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WHEN DOES THE POSITIVE SEMIDEFINITENESS CONSTRAINT HELP IN LIFTING PROCEDURES?

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We study the lift-and-project procedures of Lovász and Schrijver for 0-1 integer programming problems. We prove that the procedure using the positive semidefiniteness constraint is not better than the one without it, in the worst case. Various examples are considered. We also provide geometric conditions characterizing when the positive semidefiniteness constraint does not help.

1. Introduction. Lovász and Schrijver (1991) have proposed a very intriguing convex relaxation procedure for 0-1 integer programming problems. The procedure called N_+, to be defined shortly, when applied to a classical linear programming (LP) relaxation of the stable set problem (with only the edge and nonnegativity constraints) produces a relaxation for which many well-known inequalities are valid, including the odd-hole, odd-antihole, odd-wheel, clique, and even the orthonormal-representation inequalities of Grötschel et al. (1981). This implies that for many classes of graphs, including perfect (for which clique inequalities are sufficient) or t-perfect graphs (for which odd-hole inequalities are sufficient), one can solve the maximum stable set problem by using the N_+ procedure.

The N_+ procedure is a strengthening of another procedure, called N, also introduced by Lovász and Schrijver. The main difference between the two procedures is that N_+ involves a positive semidefinite constraint. When applied to a linear programming relaxation, N will produce another (stronger) LP relaxation while N_+ will produce a semidefinite relaxation. For the stable set problem, Lovász and Schrijver have shown that the relaxation produced by N is much weaker than the one derived from N_+.

In general, however, it is not clear in which situations the procedure N_+ is better or significantly better than N; especially when N and N_+ are applied iteratively. In this paper we try to shed some light on this question. We generalize certain properties derived by Lovász and Schrijver. We also identify certain situations in which N produces the same relaxation as N_+. Several examples are discussed throughout the paper, including one in which the number of iterations of the N_+ procedure needed to derive the convex hull of 0-1 points is equal to the dimension of the space, hence resolving a question left open by Lovász and Schrijver.

In the next section, we review the lift-and-project procedures and their basic properties. Section 3 includes upper bounds on the number of major iterations required by such procedures. Section 4 discusses techniques to prove lower bounds on the number of major iterations required. Sections 5 and 6 include geometric properties and characterizations of the convex relaxations produced by the procedures.

2. Lovász–Schrijver procedures N and N_+. First, we describe two lift-and-project procedures proposed by Lovász and Schrijver (1991) which produce tighter and tighter relaxations of the convex hull of 0-1 points in a convex set. In what follows, e_j is the jth

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unit vector and e is the vector of all ones. The sizes of e and e_j will be clear from the context. The cone generated by all 0-1 vectors $x \in \mathbb{R}^{d+1}$ with $x_0 = 1$ is called Q. Let $K \subset Q$ denote a convex cone; for example, K could be a polyhedral cone obtained from a polytope P in $[0, 1]^d$ via homogenization using a new variable x_0. That is, if

$$P = \{x \in \mathbb{R}^d : Ax \le b, 0 \le x \le e\},$$

then

$$K := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{d+1} : Ax \le x_0 b, 0 \le x \le x_0 e \right\}.$$

We are interested in determining (or approximating) K_I, the cone generated by all 0-1 vectors of K.

Let K^*, Q^* denote the dual cones of K and Q under the standard Euclidean inner product, e.g.,

$$K^* := \{s \in \mathbb{R}^{d+1} : x^T s \ge 0, \forall\, x \in K\}.$$

$\mathcal{S}^{d+1}$ denotes the space of $(d+1) \times (d+1)$ symmetric matrices and $\mathcal{S}_+^{d+1}$ denotes the cone of $(d+1) \times (d+1)$ symmetric, positive semidefinite matrices. For a matrix $A \in \mathcal{S}^{d+1}$, we denote its positive semidefiniteness by $A \succeq 0$. When we deal with the duals of convex cones in the space of $(d+1) \times (d+1)$ matrices (or in the subspace of the symmetric matrices), we always take the underlying inner product to be the trace inner product (or Frobenius inner product): $\langle A, B\rangle := \mathrm{Tr}(A^T B)$.

Let diag $: \mathcal{S}^{d+1} \to \mathbb{R}^{d+1}$ denote the linear operator which maps a symmetric matrix to its diagonal. Then its adjoint $\mathrm{diag}^* : \mathbb{R}^{d+1} \to \mathcal{S}^{d+1}$ is the linear operator $\mathrm{Diag}(\cdot)$ which maps a vector from $\mathbb{R}^{d+1}$ to the diagonal matrix in $\mathcal{S}^{d+1}$ whose (i, i)th component is the ith component of the original vector.

DEFINITION 2.1 (LOVÁSZ AND SCHRIJVER 1991). A $(d+1) \times (d+1)$ symmetric matrix, Y, with real entries is in $M(K)$ if

(i) $Ye_0 = \mathrm{diag}(Y)$, and
(ii) $u^T Y v \ge 0, \forall\, u \in Q^*, v \in K^*$.

Lovász and Schrijver note that Condition (ii) of the above definition is equivalent to $YQ^* \subseteq K$ (where $YQ^* = \{Yx : x \in Q^*\}$), or: (ii)$'$ $Ye_i \in K$ for all $i \in \{1, \ldots, d\}$ and $Y(e_0 - e_i) \in K$ for all $i \in \{1, \ldots, d\}$, since the extreme rays (after normalization) of the cone Q^* are given by $\mathrm{ext}(Q^*) = \{e_1, e_2, \ldots, e_d, (e_0 - e_1), (e_0 - e_2), \ldots, (e_0 - e_d)\}$.

DEFINITION 2.2 (LOVÁSZ AND SCHRIJVER 1991). $Y \in M_+(K)$ if $Y \in M(K)$ and Y is positive semidefinite.

Observe that if we take any $x \in K$ (not necessarily integral) and consider $Y = xx^T$, Y satisfies $Y \succeq 0$ and also (ii)$'$, but this specific Y satisfies (i) if and only if x is such that $x_i(x_0 - x_i) = 0$ for all i, i.e., x corresponds to a 0-1 vector.

Now we define the projections of these liftings M and M_+:

$$N(K) := \{\mathrm{diag}(Y) : Y \in M(K)\},$$

$$N_+(K) := \{\mathrm{diag}(Y) : Y \in M_+(K)\}.$$

The above argument regarding xx^T shows that $K_I \subseteq N_+(K) \subseteq N(K) \subseteq K$, the last inclusion following from the fact that $Y(e_0 - e_i) \in K$ and $Ye_i \in K$ imply that $x = Ye_0 \in K$.

If P is a polytope (or any convex set) in $[0, 1]^d$, then we simply write $N_+(P)$ to represent $\{x : \binom{1}{x} \in N_+(K)\}$ where K is the cone obtained via homogenization using the variable x_0, and similarly for $N(P)$. We also let $M(P) = M(K)$ and $M_+(P) = M_+(K)$.

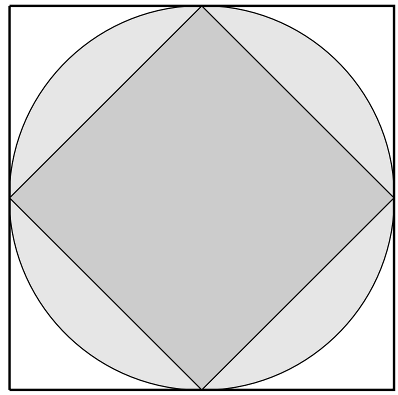

FIGURE 1. Two convex sets with the same $N_+(\cdot)$.

We should point out that the definition of M (or M_+) is such that $M(K)$ depends only on the sets $K \cap \{x : x_i = x_0\}$ and $K \cap \{x : x_i = 0\}$ for all i. In particular, we have:

LEMMA 2.1. *Let K and K' be such that $K \cap \{x : x_i = x_0\} = K' \cap \{x : x_i = x_0\}$ and $K \cap \{x : x_i = 0\} = K' \cap \{x : x_i = 0\}$ for all $i \in \{1, \ldots, d\}$. Then $M(K) = M(K')$ (and $N(K) = N(K')$) and $M_+(K) = M_+(K')$ (and $N_+(K) = N_+(K')$).*

For example, if $P = \{x \in \mathbb{R}^2 : \|x - 0.5e\|_2 \leq \frac{1}{2}\}$ and $P' = \{x \in \mathbb{R}^2 : \|x - 0.5e\|_1 \leq 0.5\}$ (see Figure 1), then $N(P) = N(P')$.

