos 3t - \sin 3t$ (2) $\sqrt{3}\cos\pi t - \sin\pi t$ (3) $3\cos(t-\phi) + 4\sin(t-\phi)$

Solution (1) $\cos 3t - \sin 3t = \sqrt{2}\cos(3t - \frac{7\pi}{4}) = \sqrt{2}\cos(3t + \frac{\pi}{4})$.

Check $t = 0 : 1 = \sqrt{2}\cos(-\frac{7\pi}{4}) = \sqrt{2}\cos(\frac{\pi}{4})$.

(2) $\sqrt{3}\cos\pi t - \sin\pi t = 2\cos(\pi t + \frac{\pi}{6})$.

Check: $(\sqrt{3})^2 + (-1)^2 = 2^2$ At $t = 0 : \sqrt{3} = 2\cos 30\,°$.

(3) $3\cos(t-\phi) + 4\sin(t-\phi) = 5\cos(t - \phi - \tan^{-1}\frac{4}{3})$.

Problems 9-15 solve real equations using the real formula (3) for M and N.

9 Solve $dy/dt = 2y + 3\cos t + 4\sin t$ after recognizing a and ω. Null solutions Ce^{2t}.

Solution $\frac{dy}{dt} = 2y + 3\cos t + 4\sin t = 2y + 5\cos(t-\phi)$ with $\tan\phi = \frac{4}{3}$.

Method 1: Look for $y = M\cos t + N\sin t$.

Method 2: Solve $\frac{dY}{dt} = 2Y + 5e^{i(t-\phi)}$ and then $y =$ real part of Y.

$Y = \frac{5}{i-2}e^{i(t-\phi)} = \frac{5}{5}(-i-2)e^{i(t-\phi)}$ and $y = -2\cos(t-\phi) + \sin(t-\phi)$.

10 Find a particular solution to $dy/dt = -y - \cos 2t$.

Solution Substitute $y = M\cos t + N\sin t$ into the equation to find M and N

$$-M\sin t + N\cos t = -M\cos t - N\sin t - \cos 2t$$

Match coefficients of $\cos t$ and $\sin t$ separately to find M and N.

$$N = -M - 1 \text{ and } -M = -N \text{ give } M = N = -\frac{1}{2}$$

Note: This is called the "method of undetermined coefficients" in Section 2.6.

11 What equation $y' - ay = A\cos\omega t + B\sin\omega t$ is solved by $y = 3\cos 2t + 4\sin 2t$?

Solution Clearly $\omega = 2$. Substitute y into the equation:

$$-6\sin 2t + 8\cos 2t - 3a\cos 2t - 4a\sin 2t = A\cos 2t + B\sin 2t.$$

Match separately the coefficients of $\cos 2t$ and $\sin 2t$:

$$A = 8 - 3a \text{ and } B = -6 - 4a$$

12 The particular solution to $y' = y + \cos t$ in Section 4 is $y_p = e^t \int e^{-s}\cos s\,ds$. Look this up or integrate by parts, from $s = 0$ to t. Compare this y_p to formula (3).

Solution That integral goes from 0 to t, and it leads to $y_p = \frac{1}{2}(\sin t - \cos t + e^t)$

If we use formula (3) with $a = 1, \omega = 1, A = 1, B = 0$ we get

$$M = -\frac{aA + \omega B}{\omega^2 + a^2} = \frac{-1}{2} \qquad N = \frac{\omega A - aB}{\omega^2 + a^2} = \frac{1}{2}$$

This solution $y = M\cos t + N\sin t = \dfrac{-\cos t + \sin t}{2}$ is a different particular solution (not starting from $y(0) = 0$). The difference is a null solution $\frac{1}{2}e^t$.

13 Find a solution $y = M\ \cos\,\omega t + N\ \sin\,\omega t$ to $y' - 4y = \cos\,3t + \sin\,3t$.

Solution Formula (3) with $a = 4, \omega = 3, A = B = 1$ gives

$$M = -\frac{4+3}{9+16} = -\frac{7}{25} \qquad N = \frac{3-4}{9+16} = -\frac{1}{25}.$$

14 Find the solution to $y' - ay = A\ \cos\,\omega t + B\ \sin\,\omega t$ **starting from $\boldsymbol{y(0) = 0}$**.

Solution One particular solution $M\cos\omega t + N\sin\omega t$ comes from formula (3). But this starts from $y_p(0) = M$. So subtract off the null solution $y_n = Me^{at}$ to get the very particular solution $y_{vp} = y_p - y_n$ that starts from $y_{vp}(0) = 0$.

15 If $a = 0$ show that M and N in equation (3) still solve $y' = A\ \cos\,\omega t + B\ \sin\,\omega t$.

Solution Formula (3) still applies with $a = 0$ and it gives

$$M = -\frac{\omega B}{\omega^2} \qquad N = \frac{\omega A}{\omega^2} \qquad y = -\frac{B}{\omega}\cos\omega t + \frac{A}{\omega}\sin\omega t.$$

This is the correct integral of $A\cos\omega t + B\sin\omega t$ in the differential equation.

Problems 16-20 solve the complex equation $y' - ay = Re^{i(\omega t-\phi)}$.

16 Write down complex solutions $y_p = Ye^{i\omega t}$ to these three equations:
(a) $y' - 3y = 5e^{2it}$ (b) $y' = Re^{i(\omega t-\phi)}$ (c) $y' = 2y - e^{it}$

Solution (a) $y' - 3y = 5e^{2it}$ has $i\omega Ye^{i\omega t} - 3Ye^{i\omega t} = 5e^{2it}$.
So $\omega = 2$ and $Y = \frac{5}{2i-3}$.

(b) $y' = Re^{i(\omega t-\phi)}$ has $i\omega Ye^{i\omega t} = Re^{i(\omega t-\phi)}$. So $Y = \frac{R}{i\omega}e^{-i\phi}$ and the solution is $y = Ye^{i\omega t} = \frac{R}{i\omega}e^{i(\omega t-\phi)}$.

(c) $y' = 2y - e^{it}$ has $\omega = 1$ and $iYe^{it} = 2Ye^{it} - e^{-it}$.
Then $Y = \frac{-1}{i-2} = \frac{1}{2-i} = \frac{2+i}{5}$ and $y = Ye^{it}$.

17 Find complex solutions $z_p = Ze^{i\omega t}$ to these complex equations:
(a) $z' + 4z = e^{8it}$ (b) $z' + 4iz = e^{8it}$ (c) $z' + 4iz = e^{8t}$

Solution (a) $z' + 4z = e^{8it}$ has $z = Ze^{8it}$ with $8iZ + 4Z = 1$ and $Z = \frac{1}{4+8i} = \frac{4-8i}{16+64} = \frac{1}{20}(1-2i)$.

(b) $z' + 4iz = e^{8it}$ is like part (a) but 4 changes to $4i$. Then $Z = \frac{1}{4i+8i} = \frac{1}{12i} = -\frac{i}{12}$.

(c) $z' + 4iz = e^{8t}$ has $z = Ze^{8t}$. Then $8Ze^{8t} + 4iZe^{8t}$ gives $Z = \frac{1}{8+4i} = \frac{8-4i}{8^2+4^2}$.

18 Start with the real equation $y' - ay = R\cos(\omega t - \phi)$. Change to the complex equation $z' - az = Re^{i(\omega t-\phi)}$. Solve for $z(t)$. Then take its real part $y_p = \text{Re}\, z$.

Solution Put $z = Ze^{i(\omega t-\phi)}$ in the complex equation to find Z:
$$i\omega Z - aZ = R \text{ gives } Z = \frac{R}{-a+i\omega} = \frac{R(-a-i\omega)}{a^2+\omega^2}.$$
The real part of $z = Z(\cos(\omega t-\phi) + i\sin(\omega t-\phi))$ is
$\frac{R}{a^2+\omega^2}(-a\cos(\omega t-\phi) + \omega\sin(\omega t-\phi))$.

19 What is the initial value $y_p(0)$ of the particular solution y_p from Problem 18? If the desired initial value is $y(0)$, how much of the null solution $y_n = Ce^{at}$ would you add to y_p?

Solution That solution to 18 starts from $y_p(0) = \frac{R}{a^2+\omega^2}(-a\cos(-\phi) + \omega\sin(-\phi))$ at $t = 0$. So subtract that number times e^{at} to get the very particular solution that starts from $y_{vp}(0) = 0$.

20 Find the real solution to $y' - 2y = \cos\omega t$ starting from $y(0) = 0$, in three steps: Solve the complex equation $z' - 2z = e^{i\omega t}$, take $y_p = \text{Re}\, z$, and add the null solution $y_n = Ce^{2t}$ with the right C.

Solution Step 1. $z' - 2Z = e^{i\omega t}$ is solved by $z = Ze^{i\omega t}$ with $i\omega Z - 2Z = 1$ and $Z = \frac{1}{-2+i\omega} = \frac{-2-i\omega}{4+\omega^2}$.

Step 2. The real part of $Ze^{i\omega t}$ is $y_p = \frac{1}{4+\omega^2}(-2\cos\omega t + \omega\sin\omega t)$.

Step 3. $y_p(0) = \frac{-2}{4+\omega^2}$ so $y_{vp} = y_p + \frac{2}{4+\omega^2}e^{2t}$ includes the right $y_n = Ce^{2t}$ for $y_{vp}(0) = 0$.

Problems 21-27 solve real equations by making them complex. First a note on α.

Example 4 was $y' - y = \cos t - \sin t$, with growth rate $a = 1$ and frequency $\omega = 1$. The magnitude of $i\omega - a$ is $\sqrt{2}$ and the polar angle has $\tan\alpha = -\omega/a = -1$. Notice: *Both* $\alpha = 3\pi/4$ *and* $\alpha = -\pi/4$ *have that tangent*! How to choose the correct angle α?

The complex number $i\omega - a = i - 1$ is in the *second quadrant*. Its angle is $\alpha = 3\pi/4$. **We had to look at the actual number and not just the tangent of its angle**.

21 Find r and α to write each $i\omega - a$ as $re^{i\alpha}$. Then write $1/re^{i\alpha}$ as $Ge^{-i\alpha}$.
(a) $\sqrt{3}\,i + 1$ (b) $\sqrt{3}\,i - 1$ (c) $i - \sqrt{3}$

Solution (a) $\sqrt{3}i + 1$ is in the first quadrant (positive quarter $0 \le \theta \le \pi/2$) of the complex plane. The angle with tangent $\sqrt{3}/1$ is $60^\circ = \pi/3$. The magnitude of $\sqrt{3}i+1$ is $r = 2$. Then $\sqrt{3}i + 1 = \boldsymbol{2e^{i\pi/3}}$.

