| **18.438 Advanced Combinatorial Optimization** | September 22, 2009 |
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| | Lecture 4 |
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In this lecture, we discuss some results on edge coloring and also introduce the notion of nowhere zero flows.

# 1 Petersen's Theorem

Recall that a graph is *cubic* if every vertex has degree exactly 3, and *bridgeless* if it cannot be disconnected by deleting any one edge (i.e., 2-edge-connected).

**Theorem 1 (Petersen)** *Any bridgeless cubic graph has a perfect matching.*

**Proof:** We will show that for any $V \subseteq U$, we have $o(G-U) \leq |U|$ (here $o(G)$ is the number of odd components of the graph $G$). The theorem will then follow from the Tutte-Berge formula.

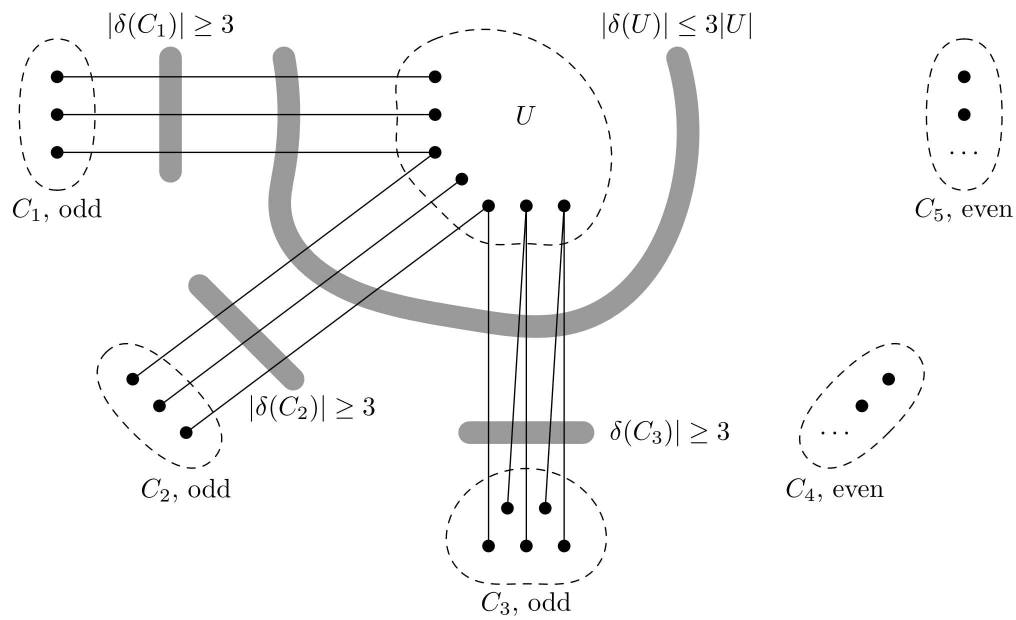


Figure 1: Illustration of the proof of Petersen's theorem. Edges inside $U$ and $C_i$, as well as between $C_4, C_5$ and $U$ are omitted.

Consider an arbitrary $U \subset V$. Each odd component of $G-U$ is left by an odd number of edges, since $G$ is cubic. Since $G$ is also bridgeless each component is left by at least 2 edges, hence by at least 3 edges. On the other hand, the set of edges leaving all odd components of $G-U$ is a subset of the edges leaving $U$, and there are at most $3|U|$ edges leaving $U$, since $G$ is cubic. Among these $3|U|$ edges, there are at least 3 edges per each odd component, therefore there are at most $|U|$ odd components. (See Figure 1.) $\square$

See Figure 2 for an example of a cubic graph along with a perfect matching.

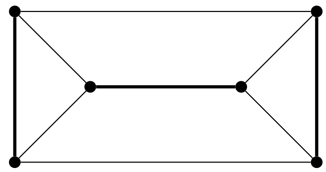

Figure 2: A bridgeless cubic graph and a perfect matching on it. Edges in the matching are bold.

