18.438 Advanced Combinatorial Optimization September 17, 2009

Lecture 3

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(* Based on notes by Robert Kleinberg and Dan Stratila.)

In this lecture, we will:

- Present the Edmonds-Gallai decomposition of a graph,
- Sketch some results regarding ear-decompositions and factor-critical graphs.

1 Edmonds-Gallai decomposition

In the previous lectures we presented Edmonds' algorithm for computing a maximal matching in a (not necessarily bipartite) graph. We also proved the following theorem

Theorem 1 (Tutte-Berge Formula) *For a graph G and a set of vertices $U \subseteq V(G)$, let $o(G \setminus U)$ denote the number of odd components of the graph $G \setminus U$, i.e. the number of components with an odd number of vertices. Then the cardinality of a maximum size matching, $\nu(G)$, satisfies:*

$$\nu(G) = \min_{U \subseteq V} \frac{1}{2} \left[|V| + |U| - o(G \setminus U) \right]. \tag{1}$$

Today, we will show that the proof of correctness of Edmonds' algorithm can be used to exhibit an interesting structure of graphs that is captured in the following theorem (which, as a by-product, gives another proof of the Tutte-Berge formula):

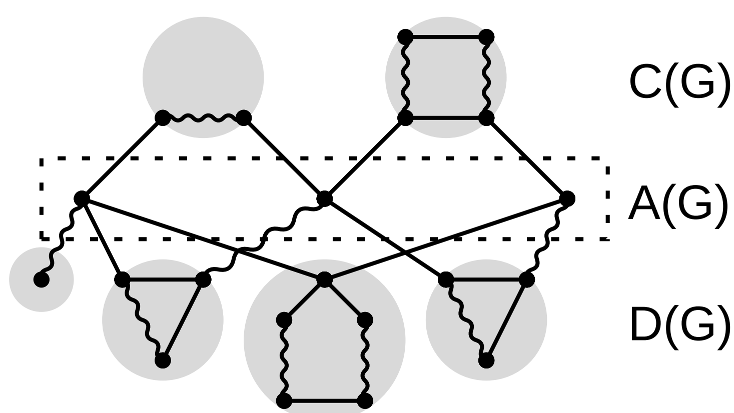

Figure 1: The Edmonds-Gallai Decomposition.

Theorem 2 (Edmonds-Gallai Decomposition) *Given a graph G, let*

$$
\begin{aligned}
D(G) &:= \{v : \text{there exists a maximum size matching missing } v\}, \\
A(G) &:= \{v : v \text{ is a neighbor of some } u \in D(G), \text{ but } v \notin D(G)\}, \\
C(G) &:= V(G) \setminus (D(G) \cup A(G)) .
\end{aligned}
$$

Then:

(i) $U = A(G)$ *achieves the minimum on the right side of the Tutte-Berge formula,*

(ii) $C(G)$ *is the union of the even-sized components of* $G \setminus A(G)$*,*

(iii) $D(G)$ *is the union of the odd-sized components of* $G \setminus A(G)$*,*

(iv) Every odd-sized component of $G \setminus A(G)$ *is factor-critical. (A graph* H *is* factor-critical *if for every vertex* v*, there is a matching in* H *whose only unmatched vertex is* v*.)*

To prove this theorem for a given graph G, let us consider the maximum-size matching M that is returned by Edmonds' algorithm executed on G. Let X be the set of vertices not matched by M.

Consider all the vertices which can be reached by an alternating path from $x \in X$. The first edge on such a path must lie outside of M, the second edge must lie in M, and so on, leading to a picture as in Figure 2.

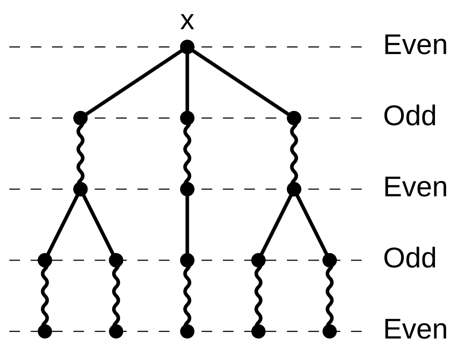

Figure 2: Vertices reachable by alternating paths from $x \in X$.

Motivated by this picture, we define the following three subsets of $V(G)$:

$$
\begin{aligned}
\textsf{Even} &:= \{v : \exists \text{ an alternating path of even length from } X \text{ to } v\}, \\
\textsf{Odd} &:= \{v : \exists \text{ an alternating path from } X \text{ to } v\} \setminus \textsf{Even}, \\
\textsf{Free} &:= \{v : \nexists \text{ an alternating path from } X \text{ to } v\}.
\end{aligned}
$$

We will sometimes refer to a vertex as being even, odd, or free, according to which of these sets it belongs to. Note that in the above definitions we are interested in alternating *paths* i.e. alternating walks in which all the vertices are distinct.

