

## Lecture 5

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In this lecture, we establish the connection between nowhere-zero (nwz)  $k$ -flow and nowhere-zero  $\mathbb{Z}_k$ -flows. Then, we present several theorems of relations between edge-connectivity of a graph, and the existence of nowhere-zero flows.

## 1 Nowhere-zero $k$ -flow

Let us first recall some definitions from the previous lecture.

**Definition 1** Let  $G = (V, E)$  be a directed graph, and  $\Gamma$  be an abelian group. A nowhere-zero  $\Gamma$ -flow is  $\phi : E \rightarrow \Gamma \setminus \{0\}$  such that

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e) \quad (\text{flow conservation}).$$

If  $G$  is undirected, then we say that it has a nowhere-zero  $\Gamma$  flow if the graph admits a nowhere-zero  $\Gamma$  flow after giving an orientation to all the edges.

As we saw, if one orientation works then any does, since inverses exist in abelian groups.

**Definition 2** Let  $G$  be an undirected graph. For integer  $k \geq 2$ , a nowhere-zero  $k$ -flow  $\phi$  is an assignment  $\phi : E \rightarrow \{1, \dots, k-1\}$  such that for some orientation of  $G$  flow conservation is achieved, i.e.,

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$

for all  $v \in V$ .

It is convenient to fix an orientation and let  $\phi$  take values in  $\{\pm 1, \dots, \pm(k-1)\}$ .

**Theorem 1 (Tutte 1950)** Let  $G$  be an undirected graph. Then  $G$  has a nowhere-zero  $k$ -flow  $\iff G$  has a nowhere-zero  $\mathbb{Z}_k$ -flow.

**Proof:** ( $\implies$ ): By definition of nowhere-zero flows.

( $\impliedby$ ): Let  $\phi$  be a nowhere-zero  $\mathbb{Z}_k$ -flow, define  $e(v) = \sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e)$  for all  $v \in V$ , under group operation in  $\mathbb{Z}$ . Observe that all  $e(v)$ 's are multiples of  $k$ . Without loss of generality, we can assume  $\phi$  is the nowhere-zero  $\mathbb{Z}_k$ -flow such that  $\sum_{v \in V} |e(v)|$  is minimized.

Suppose  $\sum_{v \in V} |e(v)| = 0$ , then we have obtained a nowhere-zero  $k$ -flow. Otherwise, let  $S = \{v : e(v) > 0\}$  and  $T = \{v : e(v) < 0\}$ . Since  $\sum_{v \in V} |e(v)| > 0$  and  $\sum_{v \in V} e(v) = 0$ , we have that both  $S$  and  $T$  are nonempty. Let  $U$  be the set of vertices reachable from  $S$ . If  $U \cap T = \emptyset$ , then  $0 < \sum_{v \in U} e(v) = \sum_{e \in \delta^+(U)} \phi(e) - \sum_{e \in \delta^-(U)} \phi(e)$ . But  $\delta^+(U) = \emptyset$ , and

thus implies  $\sum_{e \in \delta^+(U)} \phi(e) - \sum_{e \in \delta^-(U)} \phi(e) \leq 0$ , which is a contradiction. Thus, we must have  $U \cap T \neq \emptyset$ , which implies we can find a directed path  $P$  from some  $s \in S$  to  $t \in T$ .

Now, we "revert" path  $P$  to create another nowhere-zero  $\mathbb{Z}_k$ -flow. More formally, for each arc  $e \in P$ , reverse the direction of  $e$ , and name it as  $\overleftarrow{e}$ , and define

$$\begin{aligned} \phi'(\overleftarrow{e}) &= k - \phi(e) & \forall e \in P \\ \phi'(e) &= \phi(e) & \forall e \notin P \end{aligned}$$

Then  $\phi'$  is also a nowhere-zero  $\mathbb{Z}_k$ -flow. Let  $e'(v) = \sum_{e \in \delta^+(v)} \phi'(e) - \sum_{e \in \delta^-(v)} \phi'(e)$  for all  $v \in V$ . Observe that for any  $v \in V \setminus \{s, t\}$ ,  $e'(v) = e(v)$ . And  $e'(s) = e(s) - k$ ,  $e'(t) = e(t) + k$ . This implies  $\sum_{v \in V} |e'(v)| < \sum_{v \in V} |e(v)|$ , contradicting the minimality of  $\phi$ . Therefore, we have  $\sum_{v \in V} |e(v)| = 0$ , which implies we also have a nowhere-zero  $k$ -flow.  $\square$

Theorem 1 implies that if  $G$  has a nowhere-zero  $k$ -flow then  $G$  has a nowhere-zero  $k'$ -flow, for any integer  $k' \geq k$ . Combine with the theorem from previous lecture which states that the existence of a nowhere-zero  $\Gamma$ -flow depends not on the group structure of  $\Gamma$ , but only on the size of  $\Gamma$ , we have obtained the following corollary:

**Corollary 2**  $G$  has a nowhere-zero  $\Gamma$ -flow  $\implies G$  has a nowhere-zero  $\Gamma'$  flow for any  $|\Gamma'| \geq |\Gamma|$ .