(b) $\sqrt{3}i - 1$ is in the second quadrant $\pi/2 \le \theta \le \pi$. The tangent is $-\sqrt{3}$, the angle is $\theta = 2\pi/3$, the number is $\boldsymbol{2e^{2\pi i/3}}$.

(c) $i - \sqrt{3}$ is also in the second quadrant (left from zero and up). Now the tangent is $-1/\sqrt{3}$, the angle is $\theta = 150^\circ = 5\pi/6$. The magnitude is still 2, the number is $2e^{5\pi i/6}$.

22 Use G and α from Problem 21 to solve (a)-(b)-(c). Then take the real part of each equation and the real part of each solution.
(a) $y' + y = e^{i\sqrt{3}t}$ (b) $y' - y = e^{i\sqrt{3}t}$ (c) $y' - \sqrt{3}y = e^{it}$

Solution (a) $y' + y = e^{i\sqrt{3}t}$ is solved by $y = Ye^{i\sqrt{3}t}$ when $i\sqrt{3}Y + Y = 1$. Then $Y = \frac{1}{\sqrt{3}i+1} = \frac{1}{2}e^{-i\pi/3}$ from Problem 21(a). The real part $y_{\text{real}} = \frac{1}{2}\cos(\sqrt{3}t - \pi/3)$ of $Ye^{i\sqrt{3}t}$ solves the real equation $y'_{\text{real}} + y_{\text{real}} = \cos(\sqrt{3}t)$.

(b) $y' - y = e^{i\sqrt{3}t}$ is solved by $y = Ye^{i\sqrt{3}t}$ when $i\sqrt{3}Y - Y = 1$. Then $Y = \frac{1}{2}e^{-2\pi i/3}$ from Problem 21(b). the real part $y_{\text{real}} = \frac{1}{2}\cos(\sqrt{3}t - 2\pi/3)$ solves the real equation $y'_{\text{real}} - y_{\text{real}} = \cos(\sqrt{3}t)$.

(c) $y' - \sqrt{3}y = e^{it}$ is solved by $y = Ye^{it}$ when $iY - \sqrt{3}Y = 1$. Then $Y = \frac{1}{2}e^{-5\pi i/6}$ from Problem 21(c). The real part $y_{\text{real}} = \frac{1}{2}\cos(t - 5\pi/6)$ of Ye^{it} solves $y_{\text{real}} - \sqrt{3}y_{\text{real}} = \cos t$.

23 Solve $y' - y = \cos\omega t + \sin\omega t$ in three steps: real to complex, solve complex, take real part. This is an important example.

Solution **Note: I intended to choose $\boldsymbol{\omega = 1}$.** Then $y' - y = \cos t + \sin t$ has the simple solution $y = -\sin t$. I will apply the 3 steps to this case and then to the harder problem for any ω.

(1) Find R and ϕ in the sinusoidal identity to write $\cos\omega t + \sin\omega t$ as the real part of $Re^{i(\omega t - \phi)}$. This is easy for any ω.

$$\left[\tan\phi = \frac{1}{1} \text{ so } \phi = \frac{\pi}{4}\right] \quad \boldsymbol{\cos\omega t + \sin\omega t = \sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)}$$

(2) Solve $y' - y = e^{i\omega t}$ by $y = Ge^{-i\alpha}e^{i\omega t}$. Multiply by $Re^{-i\phi}$ to solve $z' - z = Re^{i(\omega t - \phi)}$.

$\boldsymbol{\omega = 1}$ $\quad y' - y = e^{it}$ has $y = Ye^{it}$ with $iY - Y = 1$. Then $Y = \frac{1}{i-1} = \frac{1}{\sqrt{2}}e^{3\pi i/4} = Ge^{-i\alpha}$.

$z = \left(\sqrt{2}e^{i(t-\pi/4)}\right)\left(\frac{1}{\sqrt{2}}e^{3\pi i/4}\right) = e^{it}e^{\pi i/2} = ie^{it}$. The real part of z is $\boldsymbol{y = -\sin t}$.

Any $\boldsymbol{\omega}$ $\quad y' - y = e^{i\omega t}$ leads to $i\omega Y - Y = 1$ and $Y = \dfrac{1}{i\omega - 1} = \dfrac{\mathbf{1}}{\boldsymbol{\sqrt{1+\omega^2}}}\boldsymbol{e^{-i\alpha}}$

with $\boldsymbol{\tan\alpha = \omega}$. Then $z(t) = \left(\frac{1}{1+\omega^2}e^{-i\alpha}\right)\left(\sqrt{2}e^{i(\omega t-\pi/4)}\right)$.

(3) Take the real part $y(t) = \text{Re}\, z(t)$. Check that $y' - y = \cos\omega t + \sin\omega t$.

$y(t) = \text{Re}\, z(t) = \frac{\sqrt{2}}{1+\omega^2}\cos(\omega t - \alpha - \frac{\pi}{4})$. Now we need $\tan\alpha = \omega, \cos\alpha = \frac{1}{\sqrt{1+\omega^2}}$, $\sin\alpha = \frac{\omega}{\sqrt{1+\omega^2}}$. Finally $y = \frac{\sqrt{2}}{1+\omega^2}[\cos(\omega t - \frac{\pi}{4})\cos\alpha + \sin(\omega t - \frac{\pi}{4})\sin\alpha]$.

24 Solve $y' - \sqrt{3}y = \cos\, t + \sin\, t$ by the same three steps with $a = \sqrt{3}$ and $\omega = 1$.

Solution (1) $\cos t + \sin t = \sqrt{2}\cos(t - \frac{\pi}{4})$.

(2) $y = Ye^{it}$ with $iY - \sqrt{3}Y = 1$ and $Y = \frac{1}{i-\sqrt{3}} = \frac{1}{2}e^{-5\pi i/6}$ from 1.5.21(c).

Then $z(t) = (\sqrt{2}e^{i(t-\pi/4)})(\frac{1}{2}e^{-5\pi i/6})$.

(3) The real part of $z(t)$ is $y(t) = \frac{1}{\sqrt{2}}\cos(t - \frac{13\pi}{12})$.

25 (**Challenge**) Solve $y' - ay = A\cos\omega t + B\sin\omega t$ in two ways. First, find R and ϕ on the right side and G and α on the left. Show that the final real solution $RG\cos(\omega t - \phi - \alpha)$ agrees with $M\cos\omega t + N\sin\omega t$ in equation (3).

Solution The first way has $R = \sqrt{A^2+B^2}$ and $\tan\phi = B/A$ from the sinusoidal identity. On the left side $1/(i\omega - a) = Ge^{-i\alpha}$ from equation (8) with $G = 1/\sqrt{\omega^2+a^2}$ and $\tan\alpha = -\omega/a$. Combining, the real solution is $y = RG\cos(\omega t - \phi - \alpha)$.

This agrees with $y = M\cos\omega t + N\sin\omega t$ (equation (3) gives M and N).

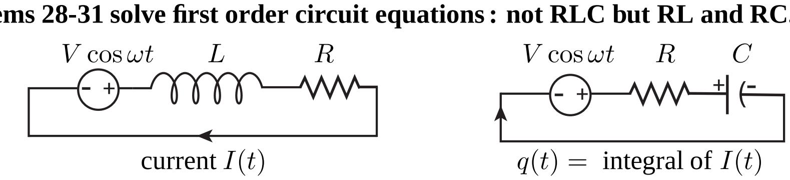
26 We don't have resonance for $y' - ay = Re^{i\omega t}$ when a and $\omega \neq 0$ are real. *Why not*? (Resonance appears when $y_n = Ce^{at}$ and $y_p = Ye^{ct}$ share the exponent $a = c$.)

Solution Resonance requires the exponents a and $i\omega$ to be equal. For real a this only happens if $a = \omega = 0$.

27 If you took the imaginary part $y = \text{Im}\, z$ of the complex solution to $z' - az = Re^{i(\omega t - \phi)}$, what equation would $y(t)$ solve? Answer first with $\phi = 0$.

Solution Assuming a is real, the imaginary part of $z' - az = Re^{i(\omega t-\phi)}$ is the equation $y' - ay = R\sin(\omega t - \phi)$. With $\phi = 0$ this is $y' - ay = R\sin\omega t$.

Problems 28-31 solve first order circuit equations: not RLC but RL and RC.



28 Solve $\boldsymbol{L\,dI/dt + RI(t) = V\ \cos\ \omega t}$ for the current $I(t) = I_n + I_p$ in the RL loop.

Solution Divide the equation by L to produce $dI/dt - aI = X\cos\omega t$ with $a = -R/L$ and $X = V/L$. In this standard form, equation (3) gives the real solution:

$$I = M\cos\omega t + N\sin\omega t \ \text{ with } \ M = -\frac{aX}{\omega^2 + a^2} \ \text{ and } \ N = \frac{\omega X}{\omega^2 + a^2}.$$

29 With $L = 0$ and $\omega = 0$, that equation is Ohm's Law $V = IR$ for direct current. The **complex impedance** $Z = R + i\omega L$ replaces R when $L \neq 0$ and $I(t) = Ie^{i\omega t}$.

$$L\,dI/dt + RI(t) = \boldsymbol{(i\omega L + R)Ie^{i\omega t} = Ve^{i\omega t}} \quad \text{gives} \quad \boldsymbol{Z\,I = V}.$$

What is the magnitude $|Z| = |R + i\omega L|$? What is the phase angle in $Z = |Z|e^{i\theta}$? Is the current $|I|$ larger or smaller because of L?

Solution $|Z| = \sqrt{R^2 + \omega^2 L^2}$ and $\tan\theta = \frac{\omega L}{R}$.

Since $|Z|$ increases with L, the current $|I|$ must decrease.

30 Solve $\boldsymbol{R\dfrac{dq}{dt} + \dfrac{1}{C}q(t) = V\ \cos\ \omega t}$ for the charge $q(t) = q_n + q_p$ in the RC loop.

Solution Dividing by R produces $\frac{dq}{dt} - aq = X\cos\omega t$ with $a = -\frac{1}{RC}$ and $X = \frac{V}{R}$.

As in Problem 28, equation (3) gives M and N from ω and this a.