The definitions of M, N, M_+, and N_+ are invariant under various operations, including flipping coordinates $x_i \to (1 - x_i)$, for any subset of the indices $\{1, 2, \ldots, d\}$. More formally,

PROPOSITION 2.2 (LOVÁSZ AND SCHRIJVER 1991). *Let L be a linear transformation mapping Q onto itself. Then*

$$N(LK) = LN(K) \quad \text{and} \quad N_+(LK) = LN_+(K).$$

One crucial feature of the operators N and N_+ is that they can be iterated. The iterated operators $N^r(K)$ and $N_+^r(K)$ are defined as follows: $N^0(K) := K$, $N_+^0(K) := K$, $N^T(K) := N(N^{r-1}(K))$, and $N_+^r(K) := N_+(N_+^{r-1}(K))$ for all integers $r \geq 1$. Lovász and Schrijver (1991) show that even without the positive semidefiniteness constraints, d iterations are sufficient to get K_I:

THEOREM 2.3 (LOVÁSZ AND SCHRIJVER 1991).

$$K \supseteq N(K) \supseteq N^2(K) \supseteq \cdots \supseteq N^d(K) = K_I,$$

and

$$K \supseteq N_+(K) \supseteq N_+^2(K) \supseteq \cdots \supseteq N_+^d(K) = K_I.$$

Let $a^T x \leq \alpha x_0$ be a valid inequality for K_I. Then, the smallest nonnegative integer r such that $a^T x \leq \alpha x_0$ is valid for $N^r(K)$ is called *the N-rank of $a^T x \leq \alpha x_0$ relative to K*. The N_+-rank of $a^T x \leq \alpha x_0$ relative to K is defined similarly. The above theorem states that these ranks are at most d for any valid inequality. The N-rank (resp., N_+-rank) of a cone K is the smallest nonnegative integer r such that $N^r(K) = K_I$ (resp., $N_+^r(K) = K_I$).

Theorem 2.3 can also be proved using the results of Balas (1974); also see Balas et al. (1993). Our interest, in this paper, mostly lies in understanding the strength of N_+ in comparison to N. Consider the stable set polytope on a graph $G = (V, E)$ defined as the convex hull of incidence vectors of sets of nonadjacent vertices (known as stable sets). Let FRAC be the relaxation defined by the edge constraints ($x_i + x_j \leq 1$ for all edges $(i, j) \in E$) and the nonnegativity constraints ($x_i \geq 0$ for all $i \in V$). Then $N(FRAC)$ is exactly equal to the relaxation obtained by adding all odd-hole inequalities, saying that $\sum_{i\in C} x_i \leq (|C|-1)/2$ for any odd cycle C with no chords. However, many more complicated inequalities have small N_+-rank. Lovász and Schrijver (1991) proved that odd-hole, odd-antihole, odd-wheel, clique, and orthogonal-representation inequalities all have N_+-rank of at most 1, relative to $FRAC$. These results are proved using Lemma 3.5 of the next section, except for the orthogonality constraints. In contrast, the N-rank of a clique inequality, for example, is equal to $p-2$ where p is the size of the clique. Note that the separation problem for the class of clique inequalities is NP-hard (and so is the problem of optimizing over the clique inequalities; see Grötschel et al. 1981). N_+, however, leads to a polynomial-time separation algorithm for a broader class of inequalities. This, and more generally the importance of N and N_+, stems from the following result.

THEOREM 2.4 (LOVÁSZ AND SCHRIJVER 1991). *If we have a weak separation oracle for K, then we have a weak separation oracle for $N^r(K)$ and $N_+^r(K)$ for any fixed constant r.*

Together with the equivalence between (weak) optimization and (weak) separation (Grötschel et al. 1981), this implies, for example, that the stable set problem can be solved in polynomial time for any graph with bounded N_+-rank (Lovász and Schrijver 1991).

Next we study the upper bounds on N-rank and N_+-rank of inequalities and convex sets.

3. Upper bounds on the N- and N_+-rank. Lovász and Schrijver give some ways to derive an upper bound for the N-rank of an inequality. They show the following:

LEMMA 3.1 (LOVÁSZ AND SCHRIJVER 1991).

$$N_+(K) \subseteq N(K) \subseteq (K \cap \{x : x_i = 0\}) + (K \cap \{x : x_i = x_0\}), \qquad \text{for all } i \in \{1, 2, \ldots, d\}.$$

Lovász and Schrijver (1991) define an operator N_0 by:

$$N_0(K) := \bigcap_{i=1,\ldots,d} \{(K \cap \{x : x_i = 0\}) + (K \cap \{x : x_i = x_0\})\}.$$

Thus, $N(K) \subseteq N_0(K)$. The iterated operator N_0^r, N_0-rank of inequalities, polytopes, and convex cones are defined analogously to the corresponding definitions of N-rank and N_+-rank.

Lemma 3.1 shows that an inequality will be valid for $N(K)$ if it is valid for $K \cap \{x : x_i = 0\}$ and $K \cap \{x : x_i = x_0\}$ for some i. To iterate Lemma 3.1, we first need the following lemma. It is stated in terms of the faces of Q, which can be obtained by intersecting Q with hyperplanes of the form $\{x : x_i = 0\}$ or $\{x : x_i = x_0\}$. Similar insights for a procedure related to the N_0-procedure were discussed by Balas (1974).

LEMMA 3.2. *Let F be any face of Q. Then*

$$N(K \cap F) = N(K) \cap F.$$

Similarly for N_+ and N_0.

PROOF. "$\subseteq$" is clear from the definitions. For the converse, let $x \in N(K) \cap F$. This means that there exists a matrix $Y \in M(K)$ with $Ye_0 = x$. Since for every $i \in \{1, 2, \ldots, d\}$, $Ye_i \in K \subseteq Q$ and $Y(e_0 - e_i) \in K \subseteq Q$ and their sum $Ye_i + Y(e_0 - e_i) = Ye_0$ belongs to the face F of Q, we have that Ye_i and $Y(e_0 - e_i)$ must belong to F, by definition of a face. Thus, $Ye_i \in K \cap F$ and $Y(e_0 - e_i) \in K \cap F$ for all i implying that $Y \in M(K \cap F)$ and $x \in N(K \cap F)$. The proofs for N_+ and N_0 are analogous. □

Iterating Lemma 3.2, we get:

COROLLARY 3.3. *Let F be any face of Q. Then, for every $r \geq 0$,*

$$N^r(K \cap F) = N^r(K) \cap F.$$

Similarly for N_+ and N_0.

Repeatedly using Lemma 3.1 and Lemma 3.2 (or Corollary 3.3), we can derive a condition that an inequality must be valid for $N^r(K)$. This in particular proves Theorem 2.3. The following result is implicit in the proof of Theorem 1.4 of Lovász and Schrijver (1991).

THEOREM 3.4 (LOVÁSZ AND SCHRIJVER 1991). $N_+^r(K) \subseteq N^r(K) \subseteq N_0^r(K) \subseteq \widetilde{N}_0^r(K)$ *where*

$$\widetilde{N}_0^r(K) := \bigcap_{\{J\subseteq\{1,\ldots,d\}:|J|=r\}} \sum_{\{(J_0,J_1)\text{partitions of } J\}} (K \cap \{x : x_i = 0 \text{ for } i \in J_0 \text{ and } x_i = x_0 \text{ for } i \in J_1\}).$$

We should point out that even though $N_0(K) = \widetilde{N}_0(K)$ and $N_+^d(K) = N^d(K) = N_0^d(K) = \widetilde{N}_0^d(K)$, $N_0^r(K)$ is not necessarily equal to $\widetilde{N}_0^r(K)$, if $2 \leq r \leq (d-1)$. For example, for $K = \{x \in Q : x_1 + x_2 + x_3 \leq 1.5x_0\}$, one can show that $(1, 0.5, 0.5, 0.5) \in (\widetilde{N}_0^2(K) \setminus N_0^2(K))$.

For $N_+(K)$, Lovász and Schrijver (1991) give a different condition for the validity of an inequality. In the statement of the next lemma, we can assume, without loss of generality, that $a \geq 0$ (by flipping coordinates if necessary, as shown in Proposition 2.2).

LEMMA 3.5 (LOVÁSZ AND SCHRIJVER 1991). *Let $a \geq 0$. If $a^T x \leq \alpha x_0$ is valid for $(K \cap \{x : x_i = x_0\})$ for all i such that $a_i > 0$, then $a^T x \leq \alpha x_0$ is valid for $N_+(K)$.*

As mentioned previously, the result that clique, odd-hole, odd-antihole, and odd-wheel inequalities for the stable set problem have N_+-rank of 1 follows from the above lemma. For the stable set problem (as for many combinatorial optimization problems), there exist several important constructions to derive facet-defining valid inequalities from other facet-defining inequalities. The simplest is *cloning* a clique at a vertex v, which consists of replacing the vertex by a clique, replacing all the edges incident to v by corresponding edges incident to all clique vertices, and substituting in the inequality the variable for v by the sum of the variables of the clique vertices. It can easily be shown that the resulting inequality is valid and facet-defining if the original inequality was a nontrivial (i.e., different from the nonnegativity constraints) facet-defining inequality. In general, it is not clear how cloning influences the N_+-rank of an inequality. However, if we perform cloning at the center vertex of an odd-wheel inequality, Lemma 3.5 implies that the N_+-rank still remains equal to 1. If we perform cloning at one or several vertices of an odd-wheel, odd-hole or odd-antihole inequality, Lemma 3.5 implies that the N_+-rank is at most 2. Indeed, if we fix any variable (of the corresponding subgraph) to 1, the resulting inequality can be seen to be a linear combination of clique inequalities, and hence valid for $N_+(FRAC)$.