We start with the following claim.

Claim 3 *If there is an edge from a vertex* $u \in \textsf{Even}$ *to some* v*, then there is an alternating walk of odd length from* X *to* v*, and there is an alternating path from* X *to* v*.*

Proof: If $e = (u, v)$ is the edge in question, and P is an alternating path of even length from X to u, then an alternating walk of odd length from X to v is constructed as follows. If $e \in M$, then we take P and delete the final edge, which is necessarily e. If $e \notin M$, then we append e to P. If this alternating walk is not a path, it can only be because v lies on P, in which case P contains a sub-path which is an alternating path from X to v. □

As a result, any vertex adjacent to a vertex from Even has to belong to either Even or Odd. This gives us the following corollary.

Corollary 4 *In G there is no edge between* Even *and* Free.

Let us now define the *shrunk graph* G_0 to be the graph obtained in the final iteration of the execution of Edmonds' algorithm on G. More precisely, G_0 is the final graph obtained from G by repeated shrinking of blossoms performed during the course of the algorithm. Let M_0 be the maximum size matching in G_0 computed by the algorithm – M_0 is just the matching M from which the edges of the blossoms shrunk in G_0 have been removed. Note that the set of the vertices of G_0 that are unmatched in M_0 is still X. Notice also that all vertices of a blossom become even whenever we expand them, since the stem is an even-length alternating path from X to the base v of the blossom, and all other vertices of the blossom are reachable from v by an even-length alternating path which goes around the blossom in one of the directions (as it is odd).

Also, we claim that the vertices in $V(G_0)$ have the same classification (as even, odd, or free) no matter whether we classify them with respect to G_0 and M_0, or G and M. Indeed, first consider an alternating path (of even or odd parity) from X to v in G_0. As we expand blossoms, if our alternating path went through the shrunk blossom then we can easily update the alternating path into the expanded graph *without modifying the parity of its length* as the alternating path will be entering the blossom through its base. Conversely, if we have an alternating path P in G from X to a vertex v which intersects a blossom B then consider the first time P visits a vertex of the corresponding flower. We can now replace the this prefix of P with part of the flower in such a way that we still have an alternating path and the parity of the length of the path has not changed.

By properties of the algorithm, G_0 has no flowers, and M_0 is a maximum matching in G_0. Therefore, G_0 has no alternating walk from X to X – if such walk existed then from the previous lecture we would know that there is either an augmenting path or a flower in G_0. This fact implies the following

Claim 5 *In G_0, there is no edge between two even vertices.*

Proof: If such an edge $e = (u, v)$ existed, then by Claim 3, G_0 contained an alternating walk P of odd length from X to v. But v is even, so there would also be an alternating path P' of even length from X to v. Concatenating P with the reverse of P', we would obtain an alternating walk from X to X, contradicting the definition of G_0. □

It is worth noting that Claim 5 doesn't necessarily hold in G. This is because, as we already mentioned above, all the vertices of a blossom are even.

We are now ready to prove that the sets Even, Odd, and Free coincide with the sets $D(G)$, $A(G)$, and $C(G)$ from definition of Edmonds-Gallai decomposition.

Claim 6 Even $= D(G) = \{v : \exists$ *a maximum-size matching missing* $v\}$.

Proof:

($\subseteq$) Certainly if v is even then there is a maximum-size matching M' missing v. Such a matching is obtained by taking an even-length alternating path P from X to v and putting $M' = M \triangle P$.

($\supseteq$) Conversely, if there exists a maximum-size matching M' missing v, then $M \triangle M'$ is a union of even-length cycles and even-length paths, and v is an endpoint of one of these paths, because it does not belong to an edge of M'. The other endpoint of this path P does not belong to an edge of M, i.e. it is an element of X. This confirms that P is an even-length alternating path from X to v. □

Claim 7 $\mathsf{Odd} = A(G) = \{v : v \text{ is a neighbor of some } u \in D(G), \text{ but } v \notin D(G)\}$.

Proof:
($\subseteq$) If v is odd, then there is an alternating path of odd length from X to v. The vertex preceding v on this path must be even, thus v is a neighbor of some vertex from Even. Moreover, since it is odd then it is not in Even. But by Claim 6 $\mathsf{Even} = D(G)$, so indeed $v \in A(G)$.