31 Why is the complex impedance now $Z = R + \frac{1}{i\omega C}$? Find its magnitude $|Z|$. **Note that mathematics prefers $\boldsymbol{i = \sqrt{-1}}$, we are not conceding yet to $\boldsymbol{j = \sqrt{-1}}$!**

Solution The physical RC equation for the current $I = \frac{dq}{dt}$ is $RI + \frac{1}{C}\int I dt = V\cos\omega t = \text{Re}\,(Ve^{i\omega t})$.

The solution I has the same frequency factor $Xe^{i\omega t}$, and the integral has the factor $e^{i\omega t}/i\omega$. Substitute into the equation and match coefficients of $e^{i\omega t}$:

$RX + \frac{1}{i\omega C}X = V$ is $ZX = V$ with impedance $Z = R + \frac{1}{i\omega C}$.

Problem Set 1.6, page 50

1 Solve the equation $dy/dt = y + 1$ up to time t, starting from $y(0) = 4$.

Solution We use the formula $y(t) = y(0)e^{at} + \frac{s}{a}(e^{at} - 1)$ with $a = 1$ and $s = 1$ and $y(0) = 4$:

$$y(t) = 4e^t + e^t - 1 = 5e^t - 1$$

2 You have \$1000 to invest at rate $a = 1 = 100\,\%$. Compare after one year the result of depositing $y(0) = 1000$ immediately with no source ($s = 0$), or choosing $y(0) = 0$ and $s = 1000$/year to deposit continually during the year. In both cases $dy/dt = y + q$.

Solution We substitute the values for the different scenarios into the solution formula:

$$y(t) = 1000e^t \qquad = 1000e \ \text{ at one year}$$

$$y(t) = 1000e^t - 1000 = 1000(e-1) \ \text{ at one year}$$

You get more for depositing immediately rather than during the year.

3 If $dy/dt = y - 1$, when does your original deposit $y(0) = \frac{1}{2}$ drop to zero?

Solution Again we use the equation $y(t) = y(0)e^{at} + \frac{s}{a}(e^{at} - 1)$ with $a = 1$ and $s = -1$. We set $y(t) = 0$ and find the time t:

$$y(t) = y(0)e^t - e^t + 1 = e^t(y(0) - 1) + 1 = 0$$

$$e^t = \frac{1}{1 - y(0)} = 2 \text{ and } t = \ln 2.$$

Notice! If $y(0) > 1$, the balance never drops to zero. Interest exceeds spending.

4 Solve $\dfrac{dy}{dt} = y + t^2$ from $y(0) = 1$ with increasing source term t^2.

Solution Solution formula (12) with $a = 1$ and $y(0) = 1$ gives

$$y(t) = e^t + \int_0^t e^{t-s}s^2 ds = e^t - t(t+2) + 2e^t - 2 = 3e^t - t(t+2) - 2$$

$$\text{Check: } \frac{dy}{dt} = 3e^t + 2t - 2 \text{ equals } y + t^2.$$

5 Solve $\dfrac{dy}{dt} = y + e^t$ (resonance $a = c$!) from $y(0) = 1$ with exponential source e^t.

Solution The solution formula with $a = 1$ and source e^t (resonance!) gives:

$$y(t) = e^t + \int_0^t e^{t-s}e^s ds = e^t + \int_0^t e^t ds = e^t(1+t)$$

$$\text{Check by the product rule: } \frac{dy}{dt} = e^t(1+t) + e^t = y + e^t.$$

6 Solve $\dfrac{dy}{dt} = y - t^2$ from an initial deposit $y(0) = 1$. The spending $q(t) = -t^2$ is growing. When (if ever) does $y(t)$ drop to zero ?

Solution

$y(t) = e^t - \displaystyle\int_0^t e^{t-s}s^2 ds = e^t + t(t+2) - 2e^t + 2 = -e^t + t(t+2)$. This definitely drops to zero (I regret there is no nice formula for that time t).

$$\text{Check: } \frac{dy}{dt} = -e^t + 2t + 2 = y - t^2.$$

7 Solve $\dfrac{dy}{dt} = y - e^t$ from an initial deposit $y(0) = 1$. This spending term $-e^t$ grows at the same e^t rate as the initial deposit (resonance). When (if ever) does y drop to zero ?

Solution $y(t) = e^t - \displaystyle\int_0^t e^{t-s}e^s ds = e^t - \int_0^t e^t ds = e^t(1-t)$ **(this is zero at $\boldsymbol{t = 1}$)**

Check by the product rule: $\frac{dy}{dt} = e^t(1-t) - e^t = y - e^t$.

8 Solve $\dfrac{dy}{dt} = y - e^{2t}$ from $y(0) = 1$. At what time T is $y(T) = 0$?

Solution $\quad y(t) = e^t - \displaystyle\int_0^t e^{t-s}e^{2s}ds = e^t - \int_0^t e^{t+s}ds = e^t + e^t(1 - e^t) = 2e^t - e^{2t}$

This solution is zero when $2e^t = e^{2t}$ and $2 = e^t$ and $t = \ln 2$.

Check that $y = 2e^t - e^{2t}$ solves the equation: $\frac{dy}{dt} = 2e^t - 2e^{2t} = y - e^{2t}$.

9 Which solution (y or Y) is eventually larger if $y(0) = 0$ and $Y(0) = 0$?

$$\frac{dy}{dt} = y + 2t \quad \text{or} \quad \frac{dY}{dt} = 2Y + t.$$

Solution

$$\begin{aligned} \frac{dy}{dt} &= y + 2t & \qquad \frac{dY}{dt} &= 2Y + t \\ y(t) &= \int_0^t e^{t-s} \cdot 2s\,ds & \qquad Y(t) &= \int_0^t e^{2t-2s} \cdot s\,ds \\ y(t) &= 2(-t + e^t - 1) & \qquad Y(t) &= \frac{e^{2t} - 1}{2} \end{aligned}$$

In the long run $Y(t)$ is larger than $y(t)$, since the exponent $2t$ is larger than t.

10 Compare the linear equation $y' = y$ to the separable equation $y' = y^2$ starting from $y(0) = 1$. Which solution $y(t)$ must grow faster ? It grows so fast that it blows up to $y(T) = \infty$ at what time T ?

Solution

$$\begin{aligned} \frac{dy}{dt} &= y & \qquad \frac{dy}{dt} &= y^2 \\ \frac{dy}{y} &= dt & \qquad \frac{dy}{y^2} &= dt \\ \int_{y(0)}^{y(t)} \frac{du}{u} &= \int_0^t dt & \qquad \int_{y(0)}^{y(t)} \frac{du}{u^2} &= \int_0^t dt \\ \ln(y(t)) - \ln(y(0)) &= t & \qquad -\frac{1}{y(t)} + \frac{1}{y(0)} &= t \\ \frac{y(t)}{y(0)} &= e^t & \qquad y(t) = \frac{1}{\frac{1}{y(0)} - t} &= \frac{\mathbf{1}}{\mathbf{1 - t}} \\ y(t) = y(0)e^t &= \boldsymbol{e^t} & & \end{aligned}$$

The second solution grows much faster, and reaches a vertical asymptote at $T = 1$.

11 $Y' = 2Y$ has a larger growth factor (because $a = 2$) than $y' = y + q(t)$. What source $q(t)$ would be needed to keep $y(t) = Y(t)$ for all time ?

Solution $\quad \frac{dY}{dt} = 2Y + 1$ with for example $Y(0) = y(0) = 0$

$$Y(t) = \int_0^t e^{2t-2s}ds = \frac{e^{2t} - 1}{2}$$

Put this solution into $\frac{dy}{dt} = y + q(t)$:

$$e^{2t} = \frac{e^{2t}-1}{2} + q(t)$$
$$\frac{e^{2t}+1}{2} = q(t)$$

12 Starting from $y(0) = Y(0) = 1$, does $y(t)$ or $Y(t)$ eventually become larger ?

$$\frac{dy}{dt} = 2y + e^t \qquad \frac{dY}{dt} = Y + e^{2t}.$$

Solution $$\frac{dy}{dt} = 2y + e^t$$

$$y(t) = e^{2t} + \int_0^t e^{2t-2s} e^s ds = e^{2t} + e^{2t} - e^t = 2e^{2t} - e^t$$

Solving the second equation:

$$\frac{dY}{dt} = Y + e^{2t}$$

$$Y(t) = e^t + \int_0^t e^{t-s} e^{2s} ds = e^t + e^{2t} - e^t = e^{2t} \text{ is always smaller than } y(t).$$

Questions 13-18 are about the growth factor $G(s, t)$ from time s to time t.

13 What is the factor $G(s, s)$ in zero time ? Find $G(s, \infty)$ if $a = -1$ and if $a = 1$.

Solution The solution doesn't change in zero time so $G(s, s) = 1$. (Note that the integral of $a(t)$ from $t = s$ to $t = s$ is zero. Then $G(s, s) = e^0 = 1$. We are talking about change in the null solution, with $y' = a(t)y$. A source term with a delta function does produce instant change.)

If $a = -1$, the solution drops to zero at $t = \infty$. So $G(s, \infty) = 0$.

If $a = 1$, the solution grows infinitely large as $t \to \infty$. So $G(s, \infty) = \infty$.

14 Explain the important statement after equation (13): *The growth factor $G(s, t)$ is the solution to $y' = a(t)y + \delta(t - s)$.* The source $\delta(t - s)$ deposits \$1 at time s.

Solution When the source term $\delta(t - s)$ deposits \$1 at time s, that deposit will grow or decay to $y(t) = G(s, t)$ at time $t > s$. This is consistent with the main solution formula (13).

15 Now explain this meaning of $G(s, t)$ when t *is less than* s. We go backwards in time. *For $t < s$, $G(s, t)$ is the value at time t that will grow to equal 1 at time s.*

When $t = 0$, $G(s, 0)$ is the "present value" of a promise to pay \$1 at time s. If the interest rate is $a = 0.1 = 10\,\%$ per year, what is the present value $G(s, 0)$ of a million dollar inheritance promised in $s = 10$ years ?

Solution In fact $G(t, s) = 1/G(s, t)$. In the simplest case $y' = y$ of exponential growth, $G(s, t)$ is the growth factor e^{t-s} from s to t. Then $G(t, s)$ is $e^{s-t} = 1/e^{t-s}$.