Lemma 3.5 can be extended to derive conditions under which the N_+-rank of an inequality is at most r.

THEOREM 3.6. *Let $a \geq 0$ and let $I_+ = \{i : a_i > 0\}$. If $a^T x \leq \alpha x_0$ is valid for $(K \cap \{x : x_i = x_0, \text{ for all } i \in I\})$ for all sets $I \subseteq I_+$ satisfying either of the following two conditions*:

(1) $|I| = r$,

(2) $|I| \leq (r-1)$ *and* $\sum_{i \in I} a_i > \alpha$,

then $a^T x \leq \alpha x_0$ is valid for $N_+^r(K)$.

Observe, however, that the result mentioned previously regarding cloning does not follow from Theorem 3.6.

PROOF. We proceed by induction on r. For $r = 1$, the result is Lemma 3.5.

Assume now that $r > 1$, that the theorem was proved for $(r-1)$ (and for every inequality and for every convex set K), and that the hypothesis is satisfied for the inequality $a^T x \leq \alpha x_0$ and r. From Corollary 3.3 and Lemma 3.5, we know that $a^T x \leq \alpha x_0$ is valid for $N_+^r(K) = N_+(N_+^{r-1}(K))$ if it is valid for $N_+^{r-1}(K) \cap \{x : x_i = x_0\} = N_+^{r-1}(K \cap \{x : x_i = x_0\})$ for all $i \in I_+$. This is equivalent to showing that $a^T x - a_i x_i \leq (\alpha - a_i)x_0$ is valid for $N_+^{r-1}(K \cap \{x : x_i = x_0\})$.

Now there are two cases. If $\alpha - a_i < 0$, then Condition 2 implies that $K \cap \{x : x_i = x_0\} = \varnothing$, and thus any inequality is valid for $N_+^{r-1}(K \cap \{x : x_i = x_0\}) = \varnothing$. On the other hand, if $\alpha - a_i \geq 0$, we can use induction to prove the result. Indeed, Conditions 1 and 2 for inequality $a^T x \leq \alpha x_0$ and r imply that Conditions 1 and 2 are satisfied for the inequality $a^T x - a_i x_i \leq (\alpha - a_i)x_0$ for $r - 1$. Thus, by the inductive hypothesis, $a^T x - a_i x_i \leq (\alpha - a_i)x_0$ is valid for $N_+^{r-1}(K \cap \{x : x_i = x_0\})$, proving the inductive statement. □

For the stable set problem, the above theorem implies that the N_+-rank of a graph is at most its stability number $\alpha(G)$, the cardinality of the largest stable set in G; this was proved in Corollary 2.19 of Lovász and Schrijver (1991). More generally, if we consider a polytope P for which P_I is only described by inequalities of the form $a^T x \leq \alpha x_0$ with $a \geq 0$ (i.e., it is lower comprehensive, see §5), then its N_+-rank is bounded above by the maximum number of variables that can be set to 1 in P to obtain a unique integral point of P_I (in which the other variables are thus set to 0). Similar, more complex, statements can be made if the polytope is not lower comprehensive.

3.1. Example 1: Matching polytope. Consider the complete undirected graph on the vertex set V; let E denote its edge set. Let

$$P := \{x \in \mathbb{R}^E : x(\delta(v)) \leq 1, \forall\, v \in V, 0 \leq x \leq e\}.$$

In the above, $\delta(v)$ is the set of edges in E that are incident to v; for $S \subseteq E$, $x(S)$ represents $\sum_{j \in S} x_j$. For $S \subseteq V$, let $E(S)$ refer to the set of edges with both endpoints in S. Then the *matching polytope* for the complete graph is

$$P_I := \operatorname{conv}\{P \cap \{0, 1\}^E\}.$$

Edmonds (1965) proved that

$$P_I = \left\{ x \in P : x(E(S)) \leq \frac{|S|-1}{2} \text{ for all } S \subseteq V \text{ such that } |S| \text{ is odd} \right\}.$$

The above inequalities are known as the blossom inequalities.

THEOREM 3.7 (STEPHEN AND TUNÇEL 1999). *The N_+-rank of the inequality*

$$x(E(S)) \leq \frac{|S|-1}{2}$$

with respect to P is $(|S|-1)/2$.

The fact that the N_+-rank is at most $(|S|-1)/2$ also follows directly from Theorem 3.6. Observe that since d is $|V|(|V|-1)/2$, we derive that the N_+-rank of P is equal to $(\sqrt{1+8d}-1)/4$ if $|V|$ is odd and $(\sqrt{1+8d}-3)/4$ if $|V|$ is even.

From Theorem 3.7, the N-rank of the blossom inequality on S is at least $(|S|-1)/2$. Furthermore, using Theorem 3.4 with J being the complement of a complete bipartite graph on $(|S|-1)/2$ and $(|S|+1)/2$ vertices on each side, we derive that the $\widetilde{N}_0$-rank of a blossom inequality is equal to $(|S|-1)^2/4$. This uses the fact that P is an integral polytope if and only if the underlying graph is bipartite. Thus, the N_0-rank and the N-rank are at most $(|S|-1)^2/4$. These bounds are to be compared with those derived from Corollary 2.8 of

Lovász and Schrijver (1991) (since a matching in a graph can be viewed as a stable set in its line graph). Their results imply a lower bound of $(|S|-2)$ and an upper bound of $\frac{1}{2}(|S|-1)^2-1$.

3.2. Example 2. Let d be an even positive integer. Consider

$$K := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{d+1} : x(S) \leq \frac{d}{2} x_0, \right.$$
$$\left. \text{for all } S \subset \{1, 2, \ldots, d\} \text{ such that } |S| = \frac{d}{2}+1, 0 \leq x \leq x_0 e \right\}.$$

Then

$$K_I = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d} x_i \leq \frac{d}{2} x_0, 0 \leq x \leq x_0 e \right\}.$$

Theorem 3.4 implies that the N-rank of $\sum_{i=1}^d x_i \leq (d/2)x_0$ is at most $(d-2)$, while Theorem 3.6 implies that the N_+-rank is at most $d/2$. These bounds are actually attained, and this is discussed in §4.2. We also show in that section that the positive semidefiniteness constraint does not help for many iterations.

4. Lower bounds on the N-rank and N_+-rank. In this section we provide lower bounds on the N-rank and N_+-rank. We also show a situation in which the positive semidefiniteness constraints do not help at all, and both the N-rank and the N_+-rank of a polytope are d.

We first provide a way to derive points in $N_+(P)$ in certain cases. For $x \in \mathbb{R}^d$ define

$$x_i^{(j)} := \begin{cases} x_i & \text{if } i \neq j; \\ 0 & \text{if } i = j. \end{cases}$$

So, $x^{(j)} = x - x_j e_j$. Throughout this section, let $K = \{\binom{\lambda}{\lambda x} : x \in P, \lambda \geq 0\}$.

THEOREM 4.1. *Let $\bar{x} \in P$ such that*

$$\bar{x}^{(j)} \textit{ and } (\bar{x}^{(j)} + e_j) \in P, \textit{ for all } j \in \{1, 2, \ldots, d\} \textit{ such that } 0 < \bar{x}_j < 1.$$

Then $\bar{x} \in N_+(P)$.

Simply stated, this result says that if we can replace any coordinate of x (strictly between 0 and 1) by 0 and 1 and remain in P, then $x \in N_+(P)$.