($\supseteq$) The reverse inclusion follows from Claim 3, which ensures that every vertex adjacent to Even belongs to $\mathsf{Even} \cup \mathsf{Odd}$, which in conjunction with Claim 6 gives us that $v \in \mathsf{Odd}$. □

Claim 8 $\mathsf{Free} = C(G) = V(G) \setminus (D(G) \cup A(G))$.

Proof: Immediate from the definition of Free, and from the preceding two claims which identify $\mathsf{Even}, \mathsf{Odd}$ with $D(G), A(G)$, respectively. □

We proceed to proving the desired properties of the decomposition. We start with property (ii) which is directly implied by the following claim asserting that not only all the vertices of $C(G)$ are matched in M, but also the edges matching them are always connecting two free vertices.

Claim 9 $|M \cap C(G)| = |C(G)|/2$.

Proof: Consider some v from $C(G)$ (which is equal to Free by Claim 8). By Corollary 4 we know that v cannot be adjacent to any even vertex, so $C(G)$ is disconnected from $D(G)$ in $G \setminus A(G)$. Moreover, v has to be matched by some edge $e = (v, u)$ in M, otherwise it would be even. However, u cannot be odd, since then we could augment the odd-length path from X to u by e which would imply that v is either odd or even. Therefore, we must have u being free as well. This implies that $M \cap C(G)$ matches all the vertices of $C(G)$ and thus has the desired size. □

To establish properties (iii) and (iv) we prove the following claim.

Claim 10 *For every connected component H of $(G \setminus A(G)) \cap D(G)$:*

(a) either $|X \cap H| = 1$ and $|M \cap \delta(H)| = 0$; or $|X \cap H| = 0$ and $|M \cap \delta(H)| = 1$, where $\delta(H)$ is the set of edges with exactly one endpoint in H.

(b) H is factor-critical.

Proof: The proof is by induction on the number of blossoms which are shrunk during the execution of Edmonds' algorithm. If no blossoms are shrunk, then $G = G_0$ and the claim follows as a consequence of Corollary 4 and Claim 5 that assert that $(G \setminus A(G)) \cap D(G)$ is a union of isolated vertices (for which both (a) and (b) trivially hold).

Now for the induction step, suppose B is a blossom in G and that the claim holds for G/B (in which B is shrunk). In this case, B corresponds to a vertex $b \in G/B$ which has to be even, since the stem of the flower containing B corresponds to an even-length alternating path from X to b in G/B. In fact, as it was already mentioned before, in G all vertices of B are even and they have all to be in the same connected component, say H_b, of $(G \setminus A(G)) \cap D(G)$.

Clearly, since the vertices of $B \setminus \{b\}$ are all matched in M by edges inside B, neither the size of $M \cap \delta(H_b)$ nor the size of $X \cap H_b$ can increase as a result of expanding B in G/B. Thus, we see that (a) holds.

Now to prove (b), we note that, by inductive assumption, all connected components of $(G \setminus A(G)) \cap D(G)$ other than H_b are factor-critical. Thus, it remains to show that H_b is factor-critical as well. To this end, assume that some vertex $v \in H_b$ was removed. If $v \notin B$ then, by inductive assumption, we know that there exists a matching M' in H_b that matches all vertices of H_b except v and $B \setminus \{b\}$. But then M' can be straight-forwardly augmented inside B to match all vertices in $B \setminus \{b\}$. Similarly, if $v \in B$ then we know that there is a matching M'' that matches all vertices of

H_b except B – this correspond to situation in which we remove b in G/B. But, if we remove any vertex of a blossom the rest of them can be easily matched within B, thus once again giving raise to matching that matches the whole $H_b \setminus \{v\}$. This concludes the proof.

□

Having proved Claim 10, property (*iii*) follows since each factor-critical graph has to be odd-sized, and property (*iv*) is implied by Claim 9 which shows that all odd-sized connected components of $G \setminus A(G)$ are in $D(G)$.

Finally, we prove property (*i*).

Claim 11 $|M| = \frac{1}{2}\left[|V| + |A(G)| - o(G \setminus A(G))\right]$.

