18.438 Advanced Combinatorial Optimization	November 12, 2009
Lecture 19	
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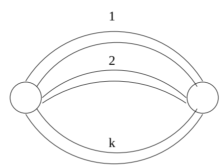
(These notes are based on notes by Jan Vondrák and Mohammed Mahdian.)

In the last lecture, we showed that every 2k-edge-connected graph has a k-arc-connected orientation. The proof was based on matroid intersection. In this lecture, we derive a result on the construction of 2k-edge-connected graphs that allows us to derive a different proof for the above orientation theorem.

In the second part of this lecture, we discuss minimization of submodular functions.

1 k-edge-connectivity

Theorem 1 *Let M_{2k} denote the graph consisting of two vertices that are connected by $2k$ parallel edges.*



Then any $2k$-connected multigraph $G = (V, E)$ can be built from M_{2k} by repeatedly

(i) adding edges

(ii) pinching k edges $(u_i, v_i) \in E$, $i = 1, \ldots, k$; that is, adding a new vertex s to V and replacing each edge (u_i, v_i) by (u_i, s) and (v_i, s).

In order to prove this theorem we need two lemmata. By minimally k-edge-connected, we mean a graph for which the removal of any edge loses the k-edge-connectivity of the graph.

Lemma 2 *Every minimally k-edge-connected graph $G = (V, E)$ has a vertex of degree k.*

Proof: Let $S \subseteq V$ be minimal such that $d(S) := |\delta(S)| = k$. Such S exists, since G is minimally k-edge-connected. If $|S| = 1$, then there is nothing to show. Suppose $|S| \geq 2$. Since $G[S]$ is connected, there exists an edge $e = (u, v) \in E(S)$. Because of the minimally k-edge-connectivity of G, there must exist a cut $\delta(T)$ of size k cutting e; i.e., $u \in T, v \notin T$ and $d(T) = k$. Note that $u \in S \cap T$, i.e., $S \cap T \neq \emptyset$. By submodularity of the cut function

$$d(S \cap T) + d(S \cup T) \leq d(S) + d(T) = k + k \ .$$

If $S \cup T \neq V$, then k-edge-connectivity of G implies $d(S \cap T) = k$. If $S \cup T = V$, then

$$\delta(S \setminus T) = \delta(T) \ ,$$

implying $d(S \setminus T) = k$. In both cases we have a contradiction to the minimality of S. $\square$

The second lemma describes a technique developed by Lovász which is very useful for connectivity augmentation and other questions concerning edge-connectivity.

Lemma 3 (*Splitting-Off-Lemma*) *Let $G = (V + s, E)$ be an undirected graph such that s has even degree and $d(s) \geq 2$. Assume that*

$$\forall \emptyset \neq U \subset V : d(U) = |\delta(U)| \geq k \ , \tag{1}$$

where $k \geq 2$. Then for every edge $(s,t) \in E$ there exists an edge $(s,u) \in E$ such that the graph $G(V + s, E')$ with $E' = E \setminus \{(s,t),(s,u)\} \cup \{(t,u)\}$ also satisfies (1).

Proof: Suppose that for some edge $(s,t) \in E$ there is no edge $(s,u) \in E$ such that $G = (V + s, E \setminus \{(s,t),(s,u)\} \cup \{(t,u)\})$ satisfies property (1). Let N denote the set of neighbors of s, i.e., $N = \{u \in V \,|\, \exists (s,v) \in E\}$. Then for every neighbor $u \in N$ there exists a set $U \subset V$ with $d'(U) < k$, where d' denotes the degree w.r.t. edge set E'. It has to hold that $u, t \in U$ and $d(U) \in \{k, k+1\}$ (see Figure 1). Now consider a minimal collection $\mathcal{C}$ of sets $U \subset V$ with $t \in U$ and $d(U) \leq k+1$ that covers N. For every $U \in \mathcal{C}$ we derive from $d(V \setminus U) \geq k$ (because of property (1) for G) that

$$\begin{aligned} 1 &\geq d(U) - d(V \setminus U) \\ &= (d(s,U) + d(U, V \setminus U)) - (d(V \setminus U, U) + d(s, V \setminus U)) = d(s,U) - d(s, V \setminus U) \ . \end{aligned}$$

Because $d(s,U) - d(s, V \setminus U)$ is an even integer (since their sum is even), we get $d(s,U) - d(s, V \setminus U) \leq 0$, which implies $d(s,U) \leq \frac{1}{2} d(s)$.