That number $G(t, s)$ would be the "present value" at the earlier time t of a promise to pay \$1 at the later time s. You wouldn't need to deposit the full \$1 because your deposit will grow by the factor $G(s, t)$. All you need to have at the earlier time is $1/G(s, t)$, which then grows to 1.

16 (a) What is the growth factor $G(s,t)$ for the equation $y' = (\sin\, t)y + Q\, \sin\, t$?

(b) What is the null solution $y_n = G(0,t)$ to $y' = (\sin\, t)y$ when $y(0) = 1$?

(c) What is the particular solution $y_p = \int_0^t G(s,t)\; Q\; \sin\; s\, ds$?

Solution (a) Growth factor: $G(s,t) = \exp\left(\int_s^t \sin T dT\right) = \exp(\cos s - \cos t)$.

(b) Null solution: $y_n = G(0,t)\, y(0) = e^{1-\cos t}$.

(c) Particular solution: $y_p = \displaystyle\int_0^t e^{\cos s - \cos t} Q \sin s\, ds$

$= Qe^{-\cos t}\left[-e^{\cos s}\right]_0^t = Q\left(e^{1-\cos t} - 1\right)$. Check $y_p(0) = Q(e^0 - 1) = 0$.

17 (a) What is the growth factor $G(s,t)$ for the equation $y' = y/(t+1) + 10$?

(b) What is the null solution $y_n = G(0,t)$ to $y' = y/(t+1)$ with $y(0) = 1$?

(c) What is the particular solution $y_p = 10 \int_0^t G(s,t)\, ds$?

Solution (a) $G(s,t) = \exp\left[\displaystyle\int_s^t \frac{dT}{T+1}\right] = \exp\left[\ln(t+1) - \ln(s+1)\right] = \dfrac{t+1}{s+1}$.

Null solution $y_n = G(0,t)\, y(0) = \exp\left[\mathbf{ln}(\boldsymbol{t}+\mathbf{1})\right] = \boldsymbol{t} + \mathbf{1}$ since $\ln(0+1) = 0$.

Particular solution $y_p = 10 \displaystyle\int_0^t \exp\left[\ln(t+1) - \ln(s+1)\right] ds = 10(t+1)\int_0^t \frac{ds}{s+1} =$ $10(t+1)\ln(t+1)$.

18 Why is $G(t,s) = 1/G(s,t)$? Why is $G(s,t) = G(s,S)G(S,t)$?

Solution Multiplying $G(s,t)\, G(t,s)$ gives the growth factor $G(s,s)$ from going up to time t and back to time s. This factor is $G(s,s) = 1$. So $G(t,s) = 1/G(s,t)$. Multiplying $G(s,S)\, G(S,t)$ gives the growth factor $G(s,t)$ from going up from s to S and continuing from S to t. In the example $y' = y$, this is $e^{S-s}e^{t-S} = e^{t-s} = G(s,t)$.

Problems 19–22 are about the "units" or "dimensions" in differential equations.

19 (recommended) If $dy/dt = ay + qe^{i\omega t}$, with t in seconds and y in meters, what are the units for a and q and ω ?

Solution a is in "inverse seconds"—for example $a = .01$ per second.

q is in meters.

ω is in "inverse seconds" or 1/seconds—for example $\omega = 2\pi$ radians per second.

20 The logistic equation $dy/dt = ay - by^2$ often measures the time t in years (and y counts people). What are the units of a and b ?

Solution a is in "inverse years"—for example $a = 1$ percent per year.

b is in "inverse people-years" as in $b = 1$ percent per person per year.

21 Newton's Law is $m\, d^2y/dt^2 + ky = F$. If the mass m is in grams, y is in meters, and t is in seconds, what are the units of the stiffness k and the force F ?

Solution ky has the same units as $m\, d^2y/dt^2$ so k is in grams per (second)^2.

F is in gram-meters per (second)^2—the units of force.

22 Why is our favorite example $y' = y + 1$ very unsatisfactory dimensionally ? Solve it anyway starting from $y(0) = -1$ and from $y(0) = 0$.

The three terms in $y' = y + 1$ seem to have different units. The rate $a = 1$ is hidden (with its units of 1/time). Also hidden are the units of the source term 1.

Solution $y(t) = y(0)e^t + \frac{1}{1}(e^t - 1)$. This is $e^t - 1$ if $y(0) = 0$. The solution stays at steady state if $y(0) = -1$.

23 The difference equation $Y_{n+1} = cY_n + Q_n$ produces $Y_1 = cY_0 + Q_0$. Show that the next step produces $Y_2 = c^2Y_0 + cQ_0 + Q_1$. After N steps, the solution formula for Y_N is like the solution formula for $y' = ay + q(t)$. Exponentials of a change to powers of c, the null solution $e^{at}y(0)$ becomes c^NY_0. The particular solution

$$Y_N = c^{N-1}Q_0 + \cdots + Q_{N-1} \quad \text{is like} \quad y(t) = \int_0^t e^{a(t-s)}q(s)ds.$$

Solution $Y_2 = cY_1 + Q_1 = c(cY_0 + Q_0) + Q_1 = c^2Y_0 + cQ_0 + Q_1$.

The particular solution $cQ_0 + Q_1$ agrees with the general formula when $N = 2$. The null solution c^2Y_0 is Step 2 in $Y_0, cY_0, c^2Y_0, c^3Y_0, \ldots$ like $e^{at}y(0)$.

24 Suppose a fungus doubles in size every day, and it weighs a pound after 10 days. If another fungus was twice as large at the start, would it weigh a pound in 5 days ?

Solution This is an ancient puzzle and the answer is 9 days. Starting twice as large cuts off 1 day.

Problem Set 1.7, page 61

1 If $y(0) = a/2b$, the halfway point on the S-curve is at $t = 0$. Show that $d = b$ and $y(t) = \dfrac{a}{d\,e^{-at} + b} = \dfrac{a}{b}\dfrac{1}{e^{-at} + 1}$. Sketch the classic S-curve — graph of $1(e^{-at} + 1)$ from $y_{-\infty} = 0$ to $y_\infty = \dfrac{a}{b}$. Mark the inflection point.

Solution $d = \dfrac{a}{y(0)} - b$ and $y(0) = \dfrac{a}{2b}$ lead to $d = \dfrac{a}{\frac{a}{2b}} - b = 2b - b = b$

$$\text{Therefore } y(t) = \frac{a}{d\,e^{-at} + b} = \frac{a}{b\,e^{-at} + b} = \frac{a}{b}\frac{1}{e^{-at} + 1}$$

2 If the carrying capacity of the Earth is $K = a/b = 14$ billion people, what will be the population at the inflection point? What is dy/dt at that point? The actual population was 7.14 billion on January 1, 2014.

Solution The inflection point comes where $y = a/2b = 7$ million. The slope dy/dt is

$$\frac{dy}{dt} = ay - by^2 = a\frac{a}{2b} - b\left(\frac{a}{2b}\right)^2 = \frac{a^2}{4b}. \text{ This is } \; b\left(\frac{a}{2b}\right)^2 = 49b.$$

3 Equation (18) must give the same formula for the solution $y(t)$ as equation (16). If the right side of (18) is called R, we can solve that equation for y:

$$y = R\left(1 - \frac{b}{a}y\right) \quad \to \quad \left(1 + R\frac{b}{a}\right)y = R \quad \to \quad y = \frac{R}{\left(1 + R\frac{b}{a}\right)}.$$

Simplify that answer by algebra to recover equation (16) for $y(t)$.

Solution This problem asks us to complete the partial fractions method which integrated $dy/(y - \frac{b}{a}y^2) = a\,dt$. The result in equation (18) can be solved for $y(t)$. The right side of (18) is called R:

$$R = e^{at}\frac{y(0)}{1 - \frac{b}{a}y(0)} = e^{at}a\frac{y(0)}{a - by(0)} = e^{at}\frac{a}{d}.$$

Then the algebra in the problem statement gives

$$y = \frac{R}{1 + R\frac{b}{a}} = \frac{e^{at}\frac{a}{d}}{1 + e^{at}\frac{b}{d}} = \text{ multiply by } \frac{de^{-at}}{de^{-at}} = \frac{a}{de^{-at} + b}.$$

4 Change the logistic equation to $y' = y + y^2$. Now the nonlinear term is positive, and *cooperation of* y *with* y promotes growth. Use $z = 1/y$ to find and solve a linear equation for z, starting from $z(0) = y(0) = 1$. Show that $y(T) = \infty$ when $e^{-T} = 1/2$. Cooperation looks bad, the population will explode at $t = T$.

Solution Put $y = 1/z$ and the chain rule $\frac{dy}{dt} = \frac{-1}{z^2}\frac{dz}{dt}$ into the cooperation equation $y' = y + y^2$:

$$-\frac{1}{z^2}\frac{dz}{dt} = \frac{1}{z} + \frac{1}{z^2} \text{ gives } \frac{dz}{dt} = -z - 1.$$

The solution starting from $z(0) = 1$ is $z(t) = 2e^{-t} - 1$. This is zero when $2e^{-T} = 1$ or $e^T = 2$ or $\boldsymbol{T = \ln 2}$.

At that time $z(T) = 0$ means $y(T) = 1/z(T)$ is infinite: blow-up at time $T = \ln 2$.

5 The US population grew from $313,873,685$ in 2012 to $316,128,839$ in 2014. If it were following a logistic S-curve, what equations would give you a, b, d in the formula (4)? Is the logistic equation reasonable and how to account for immigration?

Solution We need a third data point to find all three numbers a, b, d. **See Problem (23)**. There seems to be no simple formula for those numbers. Certainly the logistic equation is too simple for serious science. Immigration would give a negative value for h in the harvesting equation $y' = ay - by^2 - h$.

6 The **Bernoulli equation** $y' = ay - by^n$ has competition term by^n. Introduce $z = y^{1-n}$ which matches the logistic case when $n = 2$. Follow equation (4) to show that $z' = (n-1)(-az+b)$. Write $z(t)$ as in (5)-(6). Then you have $y(t)$.

Solution We make the suggested transformation:

$$\begin{aligned}
z &= y^{1-n}\\
z' &= (1-n)y^{-n}y'\\
\tfrac{dz}{dt} &= (1-n)y^{-n}(ay - by^n) = (1-n)(ay^{1-n} - b)\\
\tfrac{dz}{dt} &= (1-n)(az - b)\\
z(t) &= e^{(1-n)at}z(0) - \frac{b}{a}(e^{(1-n)at} - 1) = \frac{de^{(1-n)at} + b}{a}\\
d &= az(0) - b = \frac{a}{y(0)} - b\\
y(t) &= \frac{a}{de^{(1-n)at} + b}
\end{aligned}$$

Problems 7–13 develop better pictures of the logistic and harvesting equations.