Proof: We only need to show that $|M| \geq \frac{1}{2}\left[|V| + |A(G)| - o(G \setminus A(G))\right]$. Observe that

$$|M| \geq |M \cap C(G)| + |M \cap E(D(G))| + |M \cap \delta(A(G))|.$$

By Claim 9, the first term is $|C(G)|/2$. By Claim 10 part (a), the second term is $\frac{|D(G)|-o(G\setminus A(G))}{2}$ while the third term is $|A(G)|$ since every vertex of $A(G)$ is matched to a vertex of $D(G)$. Thus,

$$|M| \geq \frac{1}{2}(|C(G)| + |D(G)| + 2|A(G)| - o(G \setminus A(G))) = \frac{1}{2}\left[|V| + |A(G)| - o(G \setminus A(G))\right],$$

proving the claim. □

2 Ear-decompositions

An *ear decomposition* $G_0, G_1, \ldots, G_k = G$ of a graph G is a sequence of graphs with the first graph being simple (e.g. a vertex, edge, even cycle, or odd cycle), and each graph G_{i+1} obtained from G_i by *adding an ear*. Adding an ear is done as follows: take two vertices a and b of G_i and add a path P_i from a to b such that all vertices on the path except a and b are new vertices (present in G_{i+1} but not in G_i). An ear with $a \neq b$ is called proper (or open), and an ear with P_i having an odd (even) number of edges is called odd (even). (See Figure 2.) Several basic properties of graphs can be translated into the existence of an ear decomposition of a certain kind. Here are some examples.

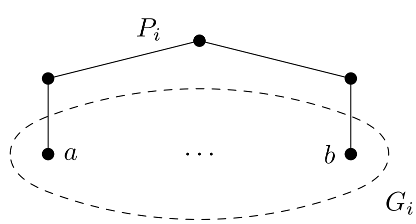

Figure 3: An even proper ear added to G_i.

Theorem 12 (Robbins, 1939 (implicit)) *G is 2-edge-connected if and only if G has an ear decomposition starting from a cycle.*

Theorem 13 (Whitney, 1932) *G is 2-connected if and only if G has a proper ear decomposition starting from a cycle.*

Proof: Obviously, any graph that has a proper ear decomposition starting from a cycle is 2-connected.

Conversely, we assume G is 2-connected, and will show by induction how to construct it starting from a cycle. First, since G is 2-connected, it contains at least one cycle, which we can take as the initial cycle.

Now, suppose we have constructed a subgraph G' of G. If $V(G') = V(G)$ and we are only missing edges, then we can add these edges as proper ears of length one. If $V(G') \subset V(G)$, then pick a vertex $v \in V(G) \setminus V(G')$. Since G is connected, there is a path P from some $a \in V(G)$ to v; since G is 2-connected, there is a path Q distinct from P from v back to some vertex $b \in V(G'), b \neq a$. Hence the paths P and Q form a proper ear from a to b containing at least one new vertex. □

Theorem 14 (Lovász, 1972) *G is factor-critical if and only if G has an odd ear decomposition starting from an odd cycle.*

Proof: If G has an odd ear decomposition, then it is factor critical, since blossoming yields a factor critical graph.

Conversely, suppose G is factor-critical. First, we establish the existence of an initial odd cycle. For any v, fix a near-perfect matching M_v that misses v. Then for an edge (u, v) the existence of M_u and M_v implies there is an alternating even path from v to u. By adding (u, v) to it we obtain an odd cycle.

Fix a vertex v. We proceed by induction; let H be the vertex set already covered by the odd ear decomposition such that no edge in M_v crosses H. Since G is connected, there is an edge $(a, b), a \in H, b \notin H, (a, b) \notin M_v$. Moreover, $M_b \triangle M_v$ contains an alternating path Q from b back to v. The first edge (w, u) to cross back into H on Q is not in M_v, by the construction of H. Therefore, we obtain an odd path from b to u, and can increase the size of H. □

The two results can be combined. One can show that G is factor-critical and 2-connected if and only it has a proper ear decomposition starting from an odd cycle.

We conclude with the following theorem

Theorem 15 *Let G be a 2-connected factor-critical graph. Then the number of near-perfect matchings is at least $|E(G)|$.*

Proof: We proceed by induction on the number of odd ears. Consider a graph G', and G obtained from G' by adding an odd ear $P = (u_0, \ldots, u_k)$ of k edges. Then $|V(G)| = |V(G')| + k - 1, |E(G)| = |E(G')| + k$.

We can obtain $|E(G')|$ near-perfect matchings by taking $(u_1, u_2), \ldots, (u_{k-2}, u_{k-1})$ into the matching, and then generating $|E(G')|$ near perfect matchings in G'. Moreover, we can obtain $k - 1$ by matching all vertices on P except $u_j, j = 1, \ldots, k$, and then taking a near-perfect matching on G' that misses either u_0 (if j is odd) or u_k (if j is even). The final matching is obtained by taking the matching missing u_k, but not u_0, removing the edge matching u_k in G' and adding the edge matching u_k in P. □