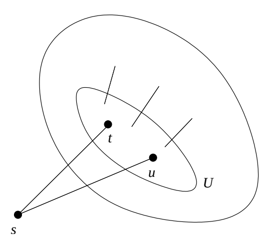


Figure 1: A set U with $d(U) \leq k+1$ and $u \in U$

The last observation clearly implies $|\mathcal{C}| \geq 2$. However, $|\mathcal{C}| = 2$ is not possible. If $\mathcal{C} = \{U_1, U_2\}$, then

$$|N \cap (U_1 \cup U_2)| \leq |N \cap U_1| + |N \cap U_2| - 1 = |N| - 1 \ ,$$

and $\mathcal{C}$ cannot cover N. Therefore, $\mathcal{C}$ contains at least three sets U_1, U_2, U_3 such that

$$\begin{aligned} t \in U_1 \cap U_2 \cap U_3& \\ U_1 \setminus (U_2 \cup U_3) &\neq \emptyset \\ U_2 \setminus (U_1 \cup U_3) &\neq \emptyset \\ U_3 \setminus (U_1 \cup U_2) &\neq \emptyset \ . \end{aligned}$$

It is easy to show (by looking at the contribution of every edge) that, for any U_1, U_2, U_3 with the above properties, *3-way-submodularity* holds, i.e.,

$$\begin{aligned} d(U_1) + d(U_2) + d(U_3) \geq d(U_1 \cap U_2 \cap U_3) \quad + \quad & d(U_1 \setminus (U_2 \cup U_3)) \\ & d(U_2 \setminus (U_1 \cup U_3)) + d(U_3 \setminus (U_1 \cup U_2)) \ . \end{aligned}$$

In our special case with $(s,t) \in E$ we can strengthen the above inequality to

$$\begin{aligned} d(U_1) + d(U_2) + d(U_3) \geq d(U_1 \cap U_2 \cap U_3) \quad + \quad & d(U_1 \setminus (U_2 \cup U_3)) \\ & d(U_2 \setminus (U_1 \cup U_3)) + d(U_3 \setminus (U_1 \cup U_2)) + 2 \ . \end{aligned}$$

This is, since (s,t) is counted three times on the left-hand side and only once on the right-hand side. With $d(U_i) \leq k+1$, $i = 1,2,3$, and property (1) for G, we obtain $3k+3 \geq 4k+2$, implying $k \leq 1$. This contradicts the assumption that $k \geq 2$. □

The last lemma states that we can "split off" a vertex s of even degree by replacing certain pairs of edges incident to s by other edges while preserving k-edge-connectivity between all vertices other than s. Now we will demonstrate its application to the construction of all $2k$-edge-connected graphs.

Proof of Theorem 1: Let $G = (V,E)$ be a 2k-edge-connected graph. We will show that by a sequence of removing edges and splitting-off vertices we obtain M_{2k}. Since these operations are the reverse operations to (i) *adding* and (ii) *pinching* edges, the statement in the theorem follows.

Starting from G, we can remove edges until there exists a vertex s of degree $2k$. The existence of such vertex is guaranteed by Lemma 2. Then by applying Lemma 3 k times, we can remove vertex s while preserving 2k-edge-connectivity. Repeating this procedure we can shrink G to a graph G' that has two vertices only and that is 2k-edge-connected. Consequently, $G' = M_{2k}$. □

Remark 1 *Theorem 1 gives another proof that any $2k$-edge-connected graph G has a k-arc-connected orientation. Starting from M_{2k} with k edges oriented each way, we build G by (i) adding edges with an arbitrary orientation and (ii) pinching edges, where an arc is replaced by two arcs oriented the same way. This procedure preserves k-arc-connectivity.*

2 Submodular function minimization

Definition 1 *A set function $f : 2^V \to \mathbb{Z}$ is called submodular if*

$$\forall\, A, B \subseteq V : f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \ .$$

Equivalently,

$$\forall\, A \subseteq B \quad \forall\, j \notin B : f(A+j) - f(A) \geq f(B+j) - f(B) \ .$$

Examples of submodular functions are the rank function of a matroid, the cut function $d(S) = |\delta(S)|$ of an undirected graph, and the cut function $d^+(S) = |\delta^+(S)|$ of a directed graph. Also, given random variables $X_1, \ldots, X_n$, the entropy function

$$H(S) = - \sum_{x_i : i \in S} p(X_i = x_i \,\forall\, i \in S) \log p(X_i = x_i \,\forall\, i \in S)$$

is submodular (here, $V = \{1, \ldots, n\}$, $S \subseteq V$). Finally, given vectors $a_1, \ldots, a_n \in \mathbb{R}^n$ in general position,

$$f(S) = \log Vol\big(\{ \sum_{i \in S} \lambda_i a_i \,|\, 0 \le \lambda_i \le 1 \quad \forall\, i \in S\}\big)$$

is a submodular function (the volume is taken in the appropriate affine dimension) .