If we were to drop the bridgeless hypothesis, then Theorem 1 would no longer be true, as shown by the example in Figure 3.

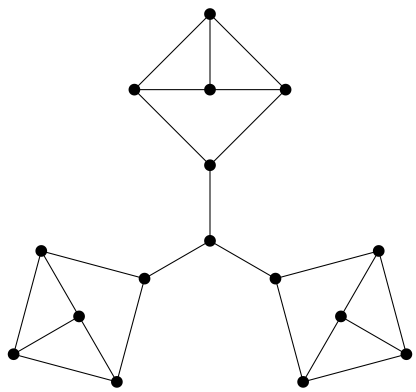

Figure 3: A cubic graph without a perfect matching.

Although any bridgeless cubic graph has a perfect matching, it is not true that any such graph can be decomposed into 3 perfect matchings. An example of this is the Petersen graph, depicted in Figure 4.

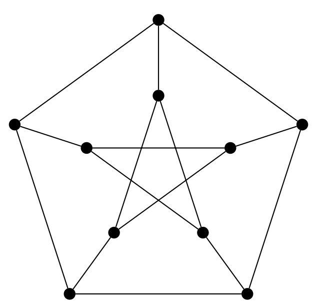

Figure 4: The Petersen graph.

However, it is true that any cubic graphs can be decomposed into 4 matchings (not necessarily perfect). Such a decomposition is equivalent to an edge coloring, which we discuss next.

## 2 Edge coloring

**Definition 1** *A graph $G$ is* $k$-edge-colorable *if we can assign an element of $\{1, 2, \ldots, k\}$ (a "color") to each edge of $G$ such that no two adjacent edges are assigned the same color. Let $\chi'(G)$, the* edge chromatic number*, be the smallest integer $k$ such that $G$ is $k$-edge-colorable.*

Bridgeless cubic graphs are not necessarily 3-edge-colorable, with the Petersen graph as a counterexample. However, it is true that any cubic graph is 4-edge-colorable. In fact, we have the following general result.

**Theorem 2 (Vizing, 1964)** *For any graph $G$ with maximum degree $\Delta$, the edge chromatic number $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$.*

It is clear that the edge chromatic number is at least $\Delta$, since all the edges incident to a vertex must have different colors. Vizing's theorem says that the graph is always $(\Delta + 1)$-colorable.

From Vizing's theorem, we know that any cubic graph $G$ has $\chi'(G) \in \{3, 4\}$. Holyer (1981) showed that it is NP-complete to decide whether a given cubic graph is 3-colorable.

The following theorem is a particularly appealing result relating matchings and colorings.

**Theorem 3 (Tait, 1878)** *The four-color theorem is equivalent to the claim that every planar cubic bridgeless graph is 3-edge-colorable.*

## 3 Nowhere zero flows

The concept of nowhere zero flows was introduced by Tutte (1954). It generalizes flows and circulations to values in abelian groups.

**Definition 2** *Let $G = (V, E)$ be a directed graph, and $\Gamma$ be an abelian group. A* $\Gamma$-circulation *$\phi$ on $G$ is an assignment $\phi : E \to \Gamma$ such that the following flow conservation condition holds for all for all $v \in V$:*

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e) \qquad (\textit{as elements of } \Gamma).$$

*A $\Gamma$-circulation $\phi$ is said to be nowhere zero if $\phi(e) \neq 0$ for all $e \in E$ (where $0$ denotes the identity for $\Gamma$). If $G$ is undirected, then we say that it has a $\Gamma$-circulation if the graph admits a $\Gamma$-circulation after giving an orientation to all the edges.*

Since inverses exist in abelian groups, the choice of edge orientations is irrelevant in the last part of the above definition; any orientation will work if one does.

The central problems regarding nowhere zero flows have the form, for a particular group $\Gamma$, which graphs have nowhere zero $\Gamma$-flows? We now give one such result.

