18.438 Advanced Combinatorial Optimization	October 27, 2009

Lecture 12

Lecturer: Michel X. Goemans | *Scribe: Chung Chan*

(See Sections 41.4 and 42.1 in Combinatorial Optimization by A. Schrijver)

1 Matroid Intersection Polytope

For matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, the intersection polytope is the convex hull of the incidence vectors of the common independent sets:

$$\mathrm{Conv}\{\chi(I), I \in \mathcal{I}_1 \cap \mathcal{I}_2\}. \tag{1}$$

Theorem 1 *Let $r_1, r_2 : 2^S \mapsto \mathbb{Z}_+$ be the rank functions of the matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ respectively. Then, the intersection polytope* (1) *is equivalent to the set of $\boldsymbol{x} \in \mathbb{R}^S$ defined by the following totally dual integral (TDI) system,*

$$x(u) := \sum_{e \in U} x_e \leq r_i(U) \qquad \forall U \subseteq S, i = 1,2 \tag{2a}$$

$$\boldsymbol{x} \geq 0 \tag{2b}$$

Proof: It suffices to show that (2) is TDI because then all extreme points are integral, and so the set of vertices are the set of incidence vectors of the common independent sets in both matroids.

To prove TDI, consider the following linear programming duality for some weight vector $\boldsymbol{w} \in \mathbb{Z}^S$,

$$\max_{\boldsymbol{x} \geq 0 : x(U) \leq r_i(U), i=1,2} \boldsymbol{w}^T \boldsymbol{x} = \text{minimize} \sum_{U \subseteq S} r_1(U) y_1(U) + \sum_{U \subseteq S} r_2(U) y_2(U) \tag{3a}$$

$$\text{subject to} \sum_{U \ni e} [y_1(U) + y_2(U)] \geq w_e \quad \forall e \in S \tag{3b}$$

$$\text{with } \boldsymbol{y}_1, \boldsymbol{y}_2 \geq 0$$

Observe that we can assume that $\boldsymbol{w} \geq 0$ since any negative entry can be deleted as it does not affect feasibility in the dual (3).

Claim 2 *There exists an optimal solution $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*)$ to the dual problem in* (3) *that satisfies,*

$$y_1^*(U) = 0 \quad \forall U \notin \mathcal{C}_1 \tag{4a}$$

$$y_2^*(U) = 0 \quad \forall U \notin \mathcal{C}_2 \tag{4b}$$

for some chains $\mathcal{C}_1, \mathcal{C}_2 \subseteq 2^S$, where a chain $\mathcal{F}$ is such that, for all $A, B \in \mathcal{F}$, either $A \subseteq B$ or $B \subseteq A$.

Proof: Let $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*)$ be an optimal solution to the dual in (3). Fixing $\boldsymbol{y}_2 = \boldsymbol{y}_2^*$, the optimization becomes

$$\begin{aligned}
&\text{minimize } \sum_{U \subseteq S} r_1(U) y_1(U) + \sum_{U \subseteq S} r_2(U) y_2^*(U) \\
&\text{subject to } \sum_{U \ni e} y_1(U) \geq \underbrace{w_e - \sum_{U \ni e} y_2^*(U)}_{w_e^{(1)} :=} \quad \forall e \in S \\
&\text{with } \boldsymbol{y}_1 \geq 0
\end{aligned}$$

This boils down to the linear programming dual of the maximum independent set problem with weight $w^{(1)}$. Thus, by the proof of the matroid polytope given in a previous lecture, the optimal solution $\boldsymbol{y}_1^*$ can be chosen to be a chain. Similarly, by now fixing $\boldsymbol{y}_1 = \boldsymbol{y}_1^*$, the optimal solution $\boldsymbol{y}_2^*$ can also be chosen to be a chain. Furthermore, they satisfy

$$\sum_{U \in \mathcal{C}_1 : U \ni e} y_1^*(U) = w_e^{(1)} \tag{5a}$$

$$\sum_{U \in \mathcal{C}_2 : U \ni e} y_2^*(U) = w_e^{(2)} \tag{5b}$$

and hence,

$$\sum_{i=1}^{2} \sum_{U \subseteq S} y_1^*(U) = w_e,$$

for all $e \in S$. □

The above claim shows that we can restrict our attention to the union of two chains. We now show that even if we restrict only to the union of two laminar familar families then the underlying matrix is totally unimodular, which will allow us to derive integrality.

Claim 3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two laminar families of 2^S. i.e. for $i = 1, 2$,*

$$A, B \in \mathcal{C}_i \implies A \cap B = \emptyset \text{ or } A \subseteq B \text{ or } B \subseteq A. \tag{6}$$

Then, $S \times (\mathcal{C}_1, \mathcal{C}_2)$-incidence matrix $\boldsymbol{M}$ is totally unimodular (TU). i.e. for all square submatrices $\boldsymbol{B}$,

$$\det(\boldsymbol{B}) \in \{0, \pm 1\}. \tag{7}$$

Proof: After elementary column operations on $\boldsymbol{M}$ (subtract from the column corresponding to a set A all columns corresponding to maximal sets contained in A), it follows from laminarity that we can transform the incidence matrix (without modifying the determinant of any square submatrix) into one such that every row has at most one "1" within columns in $\mathcal{C}_1$, and one "1" within columns in $\mathcal{C}_2$. (7) can be argued case by case as follows:

1. There is an all-zero row. The determinant is 0.

2. Every row has two "1"'s. The determinant is 0 since the sums over the columns in $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same, namely $\mathbf{1}$.
3. There is a row with one "1". Computing the determinant by expanding along that row, we have (7) by induction.

