**18.438 Advanced Combinatorial Optimization** November 10th, 2009

# Lecture 16

*Lecturer: Michel X. Goemans* *Scribe: José Soto*

The lecture started with some additional discussion of matroid matching and this was included in the previous scribe notes.

## 1 Graph Orientations

We first introduce some notation and definitions. Let $G = (V, E)$ be an undirected graph. Recall that for a non-empty subset $U \subset V$, the notation $\delta_G(U)$ denotes the set of edges with one endpoint in $U$ and the other endpoint in $V \setminus U$.

**Definition 1** *Let $\lambda_G(u, v)$ denote the maximum number of edge-disjoint $u$-$v$ paths in $G$. We say that $G$ is* $k$-edge-connected *if $\lambda_G(u, v) \geq k$ for all $u \neq v \in V$. An equivalent statement is that each cut contains at least $k$ edges, i.e., $|\delta_G(U)| \geq k$ for all non-empty $U \subset V$.*

Let $D = (V, A)$ be a directed graph. For a non-empty subset $U \subset V$, $\delta^+_D(U)$ is the set of arcs with their tail in $U$ and head in $V \setminus U$, and $\delta^-_D(U)$ is the set of arcs in the reverse direction.

**Definition 2** *Let $\lambda_D(u, v)$ denote the maximum number of edge-disjoint directed paths in $D$ from $u$ to $v$. We say that $D$ is* $k$-arc-connected *if $\lambda_D(u, v) \geq k$ for each $u, v \in V$. An equivalent statement is that $|\delta^+_D(U)| \geq k$ for all non-empty $U \subset V$. A digraph that is 1-arc-connected is also called strongly connected.*

An *orientation* of a graph $G$ is a digraph obtained by choosing a direction for each edge of $G$. We now give some results relating edge-connectivity of $G$ to arc-connectivity of orientations of $G$.

**Theorem 1 (Robbins, 1939)** *$G$ is 2-edge-connected $\iff$ there exists an orientation $D$ of $G$ that is strongly connected.*

**Proof:** $\Leftarrow$: Fix a strongly-connected orientation $D$. For any non-empty $U \subset V$, we may choose $u \in U$ and $v \in V \setminus U$. Since $D$ is strongly connected, there is a directed $u$-$v$ path and a directed $v$-$u$ path. Thus $|\delta^+_D(U)| \geq 1$ and $|\delta^-_D(U)| \geq 1$, implying $|\delta_G(U)| \geq 2$.

$\Rightarrow$: Since $G$ is 2-edge-connected, it has an ear decomposition. We proceed by induction on the number of ears. If $G$ is a cycle then we may orient the edges to form a directed cycle $D$, which is obviously strongly connected. Otherwise, $G$ consists of an ear $P$ and subgraph $G'$ with a strongly connected orientation $D'$. The ear is an undirected path with endpoints $x, y \in V(G')$ (possibly $x = y$). We orient $P$ so that it is a directed path from $x$ to $y$ and add this to $D'$, thereby obtaining an orientation $D$ of $G$.

To show that $D$ is strongly connected, consider any $u, v \in V(G)$. If $u, v \in V(G')$ then by induction there is a $u$-$v$ dipath. If $u \in P$ and $v \in V(G')$ then there is a $u$-$y$ dipath and by induction there is a $y$-$v$ dipath. Concatenating these gives a $u$-$v$ dipath. The case $u \in V(G')$ and $v \in P$ is symmetric. If both $u, v \in P$ then either a subpath of $P$ is a $u$-$v$ path, or there exist a $u$-$y$ path, a $y$-$x$ path, and a $x$-$v$ path. (The $y$-$x$ path exists by induction). Concatenating these three paths gives a $u$-$v$ path. □

The natural generalization of this theorem also holds.