7 $y' = y - y^2$ is solved by $y(t) = 1/(de^{-t}+1)$. This is an S-curve when $y(0) = 1/2$ and $d = 1$. But show that $y(t)$ is very different if $y(0) > 1$ or if $y(0) < 0$.

If $y(0) = 2$ then $d = \frac{1}{2} - 1 = -\frac{1}{2}$. Show that $y(t) \to 1$ from above.
If $y(0) = -1$ then $d = \frac{1}{-1} - 1 = -2$. At what time T is $y(T) = -\infty$?

Solution First, $y(0) = 2$ is *above* the steady-state value $y_\infty = a/b = 1/1$. Then $d = -\frac{1}{2}$ and $y(t) = 1/(1 - \frac{1}{2}e^{-t})$ is larger than 1 and approaches $y(\infty) = 1/1$ from above as e^{-t} goes to zero.

Second, $y(0) = -1$ is below the $S-$curve growing from $y(-\infty) = 0$ to $y(\infty) = 1$. The value $d = -2$ gives $y(t) = 1/(-2e^{-t}+1)$. When e^{-t} equals $\frac{1}{2}$ this is $y(t) = 1/0$ and the solution blows up. That blowup time is $\boldsymbol{t = \ln 2}$.

8 (recommended) Show those 3 solutions to $y' = y - y^2$ in one graph! They start from $y(0) = 1/2$ and 2 and -1. The S-curve climbs from $\frac{1}{2}$ to 1. Above that, $y(t)$ descends from 2 to 1. Below the S-curve, $y(t)$ drops from -1 to $-\infty$.

Can you see 3 regions in the picture? **Dropin curves above $\boldsymbol{y = 1}$ and $\boldsymbol{S}$-curves sandwiched between 0 and 1 and dropoff curves below $\boldsymbol{y = 0}$.**

Solution The three curves are drawn in Figure 3.3 on page 157. The upper curves and middle curves approach $y_\infty = a/b$. The lowest curves reach $y = -\infty$ in finite time: blow-up.

9 Graph $f(y) = y - y^2$ to see the unstable steady state $Y = 0$ and the stable $Y = 1$. Then graph $f(y) = y - y^2 - 2/9$ with harvesting $h = 2/9$. What are the steady states Y_1 and Y_2? The 3 regions in Problem 8 now have Z-curves above $y = 2/3$, S-curves sandwiched between $1/3$ and $2/3$, dropoff curves below $y = 1/3$.

Solution The steady states are the points where $Y - Y^2 = 0$ (logistic) and $Y - Y^2 - \frac{2}{9} = 0$ (harvesting). That second equation factors into $(Y - \frac{1}{3})(Y - \frac{2}{3})$ to show the steady states $\frac{1}{3}$ and $\frac{2}{3}$.

10 What equation produces an S-curve climbing to $y_\infty = K$ from $y_{-\infty} = L$?

Solution We can choose $y' = ay - by^2 - h$ with steady states K and L. Then $aK - bK^2 - h = 0$ and $aL - bL^2 - h = 0$. If we divide by h, these two linear equations give

$$\frac{a}{h} = \frac{K+L}{KL} = \frac{\mathbf{1}}{\boldsymbol{K}} + \frac{\mathbf{1}}{\boldsymbol{L}} \quad \text{and} \quad \frac{b}{h} = \frac{\mathbf{1}}{\boldsymbol{KL}}$$

$$\text{Check}: \frac{a}{h}K - \frac{b}{h}K^2 - 1 = \frac{K}{L} - \frac{K}{L} = \mathbf{0} \quad \text{and} \quad \frac{a}{h}L - \frac{b}{h}L^2 - 1 = \frac{L}{K} - \frac{L}{K} = 0$$

11 $y' = y - y^2 - \frac{1}{4} = -(y - \frac{1}{2})^2$ shows *critical harvesting* with a double steady state at $y = Y = \frac{1}{2}$. The layer of S-curves shrinks to that single line. Sketch a dropin curve that starts above $y(0) = \frac{1}{2}$ and a dropoff curve that starts below $y(0) = \frac{1}{2}$.

Solution The solution to $y' = -(y - \frac{1}{2})^2$ comes from integrating $-dy/(y - \frac{1}{2})^2 = dt$ to get $1/(y - \frac{1}{2}) = t + C$. Then $\boldsymbol{y(t) = \frac{1}{2} + \frac{1}{t+C}}$. If $y(0) > \frac{1}{2}$ then $C > 0$ and this curve approaches $y(\infty) = \frac{1}{2}$; it is a hyperbola coming down toward that horizontal line. If $y(0) < \frac{1}{2}$ then C is negative and the above solution $y = \frac{1}{2} + \frac{1}{t+C}$ blows up (or blows down! since y is negative) at the positive time $t = -C$. This is a dropoff curve below the horizontal line $y = \frac{1}{2}$. (If $y(0) = \frac{1}{2}$ the equation is $dy/dt = 0$ and the solution stays at that steady state.)

12 Solve the equation $y' = -(y - \frac{1}{2})^2$ by substituting $v = y - \frac{1}{2}$ and solving $v' = -v^2$.

Solution This approach uses the solutions we know to $dv/dt = -v^2$. Those solutions are $v(t) = \frac{1}{t+C}$. Then $v = y - \frac{1}{2}$ gives the same $y = \frac{1}{2} + \frac{1}{t+C}$ as in Problem 11.

13 With overharvesting, every curve $y(t)$ drops to $-\infty$. There are no steady states. Solve $Y - Y^2 - h = 0$ (quadratic formula) to find only complex roots if $4h > 1$.

The solutions for $h = \frac{5}{4}$ are $y(t) = \frac{1}{2} - \tan(t + C)$. Sketch that dropoff if $C = 0$. Animal populations don't normally collapse like this from overharvesting.

Solution Overharvesting is $y' = y - y^2 - h$ with h larger than $\frac{1}{4}$ (Problems 11 and 12 had $h = \frac{1}{4}$ and critical harvesting). The fixed points come from $Y - Y^2 - h = 0$. The quadratic formula gives $Y = \frac{1}{2}(1 \pm \sqrt{1 - 4h})$. These roots are complex for $h > \frac{1}{4}$: **No fixed points**.

For $h = \frac{5}{4}$ the equation is $y' = y - y^2 - \frac{5}{4} = -(y - \frac{1}{2})^2 - 1$. Then $v = y - \frac{1}{2}$ has $v' = -v^2 - 1$. Integrating $dv/(1 + v^2) = -dt$ **gives** $\tan^{-1} v = -t - C$ or $v = -\tan(t + C)$. $y = v + \frac{1}{2} = \frac{1}{2} - \tan(t + C)$. The graph of $-\tan t$ starts at zero and drops to $-\infty$ at $t = \pi/2$.

14 With **two partial fractions**, this is my preferred way to find $A = \dfrac{1}{r - s}$, $B = \dfrac{1}{s - r}$

PF2 $$\frac{\mathbf{1}}{\boldsymbol{(y - r)(y - s)}} = \frac{\mathbf{1}}{\boldsymbol{(y - r)(r - s)}} + \frac{\mathbf{1}}{\boldsymbol{(y - s)(s - r)}}$$

Check that equation: The common denominator on the right is $\boldsymbol{(y - r)(y - s)(r - s)}$. The numerator should cancel the $\boldsymbol{r - s}$ when you combine the two fractions.

Separate $\dfrac{1}{y^2-1}$ and $\dfrac{1}{y^2-y}$ into two fractions $\dfrac{A}{y-r}+\dfrac{B}{y-s}$.

Note When y approaches r, the left side of **PF2** has a blowup factor $1/(y-r)$. The other factor $1/(y-s)$ correctly approaches $A = 1/(r-s)$. So the right side **of PF2** needs the same blowup at $y = r$. The first term $A/(y-r)$ fits the bill.

Solution
$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} = \frac{1/2}{y-1} - \frac{1/2}{y+1}$$

The constants are $A = \dfrac{1}{r-s} = \dfrac{1}{1-(-1)} = -\dfrac{1}{2} = -B$

$$\frac{1}{y^2-y} = \frac{1}{(y-1)y} = \frac{A}{y-1} + \frac{B}{y} = \frac{1}{y-1} - \frac{1}{y}, \quad A = \frac{1}{r-s} = \frac{1}{1-0} = -B$$

15 The **threshold equation** is the logistic equation backward in time :

$$-\frac{dy}{dt} = ay - by^2 \quad \text{is the same as} \quad \frac{dy}{dt} = -ay + by^2.$$

Now $Y = 0$ is the stable steady state. $Y = a/b$ is the unstable state (why ?). If $y(0)$ is below the threshold a/b then $y(t) \to 0$ and the species will die out.

Graph $y(t)$ with $y(0) < a/b$ (reverse S-curve). Then graph $y(t)$ with $y(0) > a/b$.

Solution The steady states of $dy/dt = -ay + by^2$ come from $-aY + bY^2 = 0$ so again $Y = 0$ or $Y = a/b$. The stability is controlled by the **sign of df/dy at $y = Y$** :

$f = -ay + by^2$ tells how y grows $\qquad \dfrac{df}{dy} = -a + 2by$ tells how Δy grows

$Y = 0$ has $\dfrac{df}{dy} = -a$ (STABLE) $\qquad Y = \dfrac{a}{b}$ has $\dfrac{df}{dy} = -a + 2b\left(\dfrac{a}{b}\right) = a$ (UNSTABLE)

The S-curves go downward from $Y = a/b$ toward the line $Y = 0$ (never touch).

16 (Cubic nonlinearity) The equation $y' = y(1-y)(2-y)$ has **three steady states**: $Y = 0, 1, 2$. By computing the derivative df/dy at $y = 0, 1, 2$, decide whether each of these states is stable or unstable.

Draw the *stability line* for this equation, to show $y(t)$ leaving the unstable Y's.

Sketch a graph that shows $y(t)$ starting from $y(0) = \frac{1}{2}$ and $\frac{3}{2}$ and $\frac{5}{2}$.

Solution $y' = f(y) = y(1-y)(2-y) = 2y - 3y^2 + y^3$ has slope $\frac{df}{dy} = 2 - 6y + 3y^2$.

