18.434 Voltage and current in random walks

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The concepts of voltage and current in electrical networks have meaningful analogues in corresponding random walks. Given an electrical network consisting of a graph $G = (V, E)$ along with conductance c_{xy}, $\forall (x, y) \in E$, define the matrix of transition probabilities as

$$P_{xy} = \frac{c_{xy}}{\displaystyle\sum_{y:(x,y)\in E} c_{xy}}$$

if $(x, y) \in E$ and $P_{xy} = 0$ otherwise. Then the random walk defined by matrix P has properties related to the voltage and current of the original network, as given in the following theorems.

Theorem 1. Pick nodes $s, t \in V$. Define

$$h(x) = Prob(\text{starting the random walk at x, we reach s before t})$$

and define the *escape probability*

$$P_{esc}(s \to t) = Prob(\text{starting at s, we reach t before s})$$

Let $v(x)$ be the voltage at node x in the electrical network. Then if $v(s) = 1$ and $v(t) = 0$, $h(x) = v(x)$ and the effective current in the electrical network from s to t is

$$i_{eff}(s, t) = P_{esc}(s \to t) \sum_{y:(s,y)\in E} c_{sy}$$

Theorem 2. Pick nodes $s, t \in V$. For a random walk starting at s and stopping at t, define the expected *sojourn time* at x, $S_x(s \to t)$, to be the expected number of times node x is touched strictly before reaching t. Also define E_{xy} to be the expected difference between the number of times edge (x, y) is crossed from x to y and the number of times it is crossed from y to x. In the electrical network, let the voltage $v(t) = 0$, and let the effective current from s to t be $i_{eff}(s, t) = 1$. Then the voltage at x is

$$v(x) = \frac{S_x(s \to t)}{\displaystyle\sum_{y:(x,y)\in E} c_{xy}}$$

and the current on edge (x, y) is $i(x, y) = E_{xy}$.