PROOF. We define

$$Y(x) := \begin{pmatrix} 1 \\ x \end{pmatrix} (1, x^T) + \operatorname{Diag} \begin{pmatrix} 0 \\ x_1 - x_1^2 \\ x_2 - x_2^2 \\ \vdots \\ x_d - x_d^2 \end{pmatrix}.$$

By definition, $Y(\bar{x}) \in \mathcal{S}^{d+1}$, $Y(\bar{x})e_0 = \operatorname{diag}(Y(\bar{x})) = \binom{1}{\bar{x}} \in K$. Moreover,

$$Y(\bar{x})e_j = \bar{x}_j \begin{pmatrix} 1 \\ \bar{x}^{(j)} + e_j \end{pmatrix}, \qquad \text{for all } j \in \{1, 2, \ldots, d\};$$

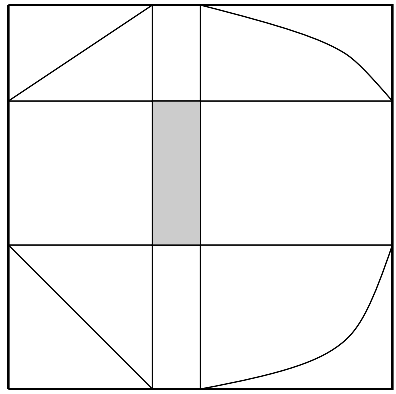

FIGURE 2. Convex set satisfying the condition of Corollary 4.2.

therefore, $Y(\bar{x})e_j \in K$ for all $j \in \{1, 2, \ldots, d\}$. Similarly,

$$Y(\bar{x})(e_0 - e_j) = (1 - \bar{x}_j)\begin{pmatrix} 1 \\ \bar{x}^{(j)} \end{pmatrix}, \qquad \text{for all } j \in \{1, 2, \ldots, d\};$$

therefore, $Y(\bar{x})(e_0 - e_j) \in K$ for all $j \in \{1, 2, \ldots, d\}$. Finally, since

$$\operatorname{Diag}\begin{pmatrix} 0 \\ x_1 - x_1^2 \\ x_2 - x_2^2 \\ \vdots \\ x_d - x_d^2 \end{pmatrix} \succeq 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ x \end{pmatrix}(1, x^T) \succeq 0,$$

for all $0 \le x \le e$, we have $Y(x) \succeq 0$. Therefore, $Y(\bar{x}) \in M_+(P)$ and $\bar{x} \in N_+(P)$ as desired. $\square$

As a corollary, we derive the following (see Figure 2 for an illustration).

COROLLARY 4.2. *Let P be such that $(P \cap \{x : x_j = 0\}) + e_j = P \cap \{x : x_j = 1\}$ for all $j \in \{1, \ldots, d\}$. Then*

$$N_+(P) = N(P) = N_0(P) = \bigcap_{j \in \{1, \ldots, d\}} \{x : x^{(j)} \in P\}.$$

PROOF. Let $C = \bigcap_{j \in \{1, \ldots, d\}} \{x : x^{(j)} \in P\}$. By Lemma 3.1, we know that $N_+(P) \subseteq N(P) \subseteq N_0(P) \subseteq C$. On the other hand, Theorem 4.1 shows that $C \subseteq N_+(P)$. $\square$

In the proof of Theorem 4.1, we constructed a $Y \in M_+(P)$ such that a certain $x \in P$ would also be in $N_+(P)$. The idea of the proof suggests a stronger technique to achieve such a goal. We define

$$Y(x) := \begin{pmatrix} 1 \\ x \end{pmatrix}(1, x^T) + \operatorname{Diag}\begin{pmatrix} 0 \\ x_1 - x_1^2 \\ \vdots \\ x_d - x_d^2 \end{pmatrix} + \begin{pmatrix} 0 & 0^T \\ 0 & B(x) \end{pmatrix},$$

where $B(x) \in \mathcal{S}^d$, $\operatorname{diag}(B) = 0$. Then clearly we have $Y(x) \in \mathcal{S}^{d+1}$, $Y(x)e_0 = \operatorname{diag}(Y(x))$. Moreover, using the Schur complement of $(Y(x))_{00}$ in $Y(x)$, we have

$$Y(x) \succeq 0 \quad \text{iff } B(x) + \operatorname{Diag}\begin{pmatrix} x_1 - x_1^2 \\ \vdots \\ x_d - x_d^2 \end{pmatrix} \succeq 0.$$

The latter can be assured in many simple ways, for example, by diagonal dominance: It suffices to choose B_{ij} such that

$$|B_{ij}| \le \min\left\{ \frac{x_i - x_i^2}{\text{\# of nonzeros in column } i}, \frac{x_j - x_j^2}{\text{\# of nonzeros in column } j} \right\}.$$

The entries of such a $B(x)$ will be further restricted by the condition $Y(x)e_i \in K$ for every $i \in \{1, 2, \ldots, d\}$ and $Y(x)(e_0 - e_i) \in K$ for every $i \in \{1, 2, \ldots, d\}$. If this condition is verified for some $B(x)$, then the above argument would imply $x \in N_+(P)$. In the case of Theorem 4.1, we utilized diagonal dominance; because of the special structure of P, we could choose $B(x) := 0$ and satisfy all the conditions for $x \in N_+(P)$.

4.1. Example 3: Infeasibility detection. We now give an example where both N and N_+ require d iterations, showing that Theorem 2.3 cannot be improved. This result was independently obtained by Cook and Dash (2001), who also show additional results regarding the rank of inequalities. Previously, the worst example known in terms of the number of repeated N_+ iterations needed to obtain K_I was the matching polytope results of Stephen and Tunçel, (1999), where the N_+-rank was of the order of $\sqrt{d}$.

Let

$$P(k) := \left\{ x \in \mathbb{R}^d : \left\| x - \frac{1}{2}e \right\|_1 \le \frac{k}{2} \right\}.$$

THEOREM 4.3. *For every $k \in \{1, 2, \ldots, d-1\}$, $N_+(P(k)) \supseteq P(k-1)$. Furthermore, $P(0) = \left\{\frac{1}{2}e\right\} \neq \varnothing$ while $P_I(d-1) = \varnothing$. Thus, the N_+ procedure requires d iterations to prove $P_I(d-1) = \varnothing$.*

PROOF. Follows from Corollary 4.2. (In fact, this corollary establishes that $N_+(P(k)) = P(k-1)$ for every $k \in \{1, 2, \ldots, d-1\}$.) $\square$

One interesting feature of the example above is that $P(d-1)$ can be described by 2^d inequalities and contains no integral point, but no inequality can be removed without creating an integral point. This is actually an extreme situation in this regard, as shown by the following result of Doignon (1973). Suppose we are given a set of m linear inequalities

$$a_i^T x \le b_i, \qquad \text{for all } i \in J,$$

where $x \in \mathbb{R}^d$ and $|J| \ge 2^d$. A theorem of Doignon (1973) implies that if this system does not contain any integer points then there is a subsystem (of this system) with at most 2^d inequalities which does not have an integer solution. Doignon's Theorem is an integer analog of Helly's Theorem.

4.2. Example 2, continued. In §3.2, we have shown that the N-rank and the N_+-rank of

$$K := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{d+1} : x(S) \le \frac{d}{2}x_0, \right.$$
$$\left. \text{for all } S \subset \{1, 2, \ldots, d\} \text{ such that } |S| = \frac{d}{2} + 1, 0 \le x \le x_0 e \right\},$$

are at most $(d-2)$ and $d/2$, respectively. Here we claim that these bounds are attained.

THEOREM 4.4. *The N-rank of $\sum_{i=1}^d x_i \le d/2$ relative to K is $(d-2)$. The N_+-rank of the same inequality relative to K is $d/2$. Moreover, for $r \le d/2 - \sqrt{d} + 3/2$, the optimum values of*

$$\max\{e^T x : x \in N^r(K)\} \quad \textit{and} \quad \max\{e^T x : x \in N_+^r(K)\}$$

are the same.

Our proof of the first statement of the theorem, saying that the N-rank is $(d-2)$, is lengthy and is not included here. The proof of the remainder of the theorem appears partly in this section and partly in the Appendix. The theorem indicates that the positive semidefiniteness constraint does not help for $(d/2 - o(d))$ iterations.

Unfortunately, neither Theorem 4.1 nor Corollary 4.2 is useful here. Instead, exploiting the symmetry (and convexity of $N(K)$ and $N_+(K)$), we will only consider points in $N^r(K)$ or $N_+^r(K)$ such that x_i takes only three possible values, 0, 1, and a constant α. Letting n_0 denote the number of x_i set to 0 and letting n_1 denote the number of x_i set to 1, we define $c(r, n_0, n_1)$ to be the largest common value α of the remaining $(d - n_0 - n_1)$ coordinates of x such that $x \in N^r(K)$. We define $c_+(r, n_0, n_1)$ similarly with respect to $N_+^r(K)$.

By symmetry, such a point x belongs to $N^r(K)$ (resp., to $N_+^r(K)$) if there exists a symmetric matrix $Y \in M^r(K)$ (resp., $Y \in M_+^r(K)$) of the form

$$Y(n_0, n_1; \alpha, \beta) := \begin{pmatrix} 1 & e^T & 0 & \alpha e^T \\ e & ee^T & 0 & \alpha ee^T \\ 0 & 0 & 0 & 0 \\ \alpha e & \alpha ee^T & 0 & (\alpha - \beta) I + \beta ee^T \end{pmatrix},$$

for some value β; here the columns of Y are partitioned in the way that the first column corresponds to the homogenizing variable x_0, the next n_1 columns correspond to those x_j that are set to one, the next n_0 columns correspond to those x_j set to zero, and the remaining $(d - n_0 - n_1)$ columns correspond to the remaining x_js (which are set to α).

