18.434 Problem Set #1, Fall 2011

The problem set is due on Monday September 26th. Solutions should be typeset and you will receive a grade for both mathematical content and for mathematical exposition. The problem set is about random spanning trees. You are welcome to brainstorm with other students in the class; however, you have to write your own solutions in your own words without looking at the write-up of other students' in the class.

1. Consider a graph $G = (V, E)$, and let $c_e \in \mathbb{Q}$ be the conductance of edge $e \in E$. For any spanning tree T in $G = (V, E)$, let
$$w(T) = \prod_{e \in T} c_e$$
be the weight of T. Let $w^* = \sum_T w(T)$ where the sum is over all spanning trees of G. We would like to show that, for the problem of generating a spanning tree T with probability equal to $w(T)/w^*$, we can reduce the problem to the case in which all conductances $c_e = 1$.

   (a) First assume that all conductances $c_e \in \mathbb{Z}_+$. Suppose, for each edge e, we replace it by c_e parallel copies each with a unit conductance, and let $G' = (V, E')$ be the resulting graph (with unit conductances). Argue that a spanning tree T of G according to the distribution $w(T)/w^*$ can be obtained from a spanning tree T' chosen uniformly at random in G'.

   (b) What if the conductances are rational (instead of being integral)?

2. Consider a graph $G = (V, E)$ and let $c_e \in \mathbb{Q}$ be the conductance of edge $e \in E$. Consider the same distribution as in the previous problem. We have seen in lecture (for unit conductances, but the previous exercise can be used to extend it to case of arbitrary conductances) that the probability $P(e \in T)$ that an edge $e = (s, t)$ is in a random spanning tree T equals the current i_{st} through the edge (s, t) when a total (or effective) current of 1 unit flows from s to t.

   Now consider two edges $e = (s, t), f = (u, v) \in E$. Prove that
$$P(e \in T | f \in T) \leq P(e \in T).$$

3. In lecture, we saw one way to generate a random (according to the distribution in the above exercises) spanning tree in a graph $G = (V, E)$ with conductances c_e for $e \in E$. In this exercise we will consider another way. Fix a vertex $s \in V$. Consider a random walk starting at v. Remember this means that when, at vertex u, one walks to v for $(u, v) \in E$ with probability equal to $\frac{c_{uv}}{C_u}$ where $C_u = \sum_{w:(u,w)\in E} c_{uw}$. For every vertex $v \in V \setminus \{s\}$, let $(f(v), v))$ be the edge used when visiting v *for the first time* (the random walk was at vertex $f(v)$ just prior to visiting v). Let $F = \{(f(v), v) : v \in V \setminus \{s\}\}$.

   (a) Prove that F is a spanning tree.

   The claim is that the probability distribution for F is precisely the one we require. You will not show this (as it is harder than it looks to prove it), but rather show that the *marginal probabilities* $p_e = P(e \in F)$ are as expected:

(b) For the spanning tree F constructed by the above random walk, for any $e = (u, v) \in E$, prove that $P(e \in F)$ are as in exercise 1.