$\square$

By Claim 2, without loss of optimality, we can restrict our dual program to the variables corresponding to the two chains $\mathcal{C}_1$ and $\mathcal{C}_2$ which gives,

$$\text{minimize} \sum_{U \in \mathcal{C}_1} r_1(U) y_1(U) + \sum_{U \in \mathcal{C}_2} r_2(U) y_2(U) \tag{8a}$$

$$\text{subject to ,} \sum_{U \in \mathcal{C}_1 : e \in U} y_1(U) + \sum_{U \in \mathcal{C}_2 : e \in U} y_2(U) \geq w_e \quad \forall e \in S \tag{8b}$$

$$\text{with } y_1(U) \geq 0 \text{ for } \forall U \in \mathcal{C}_1,\ y_2(U) \geq 0 \text{ for } \forall U \in \mathcal{C}_2.$$

We now argue that this linear program is integral. Indeed, its extreme points are obtained by solving some subsystem $\boldsymbol{B}\boldsymbol{y} = \boldsymbol{w}'$ where $\boldsymbol{B}$ is a square submatrix of the $(S \times (\mathcal{C}_1, \mathcal{C}_2))$-incidence matrix $\boldsymbol{M}$ with integral weights $\boldsymbol{w}'$. Since $\boldsymbol{M}$ is TU by Claim 3, the solution $\boldsymbol{y}$ is integral as desired. $\square$

Remark 1 *1. There is a good characterization of TU matrices. Seymour [2] gives an efficient polynomial time algorithm to verify if a matrix is TU by providing a decomposition result for regular matroids (which are precisely those matroids representable over $\mathbb{R}$ by columns of a TU matrix).*

2. The result (Claim 3) does not extend to the union of three chains, and thus not to the intersection of three matroids.

3. Since the constraints for the dual can be replaced by the equalities (5)*, we have* (8) *equal*

$$\min_{\substack{w_1, w_2 \in \mathbb{Z}_+^S : \\ w = w_1 + w_2}} \left[\max_{I \in \mathcal{I}_1} w_1(I) + \max_{I \in \mathcal{I}_2} w_2(I) \right]$$

This is known as weight-splitting, and gives a good characterization for the primal problem.

4. Cunningham [1] gives a strongly polynomial time algorithm for the separation problem: given $x \in \mathbb{Q}^S$, find U that maximizes $x(U) - r_M(U)$ for matroid M with rank function r_M. (This is a special case of submodular function minimization.) This also gives an efficient way to test if x belongs to the intersection polytope of two matroids, and to decompose x as a convex combination of the incidence vectors in the polytope.

2 Matroid Union

Consider k matroids, $M_i = (S_i, \mathcal{I}_i)$ for $i = 1, \ldots, k$. Define

$$S = \bigcup_{i=1}^{k} S_i,$$
$$\mathcal{I} = \left\{ \bigcup_{i=1}^{k} \mathcal{I}_i : I_i \in \mathcal{I}_i \quad i \in [k] \right\}.$$

Theorem 4 *$M_1 \vee M_2 \vee \cdots \vee M_k = (S, \mathcal{I})$ is a matroid with rank function*

$$r_M = \min_{T \subseteq U} \left[|U \backslash T| + \sum_{i=1}^{k} r_i(T \cap S_i) \right]. \tag{9}$$

This is proved using the following Lemma.

Lemma 5 *Let $\hat{M} = (\hat{S}, \hat{\mathcal{I}})$ be any matroid (e.g., the union of k matroids $\hat{M}_i = (\hat{S}_i, \hat{\mathcal{I}}_i)$ with $i = 1, \ldots, k$ and $\hat{S}_i$ disjoint). For any $f : \hat{S} \mapsto S$, we have $(S, \mathcal{I})$ is a matroid where*

$$\mathcal{I} := \{f(\hat{I}) : \hat{I} \in \hat{\mathcal{I}}\}. \tag{10}$$

Furthermore, its rank function is given by

$$r_M(U) = \min_{T \subseteq U} \left[r_{\hat{M}}(f^{-1}(T)) + |U \backslash T| \right]. \tag{11}$$

Proof of Lemma 5: Consider $I \in \mathcal{I}$. Then there exists some $\hat{I} \in \hat{\mathcal{I}}$ such that,

$$I = f(\hat{I}) \text{ and } |I| = |\hat{I}|.$$

Any subset of I is independent since it can be expressed as the image under f of some subset of $\hat{\mathcal{I}}$.