**Theorem 2 (Nash-Williams, 1960)** *$G$ is $2k$-edge-connected $\iff$ there exists an orientation $D$ of $G$ that is $k$-arc-connected.*

We will prove this using matroid intersection. Let $G = (V, E)$ be a $2k$-edge-connected graph and let $D = (V, A)$ denote the bidirected version of $G$, with two arcs $(u, v)$ and $(v, u)$ for each edge $\{u, v\}$. (All graphs in this lecture can be multigraphs.) We define two matroids on the ground set of arcs $A$. The first one is a partition matroid:

$$\mathcal{M}_1 = (A, \{B \subseteq A : \forall \text{ edge } \{u, v\} \in E; B \text{ contains at most one of the arcs } (u, v), (v, u)\}).$$

The bases of $\mathcal{M}_1$ are exactly the orientations of $G$. The second matroid, which will force the orientation to be $k$-arc-connected, is more involved. Define

- $H(U) = \{(v, u) \in A : u \in U\}$.
- $\mathcal{C} = \{H(U) : \emptyset \subset U \subset V\}$.
- $f(H(U)) = |E(U)| + |\delta(U)| - k = |E| - |E(V \setminus U)| - k$.

In other words, $H(U)$ is the set of arcs with their "head" in $U$ (either crossing the cut into $U$ or contained inside $U$), and $f(H(U))$ is the maximum number of edges oriented like this, so that $k$ arcs leaving $U$ are still available. We need the following definitions.

**Definition 3** *A family of sets $\mathcal{C} \subseteq 2^A$ is a* crossing family *if for all $H_1, H_2 \in \mathcal{C}$ with $H_1 \cap H_2 \neq \emptyset$ and $H_1 \cup H_2 \neq A$, both $H_1 \cup H_2$ and $H_1 \cap H_2$ are also in $\mathcal{C}$.*

**Definition 4** *Let $\mathcal{C}$ be a crossing family on $2^A$. A nonnegative function $f : \mathcal{C} \to \mathbb{Z}_+$ is* crossing submodular *on $\mathcal{C}$ if for all $H_1, H_2 \in \mathcal{C}$, with $H_1 \cap H_2 \neq \emptyset$ and $H_1 \cup H_2 \neq A$,*

$$f(H_1) + f(H_2) \geq f(H_1 \cup H_2) + f(H_1 \cap H_2).$$

The family $\mathcal{C}$ defined before is indeed a crossing family. This is simply because $H(U_1) \cap H(U_2) = H(U_1 \cap U_2)$ and $H(U_1) \cup H(U_2) = H(U_1 \cup U_2)$. Also, the function $f(H(U)) = |E| - |E(V \setminus U)| - k$ is crossing submodular on $\mathcal{C}$ since

$$|E(V \setminus U_1)| + |E(V \setminus U_2)| \leq |E(V \setminus (U_1 \cap U_2))| + |E(V \setminus (U_1 \cup U_2))|,$$

and so $f(H_1 \cap H_2) + f(H_1 \cup H_2) \leq f(H_1) + f(H_2)$. Given these properties, we shall prove that

$$\mathcal{M}_2 = (A, \{B \subseteq A : |B| \leq |E| \text{ and } \forall H \in \mathcal{C}; |B \cap H| \leq f(H)\})$$

is a matroid. This is implied by the following lemma.

**Lemma 3** *Let $\mathcal{C} \subseteq 2^A$ be a crossing family and $f : \mathcal{C} \to \mathbb{Z}_+$ a nonnegative crossing submodular function. Then for any $k \in \mathbb{Z}_+$,*

$$\mathcal{B} = \{B \subseteq A : |B| = k \text{ and } \forall H \in \mathcal{C}; |B \cap H| \leq f(H)\}$$

*are the bases of a matroid.*

**Proof:** We can prove this by checking that the exchange axiom holds. Let $B_1, B_2 \in \mathcal{B}$, and $i \in B_1 \setminus B_2$. We need to prove that there exists $j \in B_2 \setminus B_1$ such that $B_1 - i + j \in \mathcal{B}$. Observe that if $B_1 - i + j \notin \mathcal{B}$, there must exist a set $H \in \mathcal{C}$, $|B_1 \cap H_j| = f(H)$, with $i \notin H$ and $j \in H$. Assume, by contradiction, that this holds for every $j \in B_2 \setminus B_1$.