For $r = 0$ and $n_1 \le d/2$, we see by plugging x into the description of K that

$$c(0, n_0, n_1) = c_+(0, n_0, n_1) = \begin{cases} \dfrac{d/2 - n_1}{d/2 + 1 - n_1} & \text{if } n_0 \le d/2 - 1, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

For $r > 0$, the condition that $Y \in M^r(K)$ is equivalent to $\beta/\alpha \le c(r-1, n_0, n_1+1)$ (corresponding to $Ye_i \in N^{r-1}(K)$) and $(\alpha - \beta)/(1-\alpha) \le c(r-1, n_0+1, n_1)$ (corresponding to $Y(e_0 - e_i) \in N^{r-1}(K)$). Eliminating β, we derive:

$$c(r, n_0, n_1) = \frac{c(r-1, n_0+1, n_1)}{1 - c(r-1, n_0, n_1+1) + c(r-1, n_0+1, n_1)}.$$

The condition that $Y \succeq 0$ reduces to (by taking a Schur complement) $(\alpha - \beta)I + (\beta - \alpha^2)ee^T \succeq 0$ (where the matrices have size $(d - n_0 - n_1) \times (d - n_0 - n_1)$), or $\alpha - \beta \ge 0$ and $\alpha - \beta + (d - n_0 - n_1)(\beta - \alpha^2) \ge 0$. This can be seen to imply that

$$c_+(r, n_0, n_1) = \min\left(\frac{c_+(r-1, n_0+1, n_1)}{1 - c_+(r-1, n_0, n_1+1) + c_+(r-1, n_0+1, n_1)}, \right. \\ \left. \frac{(d - n_0 - n_1 - 1)c_+(r-1, n_0, n_1+1) + 1}{d - n_0 - n_1} \right).$$

Observe that the N-rank (resp., the N_+-rank) of K is the smallest integer r such that $c(r, 0, 0) = \frac{1}{2}$ (resp., $c_+(r, 0, 0) = \frac{1}{2}$). Theorem 4.4 hence follows from the following proposition.

PROPOSITION 4.5.

(1) $c(d-3,0,0) = \begin{cases} \frac{1}{2} + \frac{1}{5d-6} & \textit{if } d \textit{ is even,} \\ \frac{1}{2} + \frac{1}{10d-20} & \textit{if } d \textit{ is odd,} \end{cases}$

(2) $c_+(d/2-1,0,0) > 0.5$,

(3) *for any* r, n_0, n_1 *such that* $r+n_0+n_1 \le d/2 - \sqrt{d} + 3/2$, *we have* $c(r,n_0,n_1) = c_+(r,n_0,n_1)$.

The proof of (1) is obtained by solving explicitly the recurrence for c; a sketch of the details is in the Appendix. The proof of the rest of the proposition is also given in the Appendix.

Theorem A.3 in the Appendix actually illustrates a peculiar behavior of the N_+ operator (as well as the N operator) on this example. In cutting-plane procedures, it is usual that the improvement due to the addition of a cutting-plane (or a batch of them) decreases as the algorithm progresses. However, Theorem A.3 shows that

$$\max\{e^T x : x \in N_+^r(K)\} = dc_+(r,0,0) > d\left(1 - \frac{1}{d/2+1-r}\right).$$

Hence, as illustrated on Figure 3 for $d = 500$, the improvement in the objective function value is negligible for many iterations and only toward the end increases considerably. We should point out, however, that the procedures N and N_+ are such that the number of "important" inequalities generated in each iteration could potentially increase tremendously in later iterations.

5. Additional properties. A nonempty convex set $P \subseteq \mathbb{R}^d_+$ is called *lower comprehensive* if for every $x \in P$, every $y \in \mathbb{R}^d_+$ such that $y \le x$ is also in P.

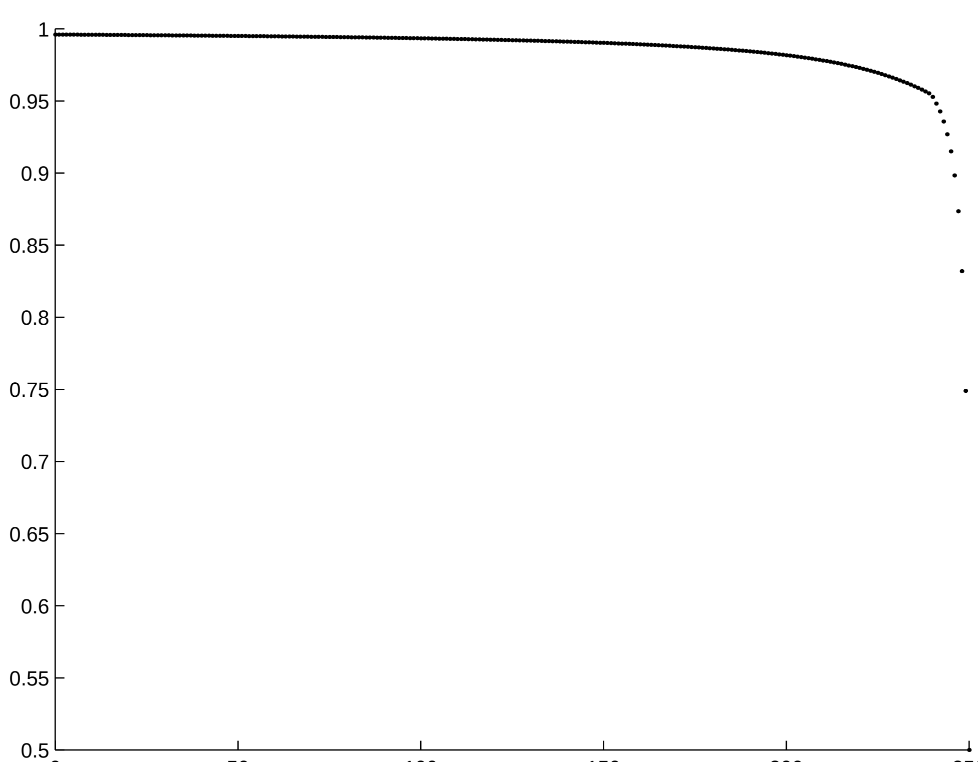

FIGURE 3. Plot of $c_+(r,0,0)$ for $d = 500$ as a function of r.

DEFINITION 5.1. Let $v \in \{0,1\}^d$. A convex set $P \subseteq [0,1]^d$ is said to be *a convex corner with respect to* v if there exists a linear transformation L of Q onto itself such that $L\binom{1}{v} = \binom{1}{0}$ and $\{x \in \mathbb{R}^d : \binom{1}{x} = L\binom{\lambda}{\lambda u}, \lambda > 0, u \in P\}$ is lower comprehensive.

THEOREM 5.1. *If P is a convex corner with respect to $v \in \{0,1\}^d$, then so are $N(P)$ and $N_+(P)$.*

PROOF. By Proposition 2.2 and the definitions, it suffices to prove that if P is lower comprehensive, then so are $N(P)$ and $N_+(P)$. Let P be lower comprehensive and $x \in N(P)$. It suffices to show that $(x - x_j e_j) \in N(P)$ for every j such that $x_j > 0$. Without loss of generality, suppose $j = 1$ and $x_1 > 0$. Then there exists $Y \in M(P)$ such that $Ye_0 = \binom{1}{x}$. Let

$$\overline{Y}_{pq} := \begin{cases} Y_{pq} & \text{if } p \neq 1 \text{ or } q \neq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then, using the fact that P is lower comprehensive, it is easy to see that $\overline{Y} \in M(P)$. Since the above argument applies to every j such that $x_j > 0$, and it can be applied recursively, we have proved that $N(P)$ is lower comprehensive.

We can prove that $N_+(P)$ is lower comprehensive by a very similar argument. We only have to note that if $Y \in M_+(P)$, then the corresponding $\overline{Y}$ constructed as above will be positive semidefinite (in addition to satisfying $\overline{Y}e_j \in K$ for every $j \in \{0, 1, 2, \ldots, d\}$ and $\overline{Y}(e_0 - e_j) \in K$ for every $j \in \{1, 2, \ldots, d\}$) since every principal minor of $\overline{Y}$ is a principal minor of Y, and Y is positive semidefinite. □

A similar fact in a less general form was observed independently by Cook and Dash (2001).

6. General conditions on the strength of the semidefinite constraint. In this section, we derive general conditions under which the positive semidefiniteness constraint is not useful. This can be expressed in several ways as

- $M(K) = M_+(K)$, or as
- $N(K) = N_+(K)$, or even as
- $\max\{c^T x : x \in N(K)\} = \max\{c^T x : x \in N_+(K)\}$ for some given c.

