How to compute the unitary dual

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Outline

Introduction

Infinitesimal characters

Hermitian forms

Calculating CR_j

Introducing FPP

Slides eventually at
`http://www-math.mit.edu/~dav/paper.html`

What's this talk about?

$G(\mathbb{R})$ real reductive algebraic group.

$\widehat{G(\mathbb{R})}_u$ = (equiv classes of) irr unitary reps of $G(\mathbb{R})$.

I'll assume that studying this set (the unitary dual problem) is the most world's best problem.

How can you approach it?

Goal for today: understand how this question might have a computable answer.

Some history

$\widehat{G(\mathbb{R})}_u$ = (equiv classes of) irr unitary reps of $G(\mathbb{R})$.

Anthony Knapp and Gregg Zuckerman in 1980s showed how a description of $\widehat{G(\mathbb{R})}_u$ could look.

Completing work of Harish-Chandra and Langlands, they gave a simple parametrization of a larger set

$\widehat{G(\mathbb{R})}_q =$ (equiv classes of) irr quasisimple reps of $G(\mathbb{R})$.

Slightly simplified version of their answer:

$\widehat{G(\mathbb{R})}_q =$countable union of cplx affine spaces $V_j(\mathbb{C})$, each with rational form $V_j(\mathbb{Q})$.

More precise: $V_j(\mathbb{Q})$ has rep of finite grp W_j, and

$$\widehat{G(\mathbb{R})}_q = \bigcup_j V_j(\mathbb{C})/W_j \qquad v \in V_j(\mathbb{C}) \mapsto J(v).$$

More history

Knapp-Zuckerman-Langlands classification $\rightsquigarrow$

$$\widehat{G(\mathbb{R})}_q = \bigcup_j V_j(\mathbb{C})/W_j.$$

Knapp-Stein on intertwining operators $\rightsquigarrow$ unitary dual is rational real polyhedron inside each $V_j(\mathbb{C})$:

$$\widehat{G(\mathbb{R})}_u = \bigcup_j C_j/W_j, \qquad C_j \subset V_j(\mathbb{C}).$$

Plan of talk:

1. Explain connection of LKZ classification with infinitesimal characters of representations.
2. Explain KZ description of hermitian reps, $\rightsquigarrow$ details about polyhedra C_j.
3. Explain fundamental parallelepiped FPP, and the FPP conjecture relating it to unitary representations.
4. Explain how FPP conjecture reduces unitary dual problem to a finite calculation that can be done (for each $G(\mathbb{R})$) by the `atlas` software.

Quasisimple representations

$G(\mathbb{R})$ real reductive, cplxified Lie algebra $\mathfrak{g} \supset \mathfrak{h}$ Cartan subalgebra, $W = W(\mathfrak{g}, \mathfrak{h})$ Weyl group

$$\begin{aligned} \mathfrak{Z}(\mathfrak{g}) &= \text{center of } U(\mathfrak{g}) \\ &\simeq S(\mathfrak{h})^W \qquad \text{(Chevalley, Harish-Chandra).} \end{aligned}$$

An infl char is algebra homomorphism $\chi\colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$.

Thm (HC, Chevalley). Infl chars are indexed by $\mathfrak{h}^*/W$.

$$(\pi, V_\pi) \text{ rep of } G(\mathbb{R}) \rightsquigarrow U(\mathfrak{g})\text{-module } V_\pi^\infty.$$

Schur's Lemma suggests

$$\pi \text{ irr} \overset{??}{\Longrightarrow} \mathfrak{Z}(\mathfrak{g}) \text{ acts on } V^\infty \text{ by infl char } \gamma(\pi). \qquad \text{(Q)}$$

(Q) fails for general π (Soergel), but holds for unitary π (Segal).

Harish-Chandra understood (Q) was characteristic of nice reps; defined π quasisimple if it has an infl char $\gamma(\pi) \in \mathfrak{h}^*$.

Infl char is called real if $\gamma \in X^*(H) \otimes_{\mathbb{Z}} \mathbb{R}$.

Real infl chars will be central in discussing unitary dual.

Langlands classif and infl chars

Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = \text{Lie}(G)$ has natural $\mathbb{Z}$-form

$$\mathfrak{h}(\mathbb{Z})_{\text{nat}} = X_*(H),$$

the lattice of cocharacters of H.

