18.310A Homework 3

Due Fri March 6th at 10AM in lecture

You may return the problem set on Monday March 9th during the quiz without penalty.

Instructions: Collaboration on homework is permitted, but you must write the solutions yourself; no copying is allowed. Please list the names of your collaborators; if you worked alone, state this. Also indicate any sources you consulted beyond the lecture notes.

1. **This is the question which was removed from problem set 2.**

   Suppose $A_1, A_2, \cdots, A_k$ are subsets of cardinality n of a finite set X. We would like to color the elements of X red or blue in such that a way that in every A_i for $i = 1, \cdots, k$, there exists at least one red element and at least one blue element. Give a condition on k (as a function of n) such that this is always possible. (For maximum credit give the greatest function of n for which you can prove the result.)

2. Let $(a_n)_{n\geq 0}$ be the series defined by $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} + 2a_{n-2}$ for all $n \geq 2$. Find an explicit expression for a_n.

3. Let $F(x) = \frac{ax+b}{(1-rx)^2}$ be the generating function for the sequence $(f_n)_{n\geq 0}$. In this case, the denominator has a double root.

   (a) Consider the case $a = 0$ and $b = 1$. Using the fact that $\frac{1}{1-rx} = \sum_{n=0}^{\infty} r^n x^n$, find the expansion for $\frac{1}{(1-rx)^2}$.

   (b) What is the expansion of $\frac{x}{(1-rx)^2}$?

   (c) Give an explicit formula for f_n (of course involving a, b and r).

4. Let a_n be the number of ways of making change on n\$ with 1\$ bills and 2\$ bills. Thus $a_0 = 1$ and for example $a_4 = 3$ ($4 = 2 \times 2, 4 = 1 \times 2 + 2 \times 1$ and $4 = 4 \times 1$).

   (a) Find the generating function $A(x)$ for the sequence $(a_n)_{n\geq 0}$.

   (b) Using the method of partial fractions, give an explicit formula for $(a_n)_{n\geq 0}$. (Check your answer on small values of n.)