18.434 Lecture - Electrical Networks and Spanning Trees

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We have shown that any solution to Kirchhoff's laws and boundary values is unique, and defined such a solution in terms of probabilities. Now we define one in terms of the number of spanning trees in the graph.

A **tree** is a graph in which there is exactly one simple path between any two vertices, i.e. a connected graph with no cycles.

A **spanning tree** of a graph $G = (V, E)$ is a subgraph of G that is a tree and contains all of V. A graph may have many spanning trees.

Given an electrical network $G = (V, E)$ with all conductances $= 1$ (result generalizes to arbitrary conductances) and $s, t \in V$.

Let $N(s, a, b, t) =$ the number of spanning trees with a path from s to t that traverses edge (a, b) from a to b and N be the total number of spanning trees of G.

Thm: The current $i_{ab} = \frac{1}{N}(N(s, a, b, t) - N(s, b, a, t))\forall(a, b) \in E$ defines a unit flow from s to t that satisfies Kirchhoff's laws.

Note that this is like $i_{ab} = Pr_T\{T$ has a path from s to t traversing (a, b) from a to $b\} - Pr_T\{T$ has a path from s to t traversing (a, b) from b to $a\}$.

The number of spanning trees relating to current then has implications for uniformly generating a random spanning tree from a graph. The probability over all spanning trees of an edge (a, b) being in a tree is i_{ab}. So include an (a, b) in the spanning tree with probability i_{ab}. Once a decision has been made for an edge, if it is not put in the tree, it can be deleted from the graph and a spanning tree generated for the remaining graph. If it is put in the tree, a and b can be combined into one node and a spanning tree generated for this new graph.