## 2 Nowhere-zero Flow and Edge Connectivity

Now, we discuss some open problems and known results of the relation between a graph's edge connectivity and the existence of its nowhere-zero flows. We begin with a famous conjecture of Tutte.

**Conjecture 1** *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

And here is a weaker version of this conjecture.

**Conjecture 2** *There exists a positive integer  $k$  such that every  $k$ -edge-connected graph has a nowhere-zero 3-flow.*

Now, we present some of the results that is known about nowhere-zero flow and edge connectivity.

**Theorem 3 (Jaeger)** *If an undirected graph  $G = (V, E)$  is 4-edge-connected, then  $G$  has a nowhere-zero 4-flow (or nowhere-zero  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow).*

It is also shown that there is a efficient algorithm for finding a nowhere-zero 4-flow of any given graph  $G$ .

**Theorem 4 (Jaeger)** *If an undirected graph  $G = (V, E)$  is 2-edge-connected, then  $G$  has a nowhere-zero 8-flow (or nowhere-zero  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -flow).*

**Theorem 5 (Seymour)** *If  $G$  is 2-edge-connected, then  $G$  has a nowhere-zero 6-flow.*

The proofs for all three theorems stated are constructive. Seymour also conjectured that if  $G$  is 2-edge-connected, then  $G$  has a nowhere-zero 5-flow.

Here, we will present the proofs of the two theorems by Jaeger. Before we present the proofs, we first state and prove a very useful proposition.

**Proposition 6** *Given an undirected graph  $G = (V, E)$ ,  $G$  has a nowhere-zero  $2^p$ -flow  $\iff$  there exists  $F_1, F_2, \dots, F_p \subset E$ , such that  $E = \cup_{i=1}^p F_i$ , and for any  $1 \leq i \leq p$ ,  $F_i$  is an even graph (every vertex of  $F_i$  has even degree).*

**Proof:** ( $\Leftarrow$ ): Given  $F_1, \dots, F_p$  define a  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$  circulation  $\phi$  as:

$$[\phi(e)]_i = \begin{cases} 1 & \text{if } e \in F_i \\ 0 & \text{otherwise} \end{cases}$$

Since  $E = \cup_{i=1}^p F_i$ ,  $\phi$  is a nowhere-zero  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ -flow.

( $\Rightarrow$ ): Let  $\phi$  be a nowhere-zero  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ -flow of  $G$  (which exists since  $G$  has a nowhere zero  $2^p$ -flow). For  $1 \leq i \leq p$ , define

$$F_i = \{e | e \in E \text{ and } [\phi(e)]_i = 1\}$$

This means  $F_i$  is an even graph for any  $i$  and  $E = \cup_{i=1}^p F_i$ . □

We also make use of a theorem by Nash and Williams, which we will state without providing its proof.

**Theorem 7 (Nash-Williams)** *If an undirected graph  $G = (V, E)$  is  $2k$ -edge-connected, then  $G$  has  $k$ -edge-disjoint spanning trees.*

Now, we have enough tools to present the proofs of Jaeger.

**Proof:** [Proof of Theorem 3] By the theorem of Nash-Williams, we can find two edge disjoint spanning trees of  $G$ , and name them as  $T_1, T_2$ .

**Claim 8**  $\exists A_1 \subset T_1$ , such that  $(E \setminus T_1) \cup A_1$  is even (again, by even, we mean for all  $v \in V$ ,  $v$  has even degree in  $(E \setminus T_1) \cup A_1$ ).