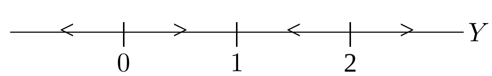
$Y = 0$ has $\frac{df}{dy} = 2$ (unstable)

$Y = 1$ has $\frac{df}{dy} = -1$ (stable)

$Y = 2$ has $\frac{df}{dy} = 2$ (unstable)

$S-$curves go up from $Y = 0$ toward $Y = 1$

$S-$curves from $Y = 2$ go down toward $Y = 1$



17 (a) Find the steady states of the **Gompertz equation** $dy/dt = y(1 - \ln y)$.

Solution (a) $Y(1 - \ln Y) = 0$ at steady states $Y = 0$ and $Y = e$.

(b) Show that $z = \ln y$ satisfies the linear equation $dz/dt = 1 - z$.

Solution (b) $z = \ln y$ has $\frac{dz}{dt} = \frac{1}{y}\frac{dy}{dt} = y(1 - \ln y)/y = 1 - \ln y = 1 - z$.

(c) The solution $z(t) = 1 + e^{-t}(z(0) - 1)$ gives what formula for $y(t)$ from $y(0)$?

Solution (c) $z' = 1/z$ gives that $z(t)$. Then set $y(t) = 1/z(t)$:

$$y(t) = \left[1 + e^{-t}(z(0) - 1)\right]^{-1} = \left[1 + e^{-t}\left(\frac{1}{y(0)} - 1\right)\right]^{-1}.$$

18 Decide stability or instability for the steady states of

(a) $dy/dt = 2(1 - y)(1 - e^y)$ (b) $dy/dt = (1 - y^2)(4 - y^2)$

Solution (a) $f(y) = 2(1 - y)(1 - e^y) = 0$ at $Y = 1$ and $Y = 0$

$\frac{df}{dy} = -2e^y(1 - y) - 2(1 - e^Y)$

At $Y = 1$ $\frac{df}{dy} = -2(1 - e) > 0$ (UNSTABLE) At $Y = 0$ $\frac{df}{dy} = -2$ (STABLE)

(b) $f(y) = (1 - y^2)(4 - y^2) = 0$ at $Y = 1, -1, 2, -2$ $\frac{df}{dy} = -10y + 4y^3$

$Y = 1$ gives $\frac{df}{dy} = -6$ (STABLE) $Y = -1$ gives $\frac{df}{dy} = 6$ (UNSTABLE)

$Y = 2$ gives $\frac{df}{dy} = 12$ (UNSTABLE) $Y = -2$ gives $\frac{df}{dy} = -12$ (STABLE)

19 Stefan's Law of Radiation is $dy/dt = K(M^4 - y^4)$. It is unusual to see fourth powers. Find all real steady states and their stability. Starting from $y(0) = M/2$, sketch a graph of $y(t)$.

Solution $f(Y) = K(M^4 - Y^4)$ equals 0 at $Y = M$ and $Y = -M$ (also $Y = \pm iM$).

$\frac{df}{dy} = -4KY^3 = -4KM^3$($Y = M$ is STABLE) $\frac{df}{dy} = 4KM^3$($Y = -M$ is UNSTABLE)

The graph starting at $y(0) = M/2$ must go upwards to approach $y(\infty) = M$.

20 $dy/dt = ay - y^3$ has how many steady states Y for $a < 0$ and then $a > 0$? Graph those values $Y(a)$ to see a *pitchfork bifurcation*—new steady states suddenly appear as a passes zero. The graph of $Y(a)$ looks like a pitchfork.

Solution $f(Y) = aY - Y^3 = Y(a - Y^2)$ has 3 steady states $Y = 0, \sqrt{a}, -\sqrt{a}$.

$\frac{df}{dy} = a - 3y^2$ equals a at $Y = 0$, $\frac{df}{dy} = -2a$ at $Y = \sqrt{a}$ and $Y = -\sqrt{a}$.

Then $Y = 0$ is UNSTABLE and $Y = \pm\sqrt{a}$ are STABLE.

21 (Recommended) The equation $dy/dt = \sin y$ has **infinitely many steady states**. What are they and which ones are stable? Draw the stability line to show whether $y(t)$ increases or decreases when $y(0)$ is between two of the steady states.

Solution $f(Y) = \sin Y$ is zero at every steady state $Y = n\pi$ $(0, \pi, -\pi, 2\pi, -2\pi, \ldots)$

$$\begin{aligned}\frac{df}{dy} &= \cos Y = 1 \text{ (UNSTABLE for } Y = 0, 2\pi, -2\pi, 4\pi, \ldots)\\ &= \cos Y = -1 \text{ (STABLE for } Y = \pi, -\pi, 3\pi, -3\pi, \ldots)\end{aligned}$$

Stability line $\quad \longleftarrow \underset{-2\pi}{|} \longrightarrow \underset{-\pi}{|} \longleftarrow \underset{0}{|} \longrightarrow \underset{\pi}{|} \longleftarrow \underset{2\pi}{|} \longrightarrow$

22 Change Problem 21 to $dy/dt = (\sin\, y)^2$. The steady states are the same, but now the derivative of $f(y) = (\sin\, y)^2$ is zero at all those states (because $\sin\, y$ is zero). What will the solution actually do if $y(0)$ is between two steady states ?

Solution $f(y) = (\sin y)^2$ has $\frac{\delta f}{\delta y} = 2 \sin y \cos y = \sin 2y$.

Now $\frac{df}{dy} = 0$ at ALL THE STEADY STATES $Y = n\pi$.

Since $\frac{dy}{dt} = (\sin y)^2$ is always positive, the solution $y(t)$ will always increase toward the next larger steady state.

We have an infinite stack of $S-$curves.

23 (*Research project*) Find actual data on the US population in the years 1950, 1980, and 2010. What values of a, b, d in the solution formula (7) will fit these values ? Is the formula accurate at 2000, and what population does it predict for 2020 and 2100 ?

You could reset $t = 0$ to the year 1950 and rescale time so that $t = 3$ is 1980.

Solution Resetting time gives $T = c(t - 1950)$. Rescaling gives $c(1980 - 1950) = 3$ so $c = \frac{1}{10}$. Then a, b, d depend on your data.

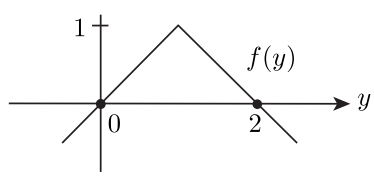
The graphs from $t = 1950$ to 1980 will show $T = \frac{1}{10}(t - 1950)$ from $T = 0$ to 3.

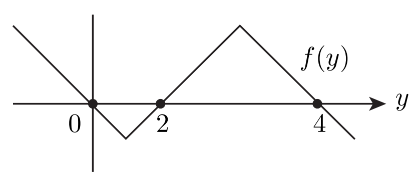
24 If $dy/dt = f(y)$, what is the limit $y(\infty)$ starting from each point $y(0)$?

Solution

$$\frac{dy}{dt} = \begin{cases} y & \text{for } y \le 1 \\ 2 - y & \text{for } y \ge 1 \end{cases} \quad \text{has fixed points } Y = \mathbf{0} \text{ and } \mathbf{2}$$

Slope $\frac{df}{dy} = 1$ at $Y = 0$ (UNSTABLE). Slope $\frac{df}{dy} = -1$ at $Y = 2$ (STABLE), $\boldsymbol{y(\infty) = 2}$.





Fixed points $Y = \mathbf{0}, \mathbf{2}, \mathbf{4}$. Slopes $\frac{df}{dy} = -1, 1, -1$.

$0, 2, 4 =$ STABLE, UNSTABLE, STABLE $\quad y(\infty) = 0$ if $y(0) < 2$ and $y(\infty) = 4$ if $y(0) > 2$.

25 (a) Draw a function $f(y)$ so that $y(t)$ approaches $y(\infty) = 3$ from every $y(0)$.

Solution The right side $f(y)$ must be zero only at $Y = 3$ which is STABLE.

Example: $\frac{dy}{dt} = f(y) = \boldsymbol{3 - y}$ has solutions $y = 3 + Ce^{-t}$.

(b) Draw $f(y)$ so that $y(\infty) = 4$ if $y(0) > 0$ and $y(\infty) = -2$ if $y(0) < 0$.

Solution This requires $Y = 4, -2$ to be stable and $Y = 0$ to be unstable.

Example: $\frac{dy}{dt} = f(y) = \boldsymbol{-y(y - 4)(y + 2)}$ Notice $\frac{df}{dy} = 8$ at $Y = 0$.

26 Which exponents n in $dy/dt = y^n$ produce blowup $y(T) = \infty$ in a finite time ? You could separate the equation into $dy/y^n = dt$ and integrate from $y(0) = 1$.

Solution $\displaystyle\int \frac{dy}{y^n} = \int dt$ gives $\displaystyle\frac{y^{1-n}}{1-n} = t + C$. The right side is zero at a finite time $t = -C$. Then y blows up at that time **if $\boldsymbol{n > 1}$**.

If $n = 1$ the integrals give $\ln y = t + C$ and $y = e^{t+C}$: NO BLOWUP in finite time.

27 Find the steady states of $dy/dt = y^2 - y^4$ and decide whether they are stable, unstable, or one-sided stable. Draw a stability line to show the final value $y(\infty)$ from each initial value $y(0)$.

Solution $f(y) = y^2 - y^4 = 0$ at $Y = 0, 1, -1$

$$\frac{df}{dy} = 2y - 4y^3 = \begin{array}{rlrl} 0 & \text{at } Y = & 0 & \text{(Double root of } f\text{)} \\ -2 & \text{at } Y = & 1 & \text{(STABLE)} \\ 2 & \text{at } Y = & -1 & \text{(UNSTABLE)} \end{array}$$

Since $Y = -1$ is unstable, $y(t)$ must go toward $Y = 0$ if $-1 < y(0) < 0$.

Since $Y = 1$ is stable, $y(t)$ must go toward $Y = 1$ if $0 < y(0) < 1$.

$$\begin{array}{cccccccc} < & | & > & | & > & | & < & \to Y \\ & -1 & & 0 & & 1 & & \end{array}$$

28 For an autonomous equation $y' = f(y)$, why is it impossible for $y(t)$ to be increasing at one time t_1 and decreasing at another time t_2 ?