This defines forms of $\mathfrak{h}$ over any other ring, for example

$$\mathfrak{h}(\mathbb{R})_{\text{nat}} = \mathbb{R} \otimes_{\mathbb{Z}} X_*(H).$$

Real Cartan $H(\mathbb{R}) \subset G(\mathbb{R})$ has a Cartan involution (over $\mathbb{Z}$)

$$\theta\colon H \to H, \qquad \theta^2 = 1.$$

DANGER OF CONFUSION: $\mathfrak{h}(\mathbb{R}) \neq \mathfrak{h}(\mathbb{R})_{\text{nat}}$ unless $H(\mathbb{R})$ is split.

Cartan decomp of $\mathfrak{h}$ is eigenspace decomp (def over $\mathbb{Q}$)

$$\mathfrak{h} = \mathfrak{h}^{\theta} \oplus \mathfrak{h}^{-\theta} = \mathfrak{t} \oplus \mathfrak{a}$$

Each affine space $V_j(\mathbb{C})$ in Langlands classification is naturally an affine subspace of $\mathfrak{h}^*$: $\iota_j\colon V_j \xrightarrow{\sim} \lambda_j + \mathfrak{a}^*, \qquad \lambda_j \in \mathfrak{t}^*(\mathbb{Q})_{\text{nat}}$.

The inclusions ι_j compute infinitesimal characters:

$$J(v) \text{ has infl char } \iota_j(v) = \lambda_j + \nu \in \mathfrak{h}^* \qquad (v \in V_j(\mathbb{C})).$$

Often just write $v = \lambda_j + \nu$, say $J(v)$ has infl char v.

Old person's complaint about terminology

Irr reps of a real reductive $G(\mathbb{R})$ that I call

quasisimple $=_{\text{def}}$ has an infinitesimal character

are often called

admissible $=_{\text{def}}$ restriction to maximal compact subgroup has finite multiplicities.

Reason saying admissible is allowed: Harish-Chandra proved irr rep is quasisimple $\iff$ admissible.

Reason saying admissible is a bad idea: admissible is important technically, but not *a priori* an *expected* property.

Reason saying quasisimple is a good idea: quasisimple is a natural property of an irr rep, motivated by Schur's Lemma. Quasisimple is used in proof of Langlands classification.

I therefore urge you to speak about quasisimple irr reps instead of admissible irr reps.

You will make at least one old person very happy!

About Hermitian forms

Any quasisimp irr π of $G(\mathbb{R})$ has Herm dual π^h.

Herm dual is an order two automorphism of $\widehat{G(\mathbb{R})}_q$.

In terms of L-K-Z classification

$$\widehat{G(\mathbb{R})}_q = \bigcup_j V_j(\mathbb{C})/W_j, \qquad \nu \in V_j(\mathbb{C}) \mapsto J(\nu),$$

K-Z: Herm dual is minus complex conjugate on $\mathfrak{a}^*$:

$$J(\lambda_j + \nu)^h = J(\lambda_j - \overline{\nu}) \quad (\lambda_j + \nu \in V_j(\mathbb{C})).$$

Easy: π has nonzero invt Herm form $\iff \pi \simeq \pi^h$.

HC thm: π unitary $\iff$ Herm and form is definite.

KZ thm: Hermitian dual of $G(\mathbb{R})$ is

$$\widehat{G(\mathbb{R})}_h = \bigcup_j \left\{ \bigcup_{w \in W_j} \{\lambda_j + \nu \in V_j(\mathbb{C}) \mid w\nu = -\overline{\nu}\} \right\} / W_j$$

Applying KZ theorem to unitary representations

Knapp-Zuckerman $\rightsquigarrow$ Hermitian dual:

$$\widehat{G(\mathbb{R})}_h = \bigcup_j \left\{ \bigcup_{w \in W_j} \{\lambda_j + \nu \in V_j(\mathbb{C}) \in V_j(\mathbb{C}) \mid w\nu = -\overline{\nu}\} \right\} / W_j$$

$$= \bigcup_j \left\{ \bigcup_{w \in W_j} \lambda_j + i\mathfrak{a}^*(\mathbb{R})^w \oplus \mathfrak{a}^*(\mathbb{R})^{-w} \right\} / W_j.$$

