Lecture 15

Lecturer: Michel X. Goemans *Scribe: Paul Christiano*

1 Matroid Matching

The Matroid matching Problem: Given a matroid $M = (S, \mathcal{I})$, let E be a set of pairs of elements of S. The matroid matching problem is to find a maximum set of djsoint pairs $F \subseteq E$, such that $\bigcup F \in \mathcal{I}$. The cardinality of this maximum matching F is denoted by $\nu(M)$.

The following are a few illustrations of the matroid matching problem.

Examples (Matroid matching):

1. Let M be the trivial matroid on a set S, i.e., $M = (S, 2^S)$. Let E be a collection of pairs on S which define a graph $G = (S, E)$. Then the matroid matching problem is equivalent to finding a maximum size matching in $G = (S, E)$.

2. Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids on the ground set S. Then the matroid intersection problem can be formulated using the matroid matching problem in the following manner. Let S' be an identical copy of S where for every $a \in S$ there is a corresponding $a' \in S'$. Define M_1 on S and M_2 on S', so that $\mathcal{I}_1$ is defined on S and $\mathcal{I}_2$ is defined on S'. Define M and E as follows.

$$M = (S \cup S', \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\})$$
$$E = \{(a, a') : a \in S\}$$

   With the above definition, the matroid matching problem for M is equivalent to finding a maximum independent set in $M_1 \cap M_2$.

3. Consider the graphic matroid $M(G)$ of a graph $G = (V, E)$. Partition the edge set E into pairs. Then the matroid matching problem is to find the maximum forest containing either both or neither element from each pair in the partition of E.

4. Finding a maximum forest in a 3-uniform hypergraph. Consider the problem of finding a maximum forest in a 3-uniform (every hyperedge connects 3 vertices) hypergraph. In other words, the problem is to find a maximum subgraph without cycles. Recall that a cycle in a hypergraph is a sequence of distinct hyperedges $h_1, h_2 \ldots, h_T$ such that there is a sequence of vertices $s_1, s_2 \ldots, s_T$ for which $s_i s_{i+1} \in h_i$ for $i = 1, 2 \ldots T$ (with $s_{T+1} = s_1$). Given a 3-uniform hypergraph G, define a graph G' on the same vertex set as G and a set E as follows. For each hyperedge (a, b, c), include edges (a, b) and (a, c) in G' (a is chosen arbitrarily) and include the pair $\{(a, b), (a, c)\}$ in E. Let M be the graphical matroid for G'. Then a matroid matching on (M, E) is a forest in G and conversely, since a set of hyperedges $\{(a, b, c)\}$ contains a cycle iff the set of edges $\{(a, b)\} \cup \{(b, c)\}$ contains a cycle.

1.1 Is the matroid matching problem solvable in polynomial time?

We will first construct an example to show that the matroid matching problem is not solvable in polynomial time when the matroid is given as an independence oracle (a black box which can determine whether a given $T \in \mathcal{I}$ is independent).

Let S be a finite set and let E be an arbitrary partition of S into pairs. Let the collection of independent sets be as follows:

$$\mathcal{I} = \{I : I \leq 2k - 1\} \cup \{I : I = 2k, I \text{ is not a union of } k \text{ pairs in } E\}.$$

It is easy to check that $M = (S, \mathcal{I})$ is a matroid. Let $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$. If $I_1 \leq 2k - 1$, then I_2 can be trivially augmented using elements from $I_1 \setminus I_2$. If $I_1 = 2k$, then I_1 intersects at least $k+1$ pairs in E; either I_2 already intersects at least $k + 1$ pairs, in which case we can trivially augment I_2, or I_1 intersects a pair which I_2 does not, in which case we can augment I_2 by an element of this intersection.

Note that $\nu(M) = k - 1$. Now take any $F \subseteq E$ such that $|F| = k$. Define M_F as

$$M_F = (S, \mathcal{I} \cup \{\bigcup F\})$$

which is a matroid for every choice of F by the same reasoning. Clearly, $\nu(M_F) = k$.

For any sequence of independence queries which does not include $\bigcup F$, the results of those queries are the same for M_F or M. Thus any sequence of queries which does not include the unions of all $\binom{|E|}{k}$ k-element subsets of E cannot distinguish between M and M_F for one of the unqueried F. In particular, no polynomial time algorithm can determine whether ν is k or $k - 1$.