Consider in addition that $J = f(\hat{J}) \in \mathcal{I}$ for some $\hat{J} \in \hat{\mathcal{I}}$ such that

$$|\hat{I}| < |\hat{J}| = |J|. \tag{12}$$

This implies

$$\hat{I} + e \in \hat{\mathcal{I}} \quad \text{ for some } e \in \hat{J} \backslash \hat{I}. \tag{13}$$

Assume further that $\hat{I}$ is chosen (among those with $\hat{I} \in \hat{\mathcal{I}}$, $f(\hat{I}) = I$, $|\hat{I}| = |I|$) to attain

$$\max_{\hat{I}} |\hat{I} \cap \hat{J}|. \tag{14}$$

Then, we will argue that $f(e) \notin I$, which gives the desired existence of $z \in J \backslash I$, namely $z = f(e)$, such that $I + z \in \mathcal{I}$. Suppose to the contrary that $f(e) \in I \cap J$. Thus there exists

$e' \in \hat{I} \setminus \hat{J} : f(e') = f(e)$. Observe that $\hat{I}' := \hat{I} + e - e' \in \hat{\mathcal{I}}$ by (13), and furthermore that $f(\hat{I}') = I$. But $|\hat{I}' \cap \hat{J}| = |\hat{I} \cap \hat{J}| + 1$ by (13), which contradicts the optimality of $\hat{I}$ in (14).

To derive the rank function r_M in (11), define the partition matroid $(\hat{S}, \mathcal{I}_p)$ induced by $\left\{f^{-1}(e) : e \in U\right\}$ for some $U \subseteq S$ as follows,

$$\mathcal{I}_p := \{I \subseteq \hat{S} : |f^{-1}(e) \cap I| \leq 1 \ \forall e \in U, \ \ |f^{-1}(e) \cap I| = 0 \ \forall e \notin U\} \tag{15}$$

which has rank function

$$r_{M_p}(T) = \left|\{e \in U : f^{-1}(e) \cap T \neq \emptyset\}\right|. \tag{16}$$

Then, I is an independent set of M in U iff there is a subset $\hat{I} \subseteq \hat{U} := f^{-1}(U)$ independent in both M_p and $\hat{M}$. Thus,

$$\max_{\substack{I \subseteq U \\ I \in \mathcal{I}}} |I| = \max_{\substack{\hat{I} \subseteq \hat{U} \\ \hat{I} \in \hat{\mathcal{I}} \cap \mathcal{I}_p}} |\hat{I}|.$$

By the matroid intersection theorem, this equality becomes,

$$r_M(U) = \min_{\hat{T} \subseteq \hat{U}} \left[r_{\hat{M}}(\hat{T}) + r_{M_p}(\hat{U} \backslash \hat{T})\right] \tag{17}$$

$$= \min_{T \subseteq U} \left[r_{\hat{M}}(f^{-1}(T)) + |U \backslash T|\right] \tag{18}$$

where the last equality is because it is optimal to have $\hat{T}$ of the form $f^{-1}(T)$ for some $T \subseteq U$. Indeed, if not, defining $\hat{T}' := \hat{U} \setminus f^{-1}(f(\hat{U} \setminus \hat{T})) = f^{-1}(U \setminus f(\hat{U} \setminus \hat{T}))$ gives a new set with (i) $\hat{T}' \subseteq \hat{T}$ (and hence, $r_{\hat{M}}(\hat{T}') \leq r_{\hat{M}}(\hat{T})$), (ii) $r_{M_p}(\hat{U} \backslash \hat{T}') = r_{M_p}(\hat{U} \backslash \hat{T})$, and (iii) $\hat{T}' = f^{-1}(T)$ for $T = U \setminus f(\hat{U} \setminus \hat{T})$. $\square$

Proof of Theorem 4: The case when S_i's are disjoint is obvious. The case when S_i's are not disjoint can reduce to this setting as follows:

1. construct $\hat{M}_i = (\hat{S}_i, \mathcal{I}_i)$ by relabeling $e \in S_i$ as (i, e).
2. define $f : \hat{S} \mapsto S$ as a selector of the second component, namely $f((i, e)) = e$.

By (11), we have,

$$r_M(U) = \min_{T \subseteq U} \left[r_{\hat{M}}(f^{-1}(T)) + |U \backslash T|\right]$$

$$= \min_{T \subseteq U} \left[\sum_{i=1}^{k} r_{\hat{M}}(f^{-1}(T) \cap \hat{S}_i) + |U \backslash T|\right]$$

since $\hat{S}_i$'s are disjoint. This gives the desired rank function (9) since $r_{\hat{M}}(f^{-1}(T) \cap \hat{S}_i) = r_M(T \cap S_i)$. $\square$

References

[1] W.H. Cunningham, "Testing maembership in matroid polyhedra", *Journal of Combinatorial Theory, Series B*, **36**, 181–188, 1984.

[2] P.D. Seymour, "Decomposition of regular matroids", *Journal of Combinatorial Theory, Series B*, **28**, 305–359, 1980.