Thm (Knapp-Stein) Suppose

$$\nu = \lambda_j + i\nu_+ + \nu_- \in \lambda_j + i\mathfrak{a}(\mathbb{R})^w \oplus \mathfrak{a}(\mathbb{R})^{-w}, \qquad w \in W_j$$

is a Langlands parameter for a Hermitian rep. Then

1. Write $L_+(\mathbb{R})$ = real Levi subgroup $G(\mathbb{R})^{\nu_+}$. Then Herm rep $J(\lambda_j + i\nu_+ + \nu_-)$ is unitarily induced from
$$J_{L_+(\mathbb{R})}(\lambda_j + \nu_-) \in \widehat{L_+(\mathbb{R})}_h.$$
2. $J(\lambda_j + i\nu_+ + \nu_-)$ unitary $\iff$ $J_{L_+(\mathbb{R})}(\lambda_j + \nu_-)$ unitary.
3. $J(\lambda_j + \nu_-)$ unitary $\iff$ ν_- belongs to a W_j-stable compact rational polyhedron $CR_j(w) \subset \mathfrak{a}^*(\mathbb{R})^{-w}$.

1984 knowledge of unitary representations

Thm

1. Any unitary irr of $G(\mathbb{R})$ is unitarily induced from a unitary of real infinitesimal character.
2. The set of unitary irrs of real infl char is

$$\widehat{G(\mathbb{R})}_{u,\mathbb{R}} = \bigcup_j \left\{ \bigcup_{w \in W_j} \{\lambda_j + \nu \mid \nu \in CR_j \subset \lambda_j + \mathfrak{a}^*(\mathbb{R})\} \right\} / W_j,$$

   with $CR_j \subset \mathfrak{a}^*(\mathbb{R})$ compact W_j-stable rational polyhedron.
3. If λ_j large enough, CR_j may be computed in a proper Levi $L_j(\mathbb{R}) \subset G(\mathbb{R})$, approximately centralizer of λ_j. Corresponding unitary reps realized by Zuckerman's cohomological induction from $L_j(\mathbb{R})$ to $G(\mathbb{R})$.

Thm reduces $\widehat{G(\mathbb{R})}_u$ to computing compact polyhedra CR_j for small enough λ_j. Still missing:

1. precise definition of small enough, and
2. method to compute any one CR_j.

Setting for the polyhedron CR_j

Real Cartan $H(\mathbb{R})$, Cartan decomp $\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{a}^*$.

Component of quasisimple dual

$$V_j(\mathbb{C}) = \lambda_j + \mathfrak{a}^* \subset \mathfrak{h}^*, \qquad (\lambda_j \in \mathfrak{t}^*(\mathbb{Q}))_{\text{nat}}.$$

In this component, the reps of real infl character are

$$V_j(\mathbb{R}) = \lambda_j + \mathfrak{a}^*(\mathbb{R}) \subset \mathfrak{h}^*(\mathbb{R})_{\text{nat}}.$$

Space $\mathfrak{h}^*(\mathbb{R})_{\text{nat}}$ is very familiar: it is the real vector space containing the root system $\Delta(\mathfrak{g}, \mathfrak{h})$.

Def The affine coroot hyperplanes in $\mathfrak{h}^*(\mathbb{R})_{\text{nat}}$ are

$$H_{\alpha^\vee, m} = \{\gamma \in \mathfrak{h}^*(\mathbb{R})_{\text{nat}} \mid \gamma(\alpha^\vee) + m = 0\}$$

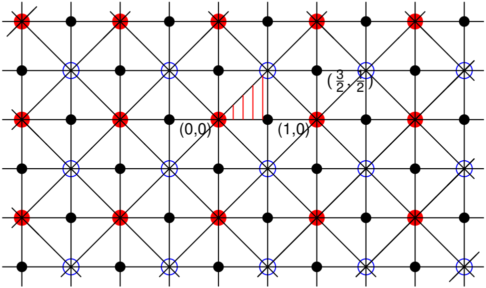
for $\alpha^\vee \in \mathfrak{h}(\mathbb{R})_{\text{nat}}$ any coroot and $m \in \mathbb{Z}$.