**Proposition 4** *Let $G$ be a cubic graph. Then $\chi'(G) = 3$ if and only if $G$ has a nowhere zero $\Gamma$-flow for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

**Proof:** ($\Rightarrow$) If $\chi'(G) = 3$, then $G$ is 3-edge-colorable, so let us color the edges with $\{(0,1),(1,0),(1,1)\}$ (as elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$). Note that $x = -x$ for any $x \in \Gamma$, so the orientation of the edges does not matter. Then the resulting coloring gives a nowhere zero $\Gamma$-flow. Indeed, each vertex is incident to three edges colors with $(0,1),(1,0),(1,1)$, so their sum is zero in $\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence flow conservation is satisfied.

($\Leftarrow$) If $G$ has a nowhere zero $\Gamma$-flow, then every vertex is incident to three edges of values $(0,1),(1,0),(1,1)$ (as no other combination of three nonzero elements of $\Gamma$ sum to zero). Hence we obtain a 3-edge-coloring of $G$ with colors corresponding to the value of the flow. □

We mentioned earlier that it is NP-complete to decide whether a given cubic graph is 3-colorable. It follows as a corollary that it is NP-complete to decide whether a given cubic graph has a nowhere zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow.

The following remarkable result of Tutte says that the existence of a nowhere zero $\Gamma$-flow depends not on the group structure of $\Gamma$, but only on the size of $\Gamma$.

**Theorem 5 (Tutte 1954)** *Let $\Gamma$ and $\Gamma'$ be two finite abelian groups with $|\Gamma| = |\Gamma'|$. Then a graph $G$ has a nowhere zero $\Gamma$-flow if and only if $G$ has a nowhere zero $\Gamma'$ flow.*

In fact, we will prove something stronger. For a directed graph $G$, allowing multiple edges and loops, let $n_\Gamma(G)$ denote the number of nowhere zero $\Gamma$-flows on $G$.

**Claim 6** *Let $G$ be a directed graph, allowing multiple edges and loops. If $\Gamma$ and $\Gamma'$ are two finite abelian groups with $|\Gamma| = |\Gamma'|$, then $n_\Gamma(G) = n_{\Gamma'}(G)$.*

**Proof:** Let us induct on the number of non-loop edges of $G$. When $G$ only has loops, any assignment $\phi : E \to \Gamma \setminus \{0\}$ gives a valid flow, since flow is always conserved. Thus, the number of nowhere zero $\Gamma$-flows in this case is $n_\Gamma(G) = (|\Gamma| - 1)^{|E(G)|}$, which depends only on the size of $\Gamma$. Hence $n_\Gamma(G) = n_{\Gamma'}(G)$. This proves the base case.

Now let $G$ be any graph with at least one non-loop edge $e$. Let $G/\{e\}$ denote the graph constructed from $G$ by contracting the edge $e$ in the following sense. Suppose that $e$ connects vertices $x$ and $y$, then in the new graph, the vertices $x$ and $y$ will be merged into a single vertex $z$; any edge in $G$ (other than $e$) with an endpoint in $\{x, y\}$ will now have the corresponding endpoint at $z$.

Let $G \setminus \{e\}$ be the graph obtained from $G$ by removing the edge $e$. We claim that

$$n_\Gamma(G/\{e\}) = n_\Gamma(G) + n_\Gamma(G \setminus \{e\}). \tag{1}$$

Indeed, let us construct a bijection between the set of nowhere zero $\Gamma$-flows on $G/\{e\}$ and the set of nowhere zero $\Gamma$-flows on $G$ or $G \setminus \{e\}$, or equivalently, the set of flows on $G$ which is nowhere zero except for possibly the edge $e$.