**Proof:** [Proof of the Claim]: We prove the claim by describe an algorithm to find  $A_1$ . Start the algorithm with  $U = E \setminus T_1$ ,  $A_1 = \emptyset$  and  $T = T_1$ . At every iteration, find a leaf vertex  $v$  of  $T$ . If the degree of  $v$  in  $U$  is even, leave  $U$  as it is, and if degree of  $v$  in  $U$  is odd, add the edge of  $T$  incident to  $v$  into  $U$  and  $A_1$ . Next, delete  $v$ , update  $T$  and start another iteration. At the end of the algorithm (when  $T$  has exactly one vertex left), we have that  $U$  has all even degrees except possibly at that last vertex. But the sum of degrees (in  $U$ ) of all vertices must be even, so all vertices in  $U$  must have even degrees. So we have  $U = (E \setminus T_1) \cup A_1$  and  $U$  is even. □

By similar argument, there also exists  $A_2 \subset T_2$ , such that  $(E \setminus T_2) \cup A_2$  is even. Now let  $F_1 = (E \setminus T_1) \cup A_1$ ,  $F_2 = (E \setminus T_2) \cup A_2$ . Then  $T_1 \cap T_2 = \emptyset$  implies that  $F_1 \cup F_2 = E$ . By proposition 6, we have that  $G$  has a nowhere-zero 4-flow. □

Now, we prove Theorem 4, which is similar to the previous proof.

**Proof:** [Proof of Theorem 4] Suppose  $G$  is 2-edge-connected but not 3-edge-connected. Then, there exists a cut  $\{e_1, e_2\}$ . Without loss of generality, assume that, in our orientation of  $G$ ,  $e_1$  and  $e_2$  disagree in orientation within the cut  $\{e_1, e_2\}$ . Let  $G' = G/e_1$ , and by induction (on the number of vertices),  $G'$  has a nowhere-zero 8-flow  $\phi'$ . Define

$$[\phi(e)] = \begin{cases} \phi'(e) & \text{if } e \neq e_1 \\ \phi'(e_2) & \text{if } e = e_1 \end{cases}$$

Then  $\phi(e)$  is a nowhere-zero 8-flow of  $G$  (by flow conservation across the cut  $\{e_1, e_2\}$ ).

Now, suppose  $G$  is 3-edge-connected, then we duplicate every edge of  $G$  to create  $G' = (V, E')$ . Clearly,  $G'$  is 6-edge-connected. Again, by Nash-Williams, there exists pairwise disjoint sets  $T'_1, T'_2, T'_3 \subset E'$ , such that  $T'_1 \cup T'_2 \cup T'_3 = E'$ . Define  $T_i = \{e | e \text{ is in } T'_i \text{ or the duplicate of } e \text{ is in } T'_i\}$ . Then  $T_1, T_2, T_3 \subset E$ , such that for any  $e \in E$ , there is  $i$  such that  $e \notin T_i$ . By similar arguments as the last proof, we can find  $A_1, A_2, A_3$  such that  $(E \setminus T_i) \cup A_i$  is even. Let  $F_i = (E \setminus T_i) \cup A_i$ . Since  $\forall e \in E$ , there is some  $i$  such that  $e \notin T_i$  (and thus  $e \in F_i$ ), we have  $E = F_1 \cup F_2 \cup F_3$ . Now, the proof is completed by applying Proposition 6.  $\square$

### 3 Finding Nowhere-zero $k$ -flow

The proof we gave of Tutte's theorem stating that the existence of a nowhere zero  $\Gamma$ -flow in  $G$  depends only on  $|\Gamma|$  is non-constructive. In this section, we prove it algorithmically by going from any nowhere zero  $\Gamma$ -flow to a nowhere zero  $\Gamma'$ -flow where  $|\Gamma| = |\Gamma'|$ . This does not seem to be known (or at least widely known); Jensen and Toft [1, p. 210] write:

No constructive proof has been published so far as we know. However, the arguments given by Minty [1967] for the case  $k = 4$  seem to provide a key to a constructive proof also in general.

For this purpose, we use the fundamental theorem of finite abelian groups which says that any finite abelian group is isomorphic to  $\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_\ell}$  where the  $q_i$ 's are prime powers (the constructions below will work even for general  $q_i$ 's). Throughout this section, we consider a directed graph  $G = (V, E)$ , so when we talk about a nowhere zero  $k$ -flow  $\phi$ , we assume that  $\phi$  takes values in  $\{\pm 1, \dots, \pm(k-1)\}$ .