For each $j \in B_2 \setminus B_1$, let $H_j \in \mathcal{C}$ be the maximal set such that $|B_1 \cap H_j| = f(H_j)$, $i \notin H_j$, and $j \in H_j$. We claim that these sets are either pairwise equal or disjoint. Indeed, if $H_j \neq H_{j'}$ and $H_j \cap H_{j'} \neq \emptyset$, we have, by crossing submodularity of $f$ and the definition of $\mathcal{B}$ that

$$\begin{aligned} |B_1 \cap (H_j \cup H_{j'})| + |B_1 \cap (H_j \cap H_{j'})| &= |B_1 \cap H_j| + |B_1 \cap H_{j'}| = f(H_j) + f(H_{j'}) \\ &\geq f(H_j \cup H_{j'}) + f(H_j \cap H_{j'}) \\ &\geq |B_1 \cap (H_j \cup H_{j'})| + |B_1 \cap (H_j \cap H_{j'})|. \end{aligned}$$

We deduce from here that $|B_1 \cap (H_j \cup H_{j'})| = f(H_j \cup H_{j'})$. But then, we can replace both $H_j$ and $H_{j'}$ by $H_j \cup H_{j'}$, which contradicts the maximality of both sets.

Let $\mathcal{P} = \{H_j : j \in B_2 \setminus B_1\}$ denote the collection of these disjoint sets, and $W = A \setminus \bigcup \mathcal{P}$ the set of remaining uncovered elements. For each $H_j \in \mathcal{P}$, we have $|B_2 \cap H_j| \leq f(H_j) = |B_1 \cap H_j|$. All the elements of $B_2 \setminus B_1$ are covered by $\mathcal{P}$, so $B_2 \cap W \subseteq B_1 \cap W$, and there is an element $i \in W$ which belongs to $B_1$ but not $B_2$. Therefore $|B_2 \cap W| < |B_1 \cap W|$ and $|B_2| < |B_1|$ which is a contradiction. □

Recall that the bases of $\mathcal{M}_1$ correspond to orientations of $G$ and that an orientation $I$ of $G$ is a base of $\mathcal{M}_2$ if and only if for every $\emptyset \subset U \subset V$, $|I \cap \delta_D^-(U)| \leq |\delta_G(U)| - k$. Or equivalently, if for every such $U$, $|I \cap \delta_D^+(U)| \geq k$. From here we get that the collection of $k$-arc-connected orientations of $G$ corresponds exactly to the set of common bases of $\mathcal{M}_1$ and $\mathcal{M}_2$. In particular, if one such base exists, it can be found using matroid intersection[1].

It remains to prove that there exists a base common to both matroids. Let $P(\mathcal{M}_1), P(\mathcal{M}_2)$ be the matroid polytopes of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, and $P(\mathcal{M}_1 \cap \mathcal{M}_2)$ be the convex hull of all indicator vectors of sets that are independent in both matroids. We have seen that the polytope $P(\mathcal{M}_1 \cap \mathcal{M}_2)$ is integral and equal to $P(\mathcal{M}_1) \cap P(\mathcal{M}_2)$. Consider the vector $x \in \mathbb{R}^A$ such that $x_a = 1/2$ for all $a \in A$. Since for every $\{u, v\} \in E$ we have

$$x_{uv} + x_{vu} = 1,$$

we can deduce that $x \in P(\mathcal{M}_1)$. Similarly, for every $\emptyset \subset U \subset V$, we have

$$x(H(U)) = |E(U)| + |\delta_G(U)|/2 \leq |E(U)| + |\delta_G(U)| - k,$$

where the last inequality comes from the fact that $|\delta_G(U)| \geq 2k$ which holds since $G$ is $2k$-edge connected. Since we also have $x(A) \leq |E|$, we can conclude that $x \in P(\mathcal{M}_2)$. But then, $x$ is a fractional vector in $P(\mathcal{M}_1 \cap \mathcal{M}_2)$ with total weight $x(A) = |A|/2 = |E|$. By the integrality of that polytope, $x$ can be written as a convex combination of sets $I$ that are independent in both matroids. This means that at least one (and hence all) of these sets has cardinality $|E|$ and, therefore, it is a base in both matroids.

As a final remark, we should point out that there exists a stronger orientation result due to Nash-Williams which states that any graph $G$ can be oriented into a digraph $D$ such that for all $u \neq v$, we have

$$\lambda_D(u, v) \geq \left\lfloor \frac{1}{2} \lambda_G(u, v) \right\rfloor.$$

[1] Provided that membership in $\mathcal{M}_2$ can be tested efficiently, which is not explained here.