First, we rewrite Condition (ii) of Definition 2.1. Since Y is symmetric,

$$u^T Y v \geq 0, \quad \forall\, u \in Q^*, v \in K^* \iff u^T Y v + v^T Y u \geq 0, \quad \forall\, u \in Q^*, v \in K^*.$$

Using the fact that $u^T Y v + v^T Y u = \mathrm{Tr}(Y(uv^T + vu^T))$, we see that Condition (ii) is also equivalent to

(ii)″ $Y \in [T(K)]^*$,

where

$$\begin{aligned} T(K) &:= \mathrm{cone}\{uv^T + vu^T : u \in Q^*, v \in K^*\} \\ &= \mathrm{cone}\{uv^T + vu^T : u \in \mathrm{ext}(Q^*), v \in \mathrm{ext}(K^*)\}. \end{aligned}$$

Let us define

$$D := \{Y \in \mathcal{S}^{d+1} : \mathrm{diag}(Y) = Ye_0\}.$$

Note that the cone (more specifically, the subspace in this case) dual to D in the space $\mathcal{S}^{d+1}$ is the orthogonal complement of D.

$$D^* = D^{\perp} = \left\{ \sum_{i=1}^{d} \alpha_i (E_{ii} - E_{0i}) : \alpha \in \mathbb{R}^d \right\},$$

where $E_{ij} := e_i e_j^T + e_j e_i^T$. For the next theorem and its proof we need a bit more notation: $\mathcal{S}^d_{++}$ denotes the cone of $d \times d$, symmetric positive definite matrices, $\mathrm{cl}(\cdot)$ denotes the closure.

THEOREM 6.1. *Let $K \subseteq Q$ be a convex cone. Then we have*
(1) $M_+(K) = M(K)$ *if and only if* $T(K) + D^\perp \supseteq \mathcal{S}^{d+1}_{++}$.
(2) $N_+(K) = N(K)$ *if and only if for every* $s \in \mathbb{R}^{d+1}$, $\mathrm{Diag}(s) \in T(K) + D^\perp + \mathcal{S}^{d+1}_+$ *implies* $\mathrm{Diag}(s) \in \mathrm{cl}(T(K) + D^\perp)$.

PROOF. By definition of the sets $M(K), M_+(K)$, we have

$$M(K) = M_+(K) \Longleftrightarrow [T(K)]^* \cap D = [T(K)]^* \cap D \cap \mathcal{S}^{d+1}_+.$$

Since the inclusion $[T(K)]^* \cap D \supseteq [T(K)]^* \cap D \cap \mathcal{S}^{d+1}_+$ is clear, we have

$$M(K) = M_+(K) \Longleftrightarrow [T(K)]^* \cap D \subseteq \mathcal{S}^{d+1}_+.$$

Noting that

$$[T(K)]^* \cap D \subseteq \mathcal{S}^{d+1}_+ \Longleftrightarrow ([T(K)]^* \cap D)^* \supseteq \mathcal{S}^{d+1}_+$$

(we used the facts that $[T(K)]^*$ and D are closed convex cones, $\mathcal{S}^{d+1}_+$ is self dual under the trace inner product, in the space $\mathcal{S}^{d+1}$) and that

$$([T(K)]^* \cap D)^* = \mathrm{cl}(T(K) + D^*)$$

(we used Corollary 16.4.2 of Rockafellar 1970) we conclude

$$M(K) = M_+(K) \quad \text{if and only if } T(K) + D^\perp \supseteq \mathcal{S}^{d+1}_{++}.$$

This establishes Part (1). Now we turn to Part (2). Using definitions, we have

$$N(K) = N_+(K) \Longleftrightarrow \mathrm{diag}([T(K)]^* \cap D) \subseteq \mathrm{diag}([T(K)]^* \cap D \cap \mathcal{S}^{d+1}_+).$$

Now we use the fact that for any convex cone $\mathcal{K} \subseteq \mathcal{S}^{d+1}$, we have

$$[\mathrm{diag}(\mathcal{K})]^* = \{s \in \mathbb{R}^{d+1} : \mathrm{Diag}(s) \in \mathcal{K}^*\}. \tag{2}$$

Using Equation (2) and the ideas in the first part of the proof, we find

$$\begin{aligned} N(K) = N_+(K) \quad &\text{if and only if} \\ &\{s \in \mathbb{R}^{d+1} : \mathrm{Diag}(s) \in \mathrm{cl}(T(K) + D^\perp)\} \\ &\supseteq \{s \in \mathbb{R}^{d+1} : \mathrm{Diag}(s) \in \mathrm{cl}(T(K) + D^\perp + \mathcal{S}^{d+1}_+)\} \end{aligned}$$

if and only if for every $s \in \mathbb{R}^{d+1}$, $\mathrm{Diag}(s) \in T(K) + D^\perp + \mathcal{S}^{d+1}_+$ implies $\mathrm{Diag}(s) \in \mathrm{cl}(T(K) + D^\perp)$. $\square$

Note that if K is polyhedral, then in Part (1), $\mathcal{S}^{d+1}_{++}$ can be replaced by $\mathcal{S}^{d+1}_+$ and in Part (2), $\mathrm{cl}(T(K) + D^\perp)$ can be replaced by $(T(K) + D^\perp)$. This theorem completely characterizes when M and M_+ differ or are equal. To make the condition more easily tractable, we can give a more explicit description of $T(K) + D^\perp$. Define $F(K)$ to be the set of all $v = \binom{v_0}{\bar{v}} \in \mathbb{R}^{d+1}$ such that $-\bar{v}^T x \le v_0$ is a facet of P (or, more generally, for nonpolyhedral convex sets, $F(K)$ describes a set of valid inequalities exactly characterizing P). Note that $F(K)$ can be taken as the set of extreme rays of K^*. We arrive at the identity

$$\begin{aligned} T(K) + D^\perp = \mathrm{cone}\big\{&(e_i v^T + v e_i^T), i \in \{1, 2, \dots, d\}, v \in F(K); \\ &[(e_0 - e_i)v^T + v(e_0 - e_i)^T], i \in \{1, 2, \dots, d\}, v \in F(K); \\ &(E_{ii} - E_{0i}), i \in \{1, 2, \dots, d\}\big\}, \end{aligned}$$

where we have used the fact that $E_{0i} - E_{ii} \in T(K)$ since $e_i \in F(K)$. So, $M_+(K) = M(K)$ iff for every $x \in \mathbb{R}^{d+1}$, we can express xx^T as an element of the closure of the above cone $(T(K) + D^\perp)$.

Consider the clique on four vertices and the corresponding LP relaxation *FRAC* of the stable set problem (with the edge and nonnegativity constraints only). So, K is polyhedral. For this example,

$$Y := \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \in [T(K)]^* \cap D;$$

but clearly $Y \notin \mathcal{S}_+^{d+1}$. A proof of this is provided by the incidence vector of the clique inequality on the four vertices:

$$(e_0 - e_1 - e_2 - e_3 - e_4)^T Y (e_0 - e_1 - e_2 - e_3 - e_4) = -\frac{1}{3}.$$

This means, for $x := (e_0 - e_1 - e_2 - e_3 - e_4)$, xx^T is not in the convex cone $(T(K) + D^\perp)$.

Now, we relate these findings to the iterated operators $N^r(K)$ and $N_+^r(K)$.

Corollary 6.2. *If $(T(K) + D^\perp) \supseteq \mathcal{S}_{++}^{d+1}$, then $N_+^r(K) = N^r(K)$ for every $r \geq 0$.*

Proof. Trivial for $r = 0$. By Theorem 6.1, the assumption of the corollary implies $N_+(K) = N(K)$. By Theorem 2.3, $N(K) \subseteq K$. Thus,

$$(T(N(K)) + D^\perp) \supseteq (T(K) + D^\perp) \supseteq \mathcal{S}_{++}^{d+1}.$$

Now, applying Theorem 6.1 recursively, we obtain the desired result. $\square$

We should compare these results to Lemma 1.2 of Lovász and Schrijver (1991). Note that our result is also based on cone duality; we also characterize the dual cones of $N(K)$ and $N_+(K)$, but we only work in the space of symmetric matrices instead of the larger space of all matrices. As a result, the dependence of the characterization on the skew symmetric matrices is eliminated and our description is more explicit.

Our ideas in the geometric characterizations above are also applicable in comparing the weaker procedure N_0 to N. Recall

$$N_0(K) := \bigcap_{i=1,\ldots,d} \{(K \cap \{x : x_i = 0\}) + (K \cap \{x : x_i = x_0\})\}.$$

We define

$$M_0(K) := \{Y \in \mathbb{R}^{(d+1)\times(d+1)} : Ye_0 = Y^T e_0 = \operatorname{diag}(Y), u^T Y v \geq 0, \forall\, u \in Q^*, v \in K^*\};$$

the main difference with M is that Y is not necessarily symmetric. As is mentioned by Lovász and Schrijver (1991), we have

$$N_0(K) = \{Ye_0 : Y \in M_0(K)\}.$$

We further define

$$T_0(K) := \operatorname{cone}\{uv^T : u \in Q^*, v \in K^*\},$$

and

$$D_0 := \{Y \in \mathbb{R}^{(d+1)\times(d+1)} : Ye_0 = Y^T e_0 = \operatorname{diag}(Y)\}.$$

Then

$$Y \in M_0(K) \quad \text{iff } Y \in ([T_0(K)]^* \cap D_0),$$

where $[T_0(K)]^*$ is the dual of $T_0(K)$ in $\mathbb{R}^{(d+1)\times(d+1)}$ under the trace inner product.