The following construction shows that the matroid matching need not be polynomial time even for a matroid with a concise description unless P = NP. Suppose we are given a graph G whose *vertex* set is E. Let $M = (S, \mathcal{I})$ be a matroid with $\mathcal{I}$ defined as as

$$\begin{aligned} \mathcal{I} &= \{I : I \leq 2k - 1\} \cup \{I : I = 2k, I \text{ is not a union of } k \text{ pairs in } E\} \\ &\cup \{I : I = 2k, I \text{ is a union of } k \text{ pairs in } E \text{ such that the pairs form a clique in } G\}. \end{aligned}$$

Now clearly,

$$\nu(M) = \begin{cases} k - 1 & \text{if there is no clique of size } k \\ k & \text{o.w.} \end{cases}$$

Thus, checking whether $\nu(M) = k$ is not possible in polynomial time unless P = NP.

1.2 Min-max relation for matroid matching

Lovász derived a min-max relationship for matroid matching for a special class of matroids, which include all linear matroids. He also gave a polynomial time algorithm for the problem. For example, the maximum forest problem in a 3-uniform hypergraph can be solved in polynomial time using Lovász' algorithm, as a graphic matroid is linear.

We next extend the definition of matroid to allow infinite ground sets. The notion of infinite matroid is a generalization of linear spaces.

Definition 1 (Infinite matroid) *$(S, \mathcal{I})$ is an infinite matroid if the following properties hold:*

1. *$I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$,*

2. *If $J \in \mathcal{I}$ for all finite subsets J of I then $I \in \mathcal{I}$,*

3. *If $I, J \in \mathcal{I}$ and $|I| < |J| < \infty$, then $\exists j \in J \setminus I$ such that $I + j \in \mathcal{I}$.*

□

Before we state the min-max theorem, recall that a flat in a matroid $M = (S, \mathcal{I})$ is an $F \subseteq S$ such that $F = \text{span}(F)$. For linear matroids, flats are precisely the linear subspaces.

Theorem 1 (Lovász) *Let $M = (S, \mathcal{I})$ be a linear matroid (finite or infinite), let r be the rank function, and let E be a finite set of pairs in S. Then*

$$\nu(M) = \min_F \left[r(F) + \sum_{i=1}^{k} \left\lfloor \frac{1}{2}(r(F_i) - r(F)) \right\rfloor \right], \tag{1}$$

where the minimization is carried over the set

$$\{F : F \subseteq F_1 \cap F_2 \ldots \cap F_k; F_1, F_2, \ldots F_k \, \textit{are flats}; \forall (e \in E) \exists (F_i) \text{ such that } e \in F_i\}.$$

The Tutte-Berge formula is the special case of this result when the matroid is trivial.

Tutte-Berge formula: Let $M = (S, 2^S)$ be the trivial matroid (in which all sets are independent) and let the edges in the graph $G = (S, E)$ define the set of pairs E in S. Clearly,

$$\nu(M) = \text{maximum size matching in } G.$$

Now we proceed to compute the RHS of (1). In this case

$$\text{RHS of (1)} = \min_F \left[|F| + \sum_{i=1}^{k} \lfloor \frac{1}{2}(|F_i| - |F|) \rfloor \right]. \tag{2}$$

(For the trivial matroid, all sets are flats.) First, note that the minimization can be restricted to the all flats F_i's such that the sets $F_i \setminus F$ are disjoint. To see this, observe that, if for some i and j, $(F_i \cap F_j) \setminus F \neq \emptyset$, then, we can replace F_i and F_j by a single flat $F_i \cup F_j$ and that will not increase the sum in (2). Thus, we assume the minimization in (2) is carried over flats such that $F_i \setminus F$ are disjoint. Thus it means that $F, F_1 \setminus F, F_2 \setminus F \ldots F_k \setminus F$ is a partition of S. Moreover, all edges of G must belong to $E(F_i)$ for some i. If all the quantities $|F_i - F|$ were even, then (2) boils down to minimization over $(1/2)(|F| + |S|)$. Taking into account the fact that some of the $|F_i \setminus F|$ can be odd, we can write (2) as

$$\frac{1}{2} \min_F \left[|F| + |S| - \#\{i : |F_i \setminus F| \text{ odd}\} \right],$$

which is precisely the Berge-Tutte formula since $(F_i \setminus F)$ can be seen to be a connected component of $G \setminus F$.

Comments of the linearity condition in Theorem 1:

The min-max relationship given by (1) in Theorem 1 holds under a more general condition. Let $M = (S, \mathcal{I})$ be a matroid and let $\mathcal{C}$ be the set of all the circuits. Then Theorem 1 holds if M and all its contractions satisfy the relationship that

$$r\left(\bigcap_{C \in \mathcal{C}'} \text{span}(C) \right) > 0, \tag{3}$$

where

$$\mathcal{C}' = \{\text{circuit } C : C \subseteq C_1 \cup C_2, r(C) = |C_1 \cup C_2| - 2\},$$

for any two circuits C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$.