Solution Reason: The stability line shows a movement of y **in one direction**, away from one (unstable) steady state Y and toward another (stable) steady state. "One direction" means that $y(t)$ is steadily increasing or steadily decreasing.

Problem Set 1.8, page 69

1 Finally we can solve the example $dy/dt = y^2$ in Section 1.1 of this book.

Start from $y(0) = 1$. Then $\displaystyle\int_1^y \frac{dy}{y^2} = \int_0^t dt$. Notice the limits on y and t. Find $y(t)$.

Solution With those limits, integration gives $-\frac{1}{y} + 1 = t$. Then $\frac{1}{y} = 1 - t$ and $y(t) = \boldsymbol{\frac{1}{1-t}}$.

2 Start the same equation $dy/dt = y^2$ from any value $y(0)$. At what time t does the solution blow up ? For which starting values $y(0)$ does it never blow up ?

Solution $$-\frac{1}{y} + \frac{1}{y(0)} = t \text{ gives } \frac{1}{y} = \frac{1}{y(0)} - t \text{ and } \boldsymbol{y = \frac{y(0)}{1 - ty(0)}}.$$

If $y(0)$ is negative, then $1 - ty(0)$ never touches zero for $t > 0$: No blowup.

3 Solve $dy/dt = a(t)y$ as a separable equation starting from $y(0) = 1$, by choosing $f(y) = 1/y$. This equation gave the growth factor $G(0, t)$ in Section 1.6.

Solution
$$\int_{y(0)}^{y} \frac{dy}{y} = \int_0^t a(t)dt \text{ gives } \ln y(t) - \ln y(0) = \int_0^t a(t)dt$$

$$y(t) = y(0)\exp\left(\int_0^t a(t)dt\right) = \boldsymbol{G(0,t)\,y(0)}$$

4 Solve these separable equations starting from $y(0) = 0$:

(a) $\dfrac{dy}{dt} = ty$ (b) $\dfrac{dy}{dt} = t^m\,y^n$

Solution (a) $\displaystyle\int_{y(0)}^{y} \frac{dy}{y} = \int_0^t t\,dt$ and $\ln y - \ln y(0) = t^2/2$: Then $y(t) = y(0)\exp(t^2/2)$.

(b) $\frac{dy}{dt} = t^m y^n$ has $\displaystyle\int \frac{dy}{y^n} = \int t^m\,dt$ and $\frac{y^{1-n}}{1-n} = \frac{t^{m+1}}{m+1}$. Then $y = \left(\frac{1-n}{m+1}t^{m+1}\right)^{1/(1-n)}$ for $n \neq 1$.

5 Solve $\dfrac{dy}{dt} = a(t)y^2 = \dfrac{a(t)}{1/y^2}$ as a separable equation starting from $y(0) = 1$.

Solution

$$\frac{dy}{dt} = a(t)y^2$$

$$\int_1^y \frac{du}{u^2} = \int_0^t a(x)\,dx \quad (u \text{ and } x \text{ are just integration variables})$$

$$-\frac{1}{y} + 1 = \int_0^t a(x)\,dx \quad \text{gives} \quad y = \frac{1}{1 - \displaystyle\int_0^t a(x)\,dx}$$

6 The equation $\dfrac{dy}{dt} = y + t$ is not separable or exact. But it is linear and $y =$ _____.

Solution We solve the equation by taking advantage of its linearity:

Given $a = 1$, the growth factor is e^t. The source term is t. Therefore using equation (14) gives:

$$y(t) = e^t y(0) + \int_0^t e^{t-s}s\,ds = e^t y(0) - t + e^t - 1.$$

$$\text{Check: } dy/dt = e^t y(0) - 1 + e^t \text{ does equal } y + t.$$

7 The equation $\dfrac{dy}{dt} = \dfrac{y}{t}$ has the solution $y = At$ for every constant A. Find this solution by separating $f = 1/y$ from $g = 1/t$. Then integrate $dy/y = dt/t$. Where does the constant A come from?

Solution We use separation of variables to find the constant A

$$\begin{aligned}
\frac{dy}{y} &= \frac{dt}{t} \\
\int_{y(1)}^{t} \frac{du}{u} &= \int_{1}^{t} \frac{dx}{x} \\
\ln(y) - \ln(y(1)) &= \ln t \\
\frac{y}{y(\mathbf{1})} &= t \\
\boldsymbol{y = y(1)\, t}
\end{aligned}$$

Therefore we find that the constant A is equal to $y(1)$, the initial value.

8 For which number A is $\dfrac{dy}{dt} = \dfrac{ct - ay}{At + by}$ an exact equation ? For this A, solve the equation by finding a suitable function $F(y,t) + C(t)$.

Solution $f(y,t) = At + by$ and $g(y,t) = ct - ay$

The equation is exact if : $\frac{\partial f}{\partial t} = -\frac{\partial g}{\partial y}$ and $\boldsymbol{A = a}$.

We follow the three solution steps for exact equations.

1 Integrate f with respect to y :

$$\int f(y,t)\, dy = \int (At + by)\, dy = Aty + \frac{1}{2}by^2 = F(y,t)$$

2 Choose $C(t)$ so that $\frac{\partial}{\partial t}(F(y,t) + C(t)) = -g(y,t)$

$$\begin{aligned}
\frac{\partial}{\partial t}(Aty + \frac{1}{2}by^2 + C(t)) &= Ay + C'(t) = -ct + ay \\
C'(t) = -ct \;\text{ and }\; C(t) &= -\frac{1}{2}ct^2
\end{aligned}$$

3 We therefore have that :

$$\begin{aligned}
\frac{dy}{dt} = \frac{g(y,t)}{f(y,t)} \;\text{ is solved by }\; F(y,t) + C(t) &= \;\text{ constant} \\
Aty + \frac{1}{2}by^2 - \frac{1}{2}ct^2 &= \;\text{ constant}
\end{aligned}$$

9 Find a function $y(t)$ different from $y = t$ that has $dy/dt = y^2/t^2$.

Solution Using separation of variables :

$$\begin{aligned}
dy/dt &= y^2/t^2 \\
dy/y^2 &= dt/t^2 \\
\int_{y(t_0)}^{y} \frac{du}{u^2} &= \int_{t_0}^{t} \frac{dx}{x^2} \\
-\tfrac{1}{y(t)} + \tfrac{1}{y(t_0)} &= -\tfrac{1}{t} + \tfrac{1}{t_0}
\end{aligned}$$

$t_0 = 1$ and $y(t_0) = 2$ give $-\frac{1}{y(t)} + \frac{1}{2} = -\frac{1}{t} + 1$ and $y(t) = \left(\frac{1}{t} - \frac{1}{2}\right)^{-1} = \boldsymbol{\frac{2t}{2-t}}$

10 These equations are separable after factoring the right hand sides :

$$\text{Solve} \quad \frac{dy}{dt} = e^{y+t} \quad \text{and} \quad \frac{dy}{dt} = yt + y + t + 1.$$

Solution (a) $\dfrac{dy}{dt} = e^y e^t$ and $\displaystyle\int_{y_0}^{y} e^{-y} dy = \int_{t_0}^{t} e^t dt$

$$\begin{aligned} -e^{-y} + e^{-y_0} &= e^t - e^{t_0} \\ e^{-y} &= e^{-y_0} - e^t + e^{t_0} \\ y &= -\ln\left[e^{-y_0} - e^t + e^{t_0}\right] \end{aligned}$$

(b) $dy/dt = (y+1)(t+1)$

$$\int_{y_0}^{y} \frac{dy}{y+1} = \int_{t_0}^{t} (t+1)\, dt$$

$$\ln(y+1) - \ln(y_0+1) = \frac{1}{2}(t^2 - t_0^2) + (t - t_0) = G$$

$$y + 1 = (y_0 + 1)\, e^G$$

11 These equations are linear and separable : Solve $\dfrac{dy}{dt} = (y+4)\cos t$ and $\dfrac{dy}{dt} = ye^t$.

Solution (a) $\displaystyle\int_{y_0}^{y} \frac{dy}{y+4} = \int_{t_0}^{t} \cos t\, dt$

$\ln(y+4) - \ln(y_0+4) = \sin t - \sin t_0$

$y + 4 = (y_0 + 4)\exp(\sin t - \sin t_0)$

(b) $\displaystyle\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} e^t\, dt$

$\ln y - \ln y_0 = e^t - e^{t_0}$

$y = y_0 \exp(e^t - e^{t_0})$

12 Solve these three separable equations starting from $y(0) = 1$:

Solution (a) $\dfrac{dy}{dt} = -4ty$ has $\displaystyle\int_{1}^{y} \frac{dy}{y} = \int_{0}^{t} -4t\, dt$

$$\ln y = -2t^2 \quad \text{and} \quad y = \exp(-2t^2)$$

(b) $\dfrac{dy}{dt} = ty^3$ has $\displaystyle\int_{1}^{y} \frac{dy}{y^3} = \int_{0}^{t} t\, dt$ and $-\dfrac{1}{2y^2} + \dfrac{1}{2y_0^2} = \dfrac{1}{2}t^2$

$$\frac{1}{y^2} = \frac{1}{y_0^2} - t^2$$
$$y = \left(\frac{1}{y_0^2} - t^2\right)^{-1/2} = y_0\left(1 - t^2 y_0^2\right)^{-1/2}$$

(c) $(1+t)\dfrac{dy}{dt} = 4y$ has $\displaystyle\int_1^y \frac{dy}{y} = \int_0^t \frac{4\,dt}{1+t}$

$$\ln y = 4\ln(1+t) - 4\ln(1) = 4\ln(1+t)$$
$$y = \boldsymbol{(1+t)^4}$$

Check $(1+t)\frac{dy}{dt} = 4(1+t)(1+t)^3 = 4y$

Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$ and solve Problems 13-14.

13 **Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$.**

Solution (a) $g = -3t^2 - 2y^2$ has $\partial g/\partial y = -4y$

$f = 4ty + by^2$ has $-\partial f/\partial y = -4y$: EXACT

Step 1 : $\displaystyle\int f\,dy = \int \left(4ty + 6y^2\right)\,dy = 2ty^2 + 2y^3 + C(t)$

Step 2 : $\frac{\partial}{\partial t}\left(2ty^2 + 2y^3 + C(t)\right) = 2y^2 + C'(t)$.

This equals $-g$ when $C'(t) = 3t^2$ and $C(t) = t^3$.