THEOREM 6.3. *Suppose $K \subseteq Q$ is a polyhedral cone. Then we have*
(1) $M_0(K) = M(K)$ *iff* $(T_0(K) + D_0^\perp) \supseteq \{(e_i e_j^T - e_j e_i^T) : i, j \in \{1, 2, \ldots, d\}\}$.
(2) $N_0(K) = N(K)$ *if and only if for every* $s \in \mathbb{R}^{d+1}, \operatorname{Diag}(s) \in T_0(K) + D_0^\perp + \widetilde{\mathcal{S}}^{d+1}$ *implies* $\operatorname{Diag}(s) \in T_0(K) + D_0^\perp$.

PROOF. As we showed, $M_0(K) = [T_0(K)]^* \cap D_0$, and it is clear from the definitions that $M(K) = [T_0(K)]^* \cap D_0 \cap \mathcal{S}^{d+1}$. Note that

$$D_0^\perp = \operatorname{span}\{e_i e_i^T - e_0 e_i^T, e_i e_i^T - e_i e_0^T : i \in \{1, 2, \ldots, d\}\}.$$

Thus,

$$\pm(e_0 e_i^T - e_i e_0^T) \in (T_0(K) + D_0^\perp), \quad \forall i \in \{1, 2, \ldots, d\}.$$

Let $\widetilde{\mathcal{S}}^{d+1}$ denote the subspace of $(d+1)\times(d+1)$ skew-symmetric matrices with real entries. Therefore,

$$(T_0(K) + D_0^\perp) \supseteq \widetilde{\mathcal{S}}^{d+1} \quad \text{iff } (T_0(K) + D_0^\perp) \supseteq \{(e_i e_j^T - e_j e_i^T) : i, j \in \{1, 2, \ldots, d\}\}.$$

Now, using elementary cone geometry on polyhedral cones and the definitions, we have the following string of equivalences:

$$\begin{aligned}(T_0(K) + D_0^\perp) \supseteq \{(e_i e_j^T - e_j e_i^T) : i, j \in \{1, 2, \ldots, d\}\} \quad & \text{iff } (T_0(K) + D_0^\perp) \supseteq \widetilde{\mathcal{S}}^{d+1};\\ & \text{iff } [T_0(K)]^* \cap D_0 \subseteq \mathcal{S}^{d+1};\\ & \text{iff } M_0(K) = M(K).\end{aligned}$$

The second part follows from the first as in the proof of Theorem 6.1. □

COROLLARY 6.4. *If* $(T_0(K) + D_0^\perp) \supseteq \{(e_i e_j^T - e_j e_i^T) : i, j \in \{1, 2, \ldots, d\}\}$, *then* $N_0^r(K) = N^r(K)$ *for every* $r \geq 0$.

PROOF. Trivial for $r = 0$. By Theorem 6.3, the assumption of the corollary implies $N_0(K) = N(K)$. By Theorem 2.3, $N(K) \subseteq K$. Thus,

$$T_0(N(K)) \supseteq T_0(K) \supseteq \{(e_i e_j^T - e_j e_i^T) : i, j \in \{1, 2, \ldots, d\}\}.$$

Now, applying Theorem 6.3 recursively, we obtain the desired result. □

Let G denote the complete graph on d vertices, and consider the LP relaxation $FRAC$ of the stable set problem on G. For every $i, j \in \{1, 2, \ldots, d\}$ such that $i \neq j$, we have

$$(e_0 - e_i - e_j) \in K^* \quad \text{and clearly,} \quad e_i, e_j \in (K^* \cap Q^*).$$

Thus, for every $i, j \in \{1, 2, \ldots, d\}$ such that $i \neq j$, we have

$$e_i(e_0 - e_i - e_j)^T, \quad e_j e_i^T \in T_0(K) \quad \text{and} \quad (e_i e_i^T - e_i e_0^T) \in D_0^\perp.$$

This implies

$$(T_0(K) + D_0^\perp) \supseteq \{(e_i e_j^T - e_j e_i^T) : i, j \in \{1, 2, \ldots, d\}\}.$$

Therefore, the condition of Theorem 6.3 is satisfied and we have $N_0^r(FRAC) = N^r(FRAC)$ for every $r \geq 0$.

Instead of comparing $M(K)$ and $M_+(K)$, or $N(K)$ and $N_+(K)$, we might ask when the set of optimal solutions of both relaxations are the same. This is precisely when

$$[N(K)]^* + \begin{pmatrix} -z^* \\ c \end{pmatrix} \supseteq [N_+(K)]^* + \begin{pmatrix} -z^* \\ c \end{pmatrix},$$

where z^* is the optimal value of $\max\{c^T x : \binom{1}{x} \in N(K)\}$.

Sometimes we are only interested in the bound provided by the relaxation. This is equivalent to finding the smallest z for which $\binom{z}{-c} \in [N(K)]^*$ and the smallest z^+ for which $\binom{z^+}{-c} \in [N_+(K)]^*$.

Appendix. In this Appendix, we prove Proposition 4.5 and derive additional properties of c and c_+. We first start with a sketch of the proof of Part (1) of Proposition 4.5.

PROOF OF PART (1) OF PROPOSITION 4.5. We consider the case d even; the case d odd can be handled similarly. For $i \geq 0, j \geq 0, i+j \geq 3$, let $f(i,j) = c(i+j-3, d/2-i, d/2-j)$. Observe that

$$f(i,j) = \begin{cases} 0 & j = 0, \\ 1 & i = 0, \\ \dfrac{j}{j+1} & i+j = 3 \text{ and } i \geq 1, \\ \dfrac{f(i-1,j)}{1-f(i,j-1)+f(i-1,j)} & \text{otherwise.} \end{cases}$$

As a result, $f(i,j)$ is independent of d. The first few values of $f(i,j)$ are given below:

$i\backslash j$	0	1	2	3	4	5
0				1	1	1
1			2/3	3/4	4/5	5/6
2		1/2	4/7	7/11	11/16	16/22
3	0	1/3	6/13	13/24	24/40	40/62
4	0	1/4	8/21	21/45	45/85	85/147
5	0	1/5	10/31	31/76	76/161	161/308

For $i \geq 0, j \geq 0, i+j \geq 3$, define $d(i,j)$ in the following way:

$$d(i,j) = \begin{cases} 0 & j = 0, \\ 1 & i = 0, \\ 2 & i = 1, j = 2 \text{ or } i = 2, j = 1, \\ d(i-1,j) + d(i,j-1) & \text{otherwise.} \end{cases}$$

The first few values of $d(i,j)$ are given below:

$i\backslash j$	0	1	2	3	4	5
0				1	1	1
1			2	3	4	5
2		2	4	7	11	16
3	0	2	6	13	24	40
4	0	2	8	21	45	85
5	0	2	10	31	76	161

We claim that

$$f(i,j) = \frac{d(i,j)}{d(i,j+1)}.$$

To prove the claim, we verify it for

- $i = 0$: $f(0,j) = 1$ while $d(0,j)/d(0,j+1) = 1$,
- $j = 0$: $f(i,0) = 0$ while $d(i,0)/d(i,1) = 0$,
- $(i,j) = (1,2)$ or $(i,j) = (2,1)$: $f(i,j) = j/(j+1)$ while $d(i,j)/d(i,j+1) = 2/3$ if $i = 1$ and $d(i,j)/d(i,j+1) = 1/2$ if $i = 2$,
- otherwise: we proceed by induction on $i+j$. We have

$$\begin{aligned} f(i,j) &= \frac{f(i-1,j)}{1-f(i,j-1)+f(i-1,j)} = \frac{d(i-1,j)}{d(i-1,j+1)} \frac{1}{\left[1-\frac{d(i,j-1)}{d(i,j)}+\frac{d(i-1,j)}{d(i-1,j+1)}\right]} \\ &= \frac{d(i-1,j)}{d(i-1,j+1)} \frac{1}{\left[\frac{d(i,j)-d(i,j-1)}{d(i,j)}+\frac{d(i-1,j)}{d(i-1,j+1)}\right]} = \frac{d(i-1,j)}{d(i-1,j+1)} \frac{1}{\left[\frac{d(i-1,j)}{d(i,j)}+\frac{d(i-1,j)}{d(i-1,j+1)}\right]} \\ &= \frac{d(i,j)d(i-1,j+1)}{d(i-1,j+1)(d(i,j)+d(i-1,j+1))} = \frac{d(i,j)}{d(i,j+1)}. \end{aligned}$$

We now can easily check that

$$d(i,j) = \binom{i+j-3}{i} + 2\binom{i+j-2}{j-1}$$

verifies the definition for d; by convention $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. (The formula for d was obtained by computing the generating function for $d(\cdot,\cdot)$.) As a result,

$$f\left(\frac{d}{2},\frac{d}{2}\right) = \frac{\binom{d-3}{d/2}+2\binom{d-2}{d/2-1}}{\binom{d-2}{d/2}+2\binom{d-1}{d/2}} = \frac{1}{2}+\frac{1}{5d-6}$$

after simplifications. $\square$

Now, we continue with a few preliminary lemmas.