We next show that linear matroids satisfy the condition given by (3).

Proposition 2 *If $M = (S, \mathcal{I})$ is a linear matroid, then it satisfies the condition given by (3).*

Proof: Let C_1, C_2 be as in the hypotheses of the condition. Then $C_1 \setminus C_2 \in \mathcal{I}$ and $C_1 \cap C_2 \in \mathcal{I}$, since C_1 was a minimal dependent set. Since $\text{span}(C_1 \setminus C_2)$, $\text{span}(C_1 \cap C_2)$, and $\text{span}(C_1))$ are linear subspaces, and further since,

$$r(C_1 \setminus C_2) + r(C_1 \cap C_2) = C_1 \setminus C_2 + C_1 \cap C_2 = C_1 > r(C_1)$$

it follows that

$$P = \text{span}(C_1 \setminus C_2) \cap \text{span}(C_1 \cap C_2) \neq \emptyset.$$

Thus, $\exists p \neq 0 \in P$. We next argue that $p \in \text{span}(C)$ for every $C \subseteq C_1 \cup C_2$ with $r(C) = |C_1 \cup C_2| - 2$. Suppose not, i.e., $p \notin \text{span}(C)$. Now,

$$p \in \text{span}(C_1 \setminus C_2) \Rightarrow C_1 \setminus C_2 \nsubseteq C \Rightarrow \exists s \in C_1 \setminus C_2, s \notin C.$$

and similarly

$$p \in \text{span}(C_1 \cap C_2) \Rightarrow \exists t \in C_1 \cap C_2, t \notin C.$$

Now, $\text{span}(C_2) = \text{span}(C_2 - t)$ (this is always true for an element of a circuit) implies

$$t \in \text{span}(C_2 - t) \subseteq \text{span}(C_1 \cup C_2 \setminus \{s, t\}),$$

as $C_2 \setminus \{t\} \subseteq (C_1 \cup C_2) \setminus \{s, t\}$. Therefore

$$s \in \text{span}(C_1 - s) \subseteq \text{span}((C_1 \cup C_2) \setminus \{s\}) = \text{span}((C_1 \cup C_2) \setminus \{s, t\}),$$

as $t \in \text{span}((C_1 \cup C_2) \setminus \{s, t\})$. Thus

$$\{s, t\} \subseteq \text{span}(C_1 \cup C_2 \setminus \{s, t\}).$$

Since $|C| > r(C)$ (as C is a circuit), $r(C) = |C_1 \cup C_2| - 2$ (by assumption) and $C \subseteq (C_1 \cup C_2) \setminus \{s, t\}$, we obtain

$$|C| > r(C) = |C_1 \cup C_2| - 2 = |(C_1 \cup C_2) \setminus \{s, t\}| \geq |C|,$$

and we have reached a contradiction. □

Although condition (3) is satisfied by matroids coming from linear spaces, it is not necessarily satisfied by all representable matroids. For example, let M be the graphic matroid associated to graph G in Figure 1. We know that M is representable, so it is a restriction of a matroid coming

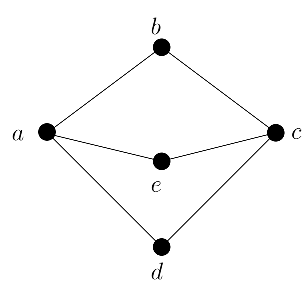

Figure 1: A graph G.

from a linear space. The only circuits of M are $C_1 = \{ab, bc, ce, ea\}$, $C_2 = \{ae, ec, cd, da\}$ and $C_3 = \{ab, bc, cd, da\}$. But note that $r(C_1 \cup C_2) = r(E) = 4 = |C_1 \cup C_2| - 2$, but no edge is in the span of all three circuits. Note that if edge ac were present, this would not happen.

Lovász also gave a polynomial time algorithm to find a maximum cardinality matroid matching for every matroid satisfying the min-max relation (3). We can also use this algorithm to solve the matroid matching problem on restrictions of those matroids. Indeed, if E is a set of pairs of elements of the smaller matroid then any maximum matching of the bigger matroid over this set of pairs will also be a maximum matching of the smaller one. In particular, we can efficiently solve the problem for any linear matroid. Observe that, in order to get this result for finite matroids that are representable over infinite fields, the proof required the introduction of an infinite matroid.