Hyperplanes partition $\mathfrak{h}^*(\mathbb{R})_{\text{nat}}$ into open cvx alcoves and cvx facets; (alcove = top-dim facet).

Facets for $SO(5)$, $\mathfrak{h}^*(\mathbb{R}) = \mathbb{R}^2$.

Affine coroot hyperplanes $\{v_1 \pm v_2 = m\}$, $\{2v_i = m'\}$ each divide $\mathbb{R}^2$ into three pieces: the hyperplane itself, and two open pieces.

Facets are intersections over all affine coroots of such pieces.



Each open triangle is a facet, called an alcove. An alcove has three kinds of 1-diml facets as edges, and three kinds of 0-diml facets as vertices.

3 kinds of 0-diml facets: integral; half-integral $(p + 1/2, q + 1/2)$; and mixed $(p + 1/2, q)$ or $(p, q + 1/2)$.

3 kinds of 1-diml facets (black open intervals): horiz or vert, red to black; horiz or vert, black to blue; and diagonal, blue to red.

1 kind of 2-diml facets: open red to black to blue triangles.

G simple rk n: d-facets are $\binom{n+1}{d+1}$ kinds of open d-simplices.

Facets, hermitian forms, and unitary reps

One piece of quasisimple irrs of real infl char is indexed by

$$V_j(\mathbb{R}) = \lambda_j + \mathfrak{a}^*(\mathbb{R}) \subset \mathfrak{h}^*(\mathbb{R})_{\mathrm{nat}}.$$

Affine coroot hyperplanes partition $\mathfrak{h}^*(\mathbb{R})_{\mathrm{nat}}$ into facets F.

Intersections $F \cap (\lambda_j + \mathfrak{a}^*(\mathbb{R}))$ partition $V_j(\mathbb{R})$.

Knapp-Stein, Speh-V: intertwining operators have zeros only on (affine coroot hyperplanes) $\cap V_j(\mathbb{R})$.

To avoid technical issue, use

Observation (Adams-van Leeuwen-Trapa-V?) If $G(\mathbb{R})$ has a cpt Cartan, every irr of real infl char is hermitian.

Thm (KS,SV) Assume $G(\mathbb{R})$ has a compact Cartan, and $F \subset \mathfrak{h}^*(\mathbb{R})_{\mathrm{nat}}$ is a facet meeting $\lambda_j + \mathfrak{a}^*(\mathbb{R})$. Then signature of the invt Herm form is constant on $F \cap (\lambda_j + \mathfrak{a}^*(\mathbb{R}))$.

Calculating the polyhedron CR_j

How to calculate CR_j = unitary reps of real infl char in one piece of quasisimple dual:

$$V_j(\mathbb{R}) = \lambda_j + \mathfrak{a}^*(\mathbb{R}) \subset \mathfrak{h}^*(\mathbb{R})_{\mathrm{nat}}.$$

1. Find a compact subset X of $V_j(\mathbb{R})$ so $CR_j \subset X$.
2. List the fin many facets $F_\ell \subset \mathfrak{h}^*(\mathbb{R})_{\mathrm{nat}}$ meeting X.
3. For each facet F_ℓ, pick point $v_\ell \in F_\ell$.
4. Test whether $J(v_\ell)$ is unitary.

Then $CR_j = \bigcup_{J(v_\ell)\ \mathrm{unitary}} F_\ell$, compact rational polyhedron.

Crude answer for (1): reps with bounded matrix coeffs

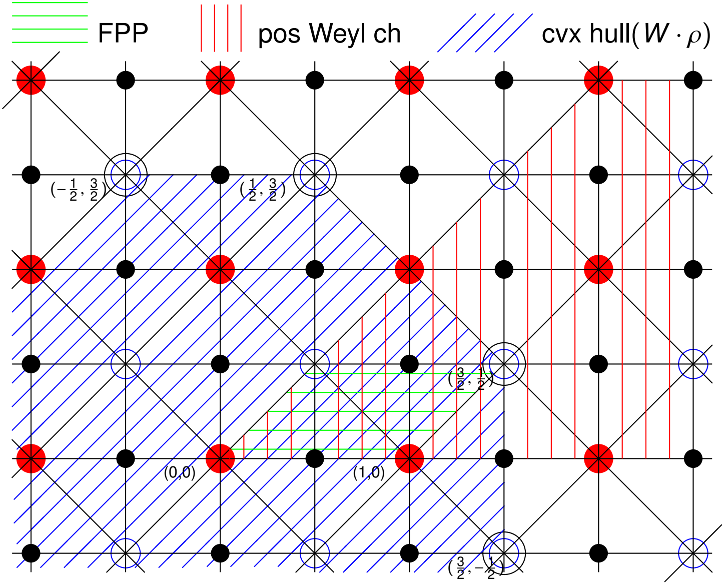
$$BR_j = \{\lambda_j + \nu \mid \nu \in \text{ cvx hull of } W \cdot \rho\}.$$