LEMMA A.1. *Assuming* $0 \le b < a \le 1$ *and* $p > 0$, *we have*

$$a > \frac{a}{1-b+a} > b,$$

and

$$a > \min\left(\frac{a}{1-b+a}, \frac{(p-1)b+1}{p}\right) > b.$$

PROOF. First, $a > a/(1-b+a)$ follows from the fact that $a > 0$ and $a > b$, and this implies also that $a > \min(a/(1-b+a), ((p-1)b+1)/p)$.

On the other hand, we have that $a/(1-b+a) > b$ iff $a > b - b^2 + ab$ iff $(a-b)(1-b) > 0$, which follows by assumption. Furthermore, $((p-1)b+1)/p > b$ iff $(p-1)b+1 > pb$ iff $1 > b$. As a result, both terms in the minimum are greater than b, and the second part of each inequality follows. $\square$

This implies the following interlacing property.

COROLLARY A.2. *For any* $r \ge 1$ *and any* $n_0, n_1 \le d/2 - r$, *we have that*

$$c(r-1, n_0, n_1+1) < c(r, n_0, n_1) < c(r-1, n_0+1, n_1)$$

and

$$c_+(r-1, n_0, n_1+1) < c_+(r, n_0, n_1) < c_+(r-1, n_0+1, n_1).$$

PROOF. For $r = 1$ and $n_0, n_1 \leq d/2 - 1$, we have that $0 \leq f(r-1, n_0, n_1+1) < f(r-1, n_0+1, n_1) \leq 1$ where $f = c$ or $f = c_+$ by (1). Lemma A.1 now implies the result for $r = 1$.

Proceeding by induction on r and assuming the result for $r-1$ is true, we derive that $f(r-1, n_0, n_1+1) < f(r-2, n_0+1, n_1+1) < f(r-1, n_0+1, n_1)$, which implies the result for r by Lemma A.1. □

We can now get a lower bound on the coefficients c and c_+.

THEOREM A.3. *For any r, n_0, n_1 such that $s = r + n_0 + n_1 \leq d/2$, we have that*

$$c(r, n_0, n_1) \geq c_+(r, n_0, n_1) > c(0, 0, s) = \frac{d/2 - s}{d/2 + 1 - s}.$$

In particular, $c_+(d/2 - 1, 0, 0) > 0.5$.

This shows that the N_+-rank of K is $d/2$.

PROOF. For $s \leq d/2$, we have

$$c(r, n_0, n_1) \geq c_+(r, n_0, n_1) > c_+(r, 0, n_0 + n_1) > c_+(0, 0, r + n_0 + n_1) = c(0, 0, s),$$

where we have used Corollary A.2 twice. □

LEMMA A.4. *Let $a > b > c > d \geq 0$ with $1 > b$ be such that $a - b < b - c < c - d$. Then*

$$\frac{a}{1 - b + a} - \frac{b}{1 - c + b} < \frac{b}{1 - c + b} - \frac{c}{1 - d + c}.$$

PROOF. Since $a/(1 - b + a)$ is an increasing function of a for $1 > b$, we have that $a/(1 - b + a) < a'/(1 - b + a')$, where $a' - b = b - c$. Hence,

$$\begin{aligned} \frac{a}{1 - b + a} - \frac{b}{1 - c + b} &< \frac{2b - c}{1 - c + b} - \frac{b}{1 - c + b} \\ &= \frac{b - c}{1 - c + b} < \frac{b}{1 - c + b} - \frac{c}{1 - d + c}. \end{aligned}$$ □

This implies the following corollary.

COROLLARY A.5. *For any $r \geq 1$, any $0 \leq n_0 \leq d/2 - r - 2$, any $1 \leq n_1 \leq d/2 - r$, we have $c(r-1, n_0+2, n_1-1) - c(r-1, n_0+1, n_1) < c(r-1, n_0+1, n_1) - c(r-1, n_0, n_1+1)$.*

PROOF. For $r = 1$, $1 \leq n_1 \leq d/2 - 1$ and $n_0 \leq d/2 - 3$, let $a = c(0, n_0+2, n_1-1)$, $b = c(0, n_0+1, n_1)$, and $c = c(0, n_0, n_1+1)$. Observe that $a = 1 - 1/(d/2 + 2 - n_1)$, $b = 1 - 1/(d/2 + 1 - n_1)$, and $c = 1 - 1/(d/2 - n_1)$, implying that $a > b > c$ and $a - b < b - c$.

We now proceed by induction and assume the result true for $r - 1 \geq 1$. Let $a = c(r-2, n_0+3, n_1-1)$, $b = c(r-2, n_0+2, n_1)$, $c = c(r-2, n_0+1, n_1+1)$, and $d = c(r-2, n_0, n_1+2)$. We know from Corollary A.2 that $a > b > c > d$ and from the inductive hypothesis that $(a-b) < (b-c) < (c-d)$. Lemma A.4 then implies the result for r. □

We need an additional lemma.

LEMMA A.6. *Let $1 \geq a > b > c \geq 0$ be such that $a - b < b - c$. Then*

$$\frac{a}{1 - b + a} - \frac{b}{1 - c + b} < b - c.$$

PROOF. We have that

$$\left(\frac{b}{1 - c + b} - c\right) - \left(\frac{a}{1 - b + a} - b\right) = \frac{(b - c)(1 - c)}{1 - c + b} - \frac{(a - b)(1 - b)}{1 - b + a}.$$

Moreover, we know that $1-c>1-b>0$ and $(b-c)/(1-c+b)>(a-b)/(1-b+a)>0$ since $0<a-b<b-c$. Multiplying these two inequalities together, we get the desired inequality. $\square$

Lemma A.6 implies the following corollary.

COROLLARY A.7. *For any $r \geq 1$, any $0 \leq n_0 \leq d/2-r-2$, any $1 \leq n_1 \leq d/2-r$, we have that $c(r,n_0+1,n_1-1)-c(r,n_0,n_1)<c(r-1,n_0+1,n_1)-c(r-1,n_0,n_1+1)$.*

PROOF. Setting $a=c(r-1,n_0+2,n_1-1)$, $b=c(r-1,n_0+1,n_1)$, and $c=c(r-1,n_0,n_1+1)$, we need to prove that $a/(1-b+a)-b/(1-c+b)<b-c$. From Corollary A.2 we know that $a>b>c$ and from Corollary A.5 that $a-b<b-c$. The result now follows from Lemma A.6. $\square$

Using Corollaries A.5 and A.7 repeatedly, we derive the following corollary.

COROLLARY A.8. *For any $r \geq 1$, $n_0, n_1 \geq 0$ such that $s=r+n_0+n_1 \leq d/2$, we have that*

$$\begin{aligned} &c(r-1,n_0+1,n_1)-c(r-1,n_0,n_1+1) \\ &\quad < c(0,1,s-1)-c(0,0,s)=\frac{1}{(d/2+1-s)(d/2+2-s)}. \end{aligned}$$

PROOF. Using Corollaries A.5 and A.7, we derive

$$\begin{aligned} &c(r-1,n_0+1,n_1)-c(r-1,n_0,n_1+1) \\ &\quad < c(r-1,1,n_0+n_1)-c(r-1,0,n_0+n_1+1) \\ &\quad < c(0,1,s-1)-c(0,0,s). \end{aligned}$$ $\square$

THEOREM A.9. *For any $r, n_0, n_1 \geq 0$ such that $s=r+n_0+n_1 \leq d/2-\sqrt{d}+3/2$, we have that $c(r,n_0,n_1)=c_+(r,n_0,n_1)$.*

PROOF. The proof is by induction on r. The base case is obvious. Assume the result is true for $r-1$. This implies that $c(r-1,n_0+1,n_1)=c_+(r-1,n_0+1,n_1)$ and $c(r-1,n_0,n_1+1)=c_+(r-1,n_0,n_1+1)$; we denote these quantities by a and b, respectively. The result would then follow if we can show that

$$\frac{a}{1-b+a} \leq \frac{(p-1)b+1}{p},$$

where $p=d-n_0-n_1$. This inequality is equivalent to $pa \leq 1-b+a+(p-1)b-(p-1)b^2+(p-1)ab$, or to $(1-b)(a-b)(p-1) \leq 1-b$. Since $b \leq 1$, we need to prove that $a-b \leq 1/(p-1)=1/(d-n_0-n_1-1)$. This follows from Corollary A.8 since we have that

$$\begin{aligned} a-b &< \frac{1}{(d/2+1-s)(d/2+2-s)} \leq \frac{1}{(\sqrt{d}-0.5)(\sqrt{d}+0.5)} \\ &< \frac{1}{d-1} \leq \frac{1}{d-n_0-n_1-1}. \end{aligned}$$ $\square$

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