Spring 2015

18.310A Homework 1

Due February 13th at 10AM in lecture

Instructions: Collaboration on homework is permitted, but you must write the solutions yourself; no copying is allowed. Please list the names of your collaborators; if you worked alone, state this. Also indicate any sources you consulted beyond the lecture notes.

1. Give an example of 3 events A_1, A_2 and A_3 which are *pairwise independent*, i.e. such that any 2 of them are independent, but which are not independent.

2. Let X be a uniformly random subset of $[n] := \{1, 2, \cdots, n\}$; as there are 2^n subsets on n elements, each subset is chosen with probability $\frac{1}{2^n}$.

   (a) Let A_i be the event that $i \in X$. Show that the events A_i for $i = 1, \cdots, n$ are independent.

   (b) Let Y be a random subset chosen independently from X. What is $\mathbb{E}[|X \cup Y|]$?

   (c) What is the probability that $X \cup Y = [n]$ (again assuming that X and Y are independent uniformly random sets). Justify your answer.

3. One hundred people line up to board a plane, but the first person has lost his boarding passand takes a uniformly random seat instead. Each subsequent passenger takes his or her assigned seat if available, and otherwise takes a uniformly random seat among the remaining seats. What is the probability that the last passenger ends up in his/her own seat.

4. Let A_i be the event that it snows on day i of February (with $1 \leq i \leq 28$). Assume that these events A_i are independent and that $\mathbb{P}[A_i] = 0.5$.

   (a) Let p be the probability that, during the month of February, there exist 6 consecutive days with snow followed by a day without snow. Give an expression for p and simplify as much as possible.

   (b) Let X be the random variable equal to the number of occurrences of such sequences of 6 snowy days followed by a day with no snow. What is $\mathbb{E}[X]$? What is $\mathrm{Var}[X]$?

5. Consider a uniformly random permutation σ on $[n] = \{1, 2, \cdots, n\}$. Call i a fixed point if $\sigma(i) = i$. Let X be the random variable denoting the number of fixed points in a uniformly random permutation σ. (When discussing absent-minded math professors, we saw in lecture that $\mathbb{P}[X = 0] \sim \frac{1}{e}$ as n tends to infinity.

   (a) What is $\mathbb{E}[X]$? What is $\mathrm{Var}(X)$?

   (b) Use Chebyshev's inequality to give an upper bound on $\mathbb{P}[X \geq t]$ for any given integer $t \geq 2$.)