Problem Statement. Let $f : 2^V \to \mathbb{Z}$ be a submodular function given by an oracle. The task is to find a set $S \subseteq V$ that minimizes $f(S)$ over all subsets S of V. We can assume without loss of generality that $f(\emptyset) = 0$ (by adding a constant, if necessary).

This problem has many applications. As an example, consider the matroid intersection problem that we discussed in previous lectures. We know that

$$\max_{U \in \mathcal{I}_1 \cap \mathcal{I}_2} |U| = \min_{T \subseteq S} (r_1(T) + r_2(S \setminus T)) \ .$$

Since the rank function of a matroid is submodular and the sum of two submodular function is submodular, finding a set of maximum size that is independent for both matroids is equivalent to minimizing a submodular function. As another example, for the Shannon switching game, the opimum strategy required being able to minimize $2r(X) - |X|$ over all sets X.

The obvious way to minimize any set function is to evaluate f for all possible sets. However, this requires an exponential number of oracle calls. If the function f has no particular structure, then there is no better way to find the minimum than calling the oracle on each of the $2^{|V|}$ sets. However, in the case of submodular functions several algorithms that use only a polynomial number of calls to the oracle have been developed. A first important question that arises in this context is the question of a compact certificate of optimality. That is, given $S \subseteq V$ that minimizes f over all subsets of V, how can we certify that S is indeed a minimizing set?

Definition 2 *Let $f : 2^V \to \mathbb{Z}$ be a submodular function such that $f(\emptyset) = 0$. We define the submodular polyhedron of f by*

$$P(f) = \{x \in \mathbb{R}^V : x(S) \le f(S) \quad \forall S \subseteq V\} \ ,$$

and the base polyhedron of f by

$$B(f) = \{x \in \mathbb{R}^V : x \in P(f),\, x(V) = f(V)\} \ .$$

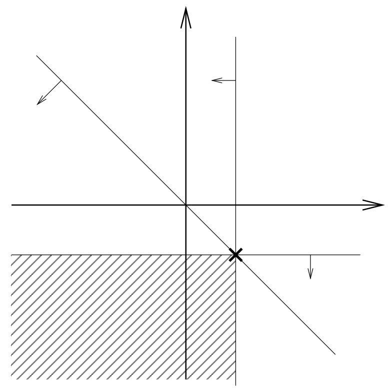


Figure 2: $P(f)$ for $V = \{1, 2\}$, $f(\emptyset) = 0, f(\{1\}) = 1, f(\{2\}) = -1, f(\{1, 2\}) = 0$

Notice that this definition does not require $x \geq 0$. Figure 2 gives an example of the above definitions. The shaded area shows $P(f)$, the point marked by a cross represents $B(f)$.

A main insight for deriving a polynomial-size certificate of optimality is given in the following theorem.

Theorem 4 *Let $f : 2^V \to \mathbb{Z}$ be a submodular function such that $f(\emptyset) = 0$. Then*

$$\min_{S \subseteq V} f(S) = \max\{x^-(V) \,|\, x \in B(f)\} \ ,$$

where $x^-(U) = \sum_{v \in U} x^-(v) = \sum_{x \in U} min(0, x_v)$.

The general idea for a certificate is to use an optimal solution x of the above convex program (we are maximizing a concave objective function over a polyhedral set) and show that $f(S) = x^-(V)$. The difficulty is to show efficiently that $x \in B(f)$. This issue will be discussed in detail in the next lecture. Here, we only outline the basic idea:

- Define a linear order L on V.
- For all $u \in V$, define $L(u) := \{v \in V : v \leq_L u\}$ and $y_L(u) := f(L(u)) - f(L(u) \setminus \{u\})$.
- Show that for all $u \in V : y_L(L(u)) = f(L(u))$ is an extreme point of $B(f)$.
- Show that for all extreme points x of $B(f)$ there exists an order L such that $x = y_L$.
- Provide a set of linear orders $L_1, \ldots, L_k$ (polynomially many) and corresponding multipliers $0 \leq \lambda_i \leq 1$, $i = 1, \ldots, k$ such that $x = \sum_{i=1}^{k} \lambda_i \, y_{L_i}$, i.e., x is a convex combination of the associated extreme points of $B(f)$. That will be our certificate that $x \in B(f)$.