We can identify the edges of $G/\{e\}$ with the edges of $G$ with $e$ removed. Then any nowhere zero flow on $G/\{e\}$, expressed as $\phi : E \setminus \{e\} \to \Gamma$, translates into an assignment $\phi : E \setminus \{e\} \to \Gamma$ on $G$ (which is not yet a flow). Suppose that $x$ and $y$ are the two endpoints of

$e$. By the flow conservation on $G/\{e\}$, we see that flow conservation holds on $G$ everywhere except possibly at the vertices $x$ and $y$. Furthermore, in $G$, the flow surplus of $\phi$ on $x$ equals the flow deficit on $y$. Thus there is unique element of $\Gamma$ that could be assigned to $e$ that would make $\phi$ a flow on $G$, which is nowhere zero except possibly at the edge $e$. Thus we either have a nowhere zero flow in $G$ or in $G \setminus \{e\}$, and the desired bijection proves (1).

Since $G/\{e\}$ and $G\backslash\{e\}$ both have fewer non-loop edges than $G$, we can use the induction hypothesis and obtain that

$$n_\Gamma(G) = n_\Gamma(G/\{e\}) - n_\Gamma(G \setminus \{e\}) = n_{\Gamma'}(G/\{e\}) - n_{\Gamma'}(G \setminus \{e\}) = n_{\Gamma'}(G).$$

Therefore $n_\Gamma(G) = n_{\Gamma'}(G)$ for all $G$. In particular, $n_\Gamma(G) \neq 0$ if and only if $n_{\Gamma'}(G) \neq 0$, so Theorem 5 is established. □

**Remark 1** *In fact, we have shown that there exists a polynomial $P_G$ for each graph $G$, known as the* flow polynomial*, such that the number of nowhere zero $\Gamma$-flows is exactly $P_G(|\Gamma| - 1)$.*

Next we consider integer valued flows.

**Definition 3** *Let $G$ be an undirected graph. For integer $k \geq 2$, a* nowhere zero $k$-flow $\phi$ *is an assignment $\phi : E \to \{1, \ldots, k-1\}$ such that for some orientation of $G$ flow conservation is achieved, i.e.,*

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$

*for all $v \in V$.*

A convenient way to deal with $k$-flows $\phi$ is to fix an arbitrary orientation of $G = (V, E)$, let $\phi$ take values in $\{\pm 1, \pm 2, \cdots, \pm(k-1)\}$, and impose flow conservaion. It turns out that the nowhere zero $k$-flows and nowhere zero $\Gamma$-flows are integrally related, as seen from the following theorem, which will be proved in the next lecture.

**Theorem 7 (Tutte 1950)** *Let $G$ be an undirected graph. Then $G$ has a nowhere zero $k$-flow if and only if $G$ has a nowhere zero $\mathbb{Z}_k$-flow.*

## 4 Flow-coloring duality

The following result shows the connection between nowhere zero flows and colorings in planar graphs. It says that every nowhere zero $k$-flow gives rise to a $k$-coloring of the faces, and vice versa.

**Theorem 8** *Let $G$ be a planar graph (with a drawing in the plane). Then the following are equivalent*

*(1) There exists a coloring of the faces of $G$ with $k$ colors with no two adjacent faces are assigned the same color.*

*(2) $G$ has a nowhere zero $k$-flow.*

**Proof:** (1 $\Rightarrow$ 2) Suppose that we have a coloring of the faces, with colors chosen from $\{1, 2, \ldots, k\}$. Let us assign values and orientations to edges in the unique way so that the following condition is satisfied.

> For each edge $e \in E$ (with the orientation already assigned), if we orient the diagram so that $e$ is vertical and directed upwards, then the color $c_1$ of the face to its left is greater than the color $c_2$ of the face to its right. Furthermore, $e$ is assigned flow $c_1 - c_2$. (†)

Note that $1 \leq c_1 - c_2 \leq k - 1$, so that all the edges are assigned values from $\{1, 2, \ldots, k - 1\}$.