As a warm-up, we show algorithmically how to find a nowhere-zero  $\mathbb{Z}_4$ -flow from a nowhere-zero  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow. Let  $\phi$  be a nowhere-zero  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow. Then, let  $F_i = \{e | [\phi(e)]_i \neq 0\}$ , for  $i = 1, 2$ . This implies  $(V, F_i)$  has a nowhere-zero  $\mathbb{Z}_2$ -flow for  $i = 1, 2$ . Then, we can find  $\phi_i : F_i \rightarrow \{-1, 1\}$ , where  $\phi_i$  is a 2-flow (under operations in  $\mathbb{Z}$ ) for  $i = \{1, 2\}$ , and we can extend it with 0 values to  $E$ . Finally, define

$$\phi' = 2\phi_1 + \phi_2.$$

Then  $\phi'$  satisfies the flow conservation and  $\forall e \in E$ ,  $-3 \leq \phi'(e) \leq 3$ , and  $\phi'(e) \neq 0$ . This shows  $\phi'$  is a nowhere-zero 4-flow, and taking values modulo 4, we get a  $\mathbb{Z}_4$ -flow of  $G$ .

We can extend this procedure to all finite abelian groups, that is, for any group  $\Gamma = \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_\ell}$ , where  $q_i$ 's are prime powers for  $1 \leq i \leq \ell$ . Let  $k = (q_1 \cdot q_2 \cdot \dots \cdot q_\ell)$ ,

given a nowhere-zero  $\Gamma$ -flow  $\phi$ , we can find a nowhere-zero  $\mathbb{Z}_k$ -flow  $\phi'$  algorithmically in the following way. The  $i$ th coordinate of  $\phi$  is a  $\mathbb{Z}_{q_i}$ -flow and thus, we can transform it into a  $q_i$ -flow (over  $\mathbb{Z}$ ) (instead of changing the orientations as we did in the proof of Theorem 1, we can simply decrease the flow value along  $P$  by  $q_i$ ):

$$\phi_i : E \rightarrow \{-(q_i - 1), \dots, -1, 0, 1, \dots, (q_i - 1)\}$$

for each  $i = 1, 2, \dots, \ell$ . Observe that, by construction, for every  $e$ , there exists  $i$  such that  $\phi_i(e) \neq 0$ . Then define

$$\phi'(e) = \sum_{i=1}^{\ell} \left( \prod_{j=1}^{i-1} q_j \right) \phi_i(e)$$

for each  $e \in E$ . Then, we claim that  $\phi'$  is a nowhere-zero  $k$ -flow in  $G$ . Indeed, by induction on  $h$ , we have that

$$\left| \sum_{i=1}^h \left( \prod_{j=1}^{i-1} q_j \right) \phi_i(e) \right| \leq \left( \prod_{j=1}^h q_j \right) - 1,$$

implying that (i)  $\phi'(e) \neq 0$  (by considering the largest  $i$  with  $\phi_i(e) \neq 0$ ) and (ii)  $|\phi'(e)| < k$ .

The converse can also be done. From a nowhere zero  $\mathbb{Z}_k$ -flow, one can construct a nowhere zero  $\Gamma$ -flow for  $\Gamma = \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_\ell}$  where  $k = (q_1 \cdot q_2 \cdot \dots \cdot q_\ell)$ . For this, we show how to get a nowhere zero  $\mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$  flow  $\phi'$  from a nowhere zero  $\mathbb{Z}_{q_1 q_2}$ -flow  $\phi$  for any  $q_1, q_2$ . First, define  $\phi_e \equiv \lambda_e \pmod{q_1}$  for all  $e \in E$ . Thus,  $\lambda$  is a  $\mathbb{Z}_{q_1}$ -flow; it takes value 0 on the edges of  $E_1 = \{e : \phi_e \equiv 0 \pmod{q_1}\}$ . As before, by possibly replacing  $\lambda_e$  by  $\lambda_e - q_1$ , we can get a  $q_1$ -flow  $\phi_1$  which is nowhere zero on  $E \setminus E_1$ . Let  $\mu_1$  be the  $\mathbb{Z}_{q_1}$ -flow obtained by taking the flow values  $\phi_1$  modulo  $q_1$ . Since  $\phi$  and  $\phi_1$  are both flows (over  $\mathbb{Z}$ ), we have that  $\phi - \phi_1$  is also a flow, and every flow value is a multiple of  $q_1$ . Thus  $\phi_2 = \frac{1}{q_1}(\phi - \phi_1)$  is also an integer-valued flow. Taking  $\phi_2$  modulo  $q_2$ , we get a  $\mathbb{Z}_{q_2}$ -flow. Observe now that if  $\mu_1(e) = 0$  then  $\mu_2(e) \neq 0$  since otherwise  $\phi(e)$  would be 0. Thus,  $(\mu_1, \mu_2)$  constitutes a nowhere zero  $\mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$ -flow on  $G$ .

## References

- [1] T.R. Jensen and B. Toft, "Graph Coloring Problems", Wiley, 1995.