Step 3 : Solution $\boldsymbol{2ty^2 + 2y^3 + t^3 =}$ constant

Solution (b) $g = -1 - ye^{ty}$ has $\partial g/\partial y = -yte^{ty} - e^{ty}$

$f = 2y + te^{ty}$ has $-\partial f/\partial t = -yte^{ty} - e^{ty}$: EXACT

Step 1 : $\displaystyle\int f\,dy = \int \left(2y + te^{ty}\right)\,dy = y^2 + e^{ty} + C(t) = F(y,t)$

Step 2 : $\frac{\partial}{\partial t}\left(y^2 + e^{ty} + C(t)\right) = ye^{ty} + C'(t) = -g$ where $C'(t) = 1$

Step 3 : $C'(t) = 1$ gives $C(t) = t$ and the solution is

$F(y,t) + C(t) = \boldsymbol{-yte^{ty} - e^{ty} + t}$ = constant

14 **Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$.**

Solution (a) $g = 4t - y$ and $f = t - 6y$ have $\frac{\partial g}{\partial y} = -1 = \frac{\partial f}{\partial t}$: EXACT

Step 1 : $\displaystyle\int f\,dy = ty - 3y^2 + C(t)$

Step 2 : $\frac{\partial}{\partial t}\left(ty - 3y^2 + C(t)\right) = y + C'(t) = -g = y - 4t$ when $C(t) = -2t^2$

Step 3 : Solution $\boldsymbol{ty - 3y^2 - 2t^2 =}$ **constant**

Solution (b) $g = -3t^2 - 2y^2$ and $f = 4ty + 6y^2$ have $\frac{\partial g}{\partial y} = -4y = -\frac{\partial f}{\partial t}$: EXACT

Step 1 : $\displaystyle\int f\,dy = \int \left(4ty + 6y^2\right)dy = 2ty^2 + 2y^3 + C(t)$

Step 2 : $\frac{\partial}{\partial t}\left(2ty^2+2y^3+C(t)\right)=2y^2+C'(t)=-g=3t^2+2y^2$ when $C'=3t^2$ and $C=t^3$

Step 3 : Solution $\boldsymbol{2ty^2+2y^3+t^3=}$ **constant**

15 Show that $\dfrac{dy}{dt}=-\dfrac{y^2}{2ty}$ is exact but the same equation $\dfrac{dy}{dt}=-\dfrac{y}{2t}$ is not exact. Solve both equations. (This problem suggests that many equations become exact when multiplied by an integrating factor.)

Solution $g=-y^2$ and $f=2ty$ have $\frac{\partial g}{\partial y}=-2y=-\frac{\partial f}{\partial t}$: EXACT

$g=-y$ and $f=2t$ have $\frac{\partial g}{\partial y}$ NOT EQUAL TO $-\frac{\partial f}{\partial t}$

Solve the second form which is SEPARABLE

$$\int\frac{dy}{y}=\int-\frac{dt}{2t}\ \text{ gives }\ \ln y=-\frac{1}{2}\ln t+C$$

Then $y=e^Ct^{-1/2}$ is the same as $y=ct^{-1/2}$.

The same solution must come from Steps 1, 2, 3 using the exact form.

16 Exactness is really the condition to solve two equations with the same function $H(t,y)$:

$$\frac{\partial H}{\partial y}=f(t,y)\ \text{ and }\ \frac{\partial H}{\partial t}=-g(t,y)\ \text{ can be solved if }\ \frac{\partial f}{\partial t}=-\frac{\partial g}{\partial y}.$$

Take the t derivative of $\partial H/\partial y$ and the y derivative of $\partial H/\partial t$ to show that exactness is *necessary*. It is also *sufficient* to guarantee that a solution H will exist.

Solution The point is to see the underlying idea of exactness.

$$\text{If }\ \frac{\partial H}{\partial y}=f(t,y)\quad\text{then}\quad\frac{\partial^2H}{\partial t\,\partial y}=\frac{\partial f}{\partial t}$$

$$\text{If }\ \frac{\partial H}{\partial t}=-g(t,y)\quad\text{then}\quad\frac{\partial^2H}{\partial y\,\partial t}=-\frac{\partial g}{\partial y}$$

The cross derivatives of H are always equal. **IF** a function H solves both equations then $\frac{\partial f}{\partial t}$ *must equal* $-\frac{\partial g}{\partial y}$. So behind every exact equation is a function H : exactness is a necessary and also sufficient to find H with $\partial H/\partial y=f$ and $\partial H/\partial t=-g$.

17 The linear equation $\dfrac{dy}{dt}=aty+q$ is not exact or separable. Multiply by the integrating factor $e^{-\int at\,dt}$ and solve the equation starting from $y(0)$.

Solution This problem just recalls the idea of an integrating factor :

For $\dfrac{dy}{dt}=aty+q$ the factor is $P=\exp\left(-\displaystyle\int at\,dt\right)=\exp\left(-\dfrac{1}{2}at^2\right)$.

Then $P\left(\frac{dy}{dt}-aty\right)$ agrees with $(Py)'=P\frac{dy}{dt}+\frac{dP}{dt}y$

So the original equation multiplied by P is $\frac{d}{dt}(Py)=Pq$.

Integrate both sides $P(t)y(t)-P(0)y(0)=\displaystyle\int_0^t P(t)q\,dt$. Divide by $P(t)$ to find $y(t)$.

Second order equations $F(t, y, y', y'') = 0$ involve the second derivative y''. This reduces to a first order equation for y' (not y) in two important cases:

I. When y is missing in F, set $y' = v$ and $y'' = v'$. Then $\boldsymbol{F(t, v, v') = 0}$.

II. When t is missing in F, set $y'' = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$. Then $\boldsymbol{F\left(y, v, v\dfrac{dv}{dy}\right) = 0}$.

See the website for **reduction of order** when one solution $y(t)$ is known.

18 (y is missing) Solve these differential equations for $v = y'$ with $v(0) = 1$. Then solve for y with $y(0) = 0$.

Solution (a) $y'' + y' = 0$. Set $y' = v$. Then $\boldsymbol{v' + v = 0}$ gives $v(t) = Ce^{-t}$.

Now solve $y' = v = Ce^{-t}$ to find $y = \boldsymbol{-Ce^{-t} + D}$.

Solution (b) $2ty'' - y' = 0$. Set $y' = v$. Then $\boldsymbol{2tv' - v = 0}$ is solved by

$\displaystyle\int \frac{dv}{v} = \int \frac{dt}{2t}$ and $\ln v = \ln\sqrt{t} + C$ and $v = c\sqrt{t}$. Now solve $y' = v = c\sqrt{t}$ to find

$y = c_1 t^{3/2} + c_2$.

19 Both y and t are missing in $\boldsymbol{y'' = (y')^2}$. Set $v = y'$ and go two ways:

I. Solve $\dfrac{dv}{dt} = v^2$ to find $v = \dfrac{1}{1-t}$ as in Section 1.1.

Then solve $\dfrac{dy}{dt} = v = \dfrac{1}{1-t}$ to find $y = -\dfrac{(1-t)^{-2}}{2} + \dfrac{1}{2}$ with $y(0) = 0$.

II. Solve $v\dfrac{dv}{dy} = v^2$ or $\dfrac{dv}{dy} = v$ to find $v = e^y$.

Then $\dfrac{dy}{dt} = v(y) = e^y$ gives $\displaystyle\int e^{-y}\,dy = \int dt$ satisfying $v(0) = 1, y(0) = 0$

and $-e^{-y} = t - 1$: not the same solution as part I (??)

20 An **autonomous equation** $\boldsymbol{y' = f(y)}$ has no terms that contain t (t is missing).

Explain why every autonomous equation is separable. A non-autonomous equation could be separable or not. For a linear equation we usually say LTI (**linear time-invariant**) when it is autonomous: coefficients are constant, not varying with t.

Solution Every autonomous equation separates into $\displaystyle\int \frac{dy}{f(y)} = \int dt$.

Linear equations can be $\frac{dy}{dt} = a(t)y$: Non-autonomous

LTI equations are $\frac{dy}{dt} = ay$ (linear and also a is time-invariant $\Rightarrow$ autonomous).

21 $my'' + ky = 0$ is a highly important LTI equation. Two solutions are $\cos\omega t$ and $\sin\omega t$ when $\omega^2 = k/m$. Solve differently by reducing to a first order equation for $y' = dy/dt = v$ with $y'' = v\,dv/dy$ as above:

$$mv\frac{dv}{dy} + ky = 0 \text{ integrates to } \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \text{ constant } E.$$

For a mass on a spring, kinetic energy $\frac{1}{2}mv^2$ plus potential energy $\frac{1}{2}ky^2$ is a constant energy E. What is E when $y = \cos\omega t$? What integral solves the separable $m(y')^2 = 2E - ky^2$? I would not solve the linear oscillation equation this way.

Solution With $y' = v$ and $y'' = v\frac{dv}{dy}$, the equation $my'' + ky = 0$ becomes

$mv\frac{dv}{dy} + ky = 0$. This is *nonlinear* but *separable*. Integrate $mv\,dv = -ky\,dy$ to get

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \text{ constant } E \text{ [Conservation of Energy].}$$

If $y = \cos(\omega t)$ then $v = y' = -\omega\sin(\omega t)$ and E is $\frac{1}{2}m\cos^2(\omega t) + \frac{1}{2}K\omega^2\sin^2(\omega t)$.

The separable equation $m(y')^2 = 2E - ky^2$ could be solved by $\left(\dfrac{m}{2E - Ky^2}\right)^{1/2} dy = dt$. The integral could lead to $\cos^{-1} y = \omega t$ and $y = \cos\omega t$.

22 $my'' + k\sin y = 0$ is the *nonlinear* oscillation equation: not so simple. Reduce to a first order equation as in Problem 21:

$$mv\frac{dv}{dy} + k\sin y = 0 \text{ integrates to } \frac{1}{2}mv^2 - k\cos y = \text{ constant } E.$$

With $v = dy/dt$ what impossible integral is needed for this first order separable equation? Actually that integral gives the period of a nonlinear pendulum—this integral is extremely important and well studied even if impossible.

Solution Take square roots in $\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = K\cos y + E$.

Then separate into $\left[\dfrac{m/2}{K\cos y + E}\right]^{1/2} dy = dt$.

An unpleasant integral but important for nonlinear oscillation. Chapter 1 is ending with an example that shows the reality of nonlinear differential equations: Numerical solutions possible, elementary formulas are often impossible.