It remains to check flow conservation at each vertex $v$. Let $d$ be the degree at $v$, and let the colors of the faces incident to $v$ be $c_1, c_2, \ldots, c_d$ in counterclockwise order. See Figure 5. Then, for each $i$, the net flow into $v$ on the edge between $c_i$ and $c_{i+1}$ (indices considered mod $d$) is exactly $c_i - c_{i+1}$. Then, the flow surplus at $v$ is

$$\sum_{e \in \delta^-(v)} \phi(e) - \sum_{e \in \delta^+(v)} \phi(e) = (c_1 - c_2) + (c_2 - c_3) + \cdots + (c_d - c_1) = 0.$$

Therefore, $G$ has a nowhere zero $k$-flow. See Figure 6 for an example.

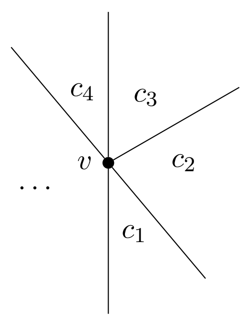


Figure 5: Showing flow conservation at $v$.

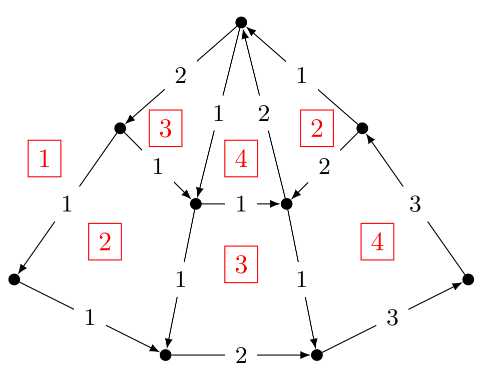


Figure 6: An example of a nowhere zero flow satisfying (†). The colors of the faces are indicated in red.

(2 $\Rightarrow$ 1) Suppose that we have nowhere zero flow $\phi$ on $G$ along with an orientation of the edges. Let us try to assign colors to the faces. We will start with colors as elements of $\mathbb{Z}$, then afterwards reduce modulo $k$ to obtain a $k$-coloring of the faces.

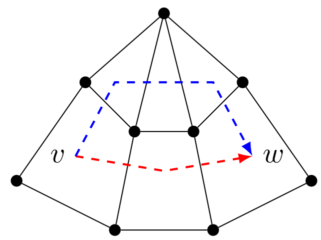


Figure 7: Illustrating the proof of the $(2 \Rightarrow 1)$ direction of Theorem 8.

Pick one face and color it arbitrarily. We claim that it is possible to color the other faces so that condition (†) is satisfied. Note that (†) in fact gives an algorithm for coloring the faces: given the color of the initial face, we can obtain the color of any other face by traversing through a sequence of adjacent faces and assigning colors following (†). We need to show that we end up with a valid coloring. It suffices to prove that the above coloring algorithm is consistent, in the sense that that color assignment of the new face does not depend on the sequence of faces that we traversed.

Let $G'$ be the planar dual to $G$, so that the vertices of $G'$ correspond to the faces of $G$. Then we need to show the following: for any two vertices $v$ and $w$ in $G'$, suppose that $v$ (as a face of $G$) has already been assigned a color and $w$ has not, then for any two paths $P_1$ and $P_2$ in $G'$ from $v$ to $w$, the color assigned to $w$ by applying (†) and traversing on $P_1$ is the same as the one obtained if we had followed $P_2$ instead.

Now, if $P_1$ and $P_2$ shared any vertex $u$ of $G'$ other than $v$ and $w$, then we can divide each $P_1$ and $P_2$ into two halves at $u$ and then argue each piece separately. Thus we will assume that $P_1$ and $P_2$ do not share any common vertex (on $G'$) except for $v$ and $w$. Then, the claimed consistency result follows from summing the flow conservation equations for vertices of $G$ lying in the bounded region between the paths $P_1$ and $P_2$ (considered as curves in the plane). Indeed, edges of $G$ lying inside the region get canceled out, and what remains are edges passing through either curve. We omit the details of the verification (since it is easier to do it yourself than to write it down). See Figure 7.

Finally, after all the colors have been assigned, we can produce a $k$-coloring of the faces by reducing these colors modulo $k$. Since no edge is assigned zero flow, by (†), no two adjacent faces have the same color. $\square$