Since facets are def by lin ineqs, (2) is linear algebra.

For (3), can take v_ℓ = barycenter of F.

For (4), paper of Adams-van Leeuwen-Trapa-V gives algorithm, implemented in `atlas` software.

Let's look at $SO(5)$ again



Why wasn't that the end of the talk?

Algorithm above uses (on each affine space piece $\widehat{G(\mathbb{R})}_q$) a unitarity test for each facet in

(pos Weyl chamber) ∩ (convex hull of $W \cdot \rho$).

In picture for $SO(5)$, red ∩ blue consists of closures of 7 alcoves: total of 29 facets.

This number of facets grows exponentially with $\mathrm{rk}(G)$.

Consequently algorithm appears to be inaccessible to existing computers for the largest exceptional groups.

We need an idea to greatly reduce the number of candidate unitary representations.

Fortunately Dan Barbasch and his collaborators have been studying unitary representations for forty years.

They have had a LOT of ideas. . .

An $SO(4,1)$ example

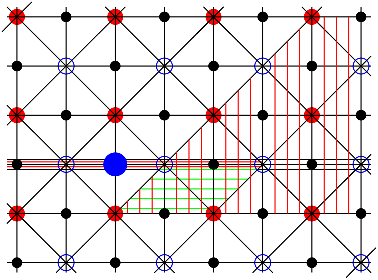
Again look at reps of real infl char in one piece of quasisimple dual:

$$V_j(\mathbb{R}) = \{\lambda_j + \nu \mid \nu \in \mathfrak{a}^*(\mathbb{R})\} \subset \mathfrak{h}^*(\mathbb{R})_{\text{nat}}.$$

Set has locally finite partition into facets, and goal is to decide which facets are unitary.

1. If λ_j large enough, unitary points are cohom ind.
2. If ν large enough, rep is not unitary

So need to study small λ_j, and (for each λ_j) all small ν.



● λ_j for spherical series

$\lambda_j + \nu$ for spherical series

unitary $\lambda_j + \nu$

Need to test ν with $\lambda_j + \nu$ in pos Weyl chamber and in convex hull of $W\rho$: altogether $1/2 \le \nu \le 3/2$. All 5 of these facets (2 open intervals of length 1/2, and their 3 endpoints) test unitary.

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What makes this too difficult?

too difficult = can't treat all exc groups.

Consider first spherical reps of a split group $G(\mathbb{R})$.

Sph reps of real infl char are indexed by $\mathfrak{h}^*(\mathbb{R})_{\text{nat}}$.

Means $\lambda_{\text{sph}} = 0$, $\theta_j = -I$, $\mathfrak{a}^*(\mathbb{R}) = \mathfrak{h}^*(\mathbb{R})_{\text{nat}}$.

For which $\nu \in \mathfrak{h}^*(\mathbb{R})_{\text{nat}}$ could $J_{\text{sph}}(\nu)$ be unitary?

Dan Barbasch, partly with Dan Ciubotaru and Alessandra Pantano, essentially determined set of unitary $J_{\text{sph}}(\nu)$. Consequence:

Thm (BCP). Suppose $G(\mathbb{R})$ split, $\nu \in \mathfrak{h}^*(\mathbb{R})_{\text{nat}}$ dominant, and $J_{\text{sph}}(\nu)$ unitary. Then ν must belong to the fundamental parallelepiped

$$FPP =_{\text{def}} \{\nu \in \mathfrak{h}^*(\mathbb{R})_{\text{nat}} \mid 0 \le \langle \nu, \alpha^\vee \rangle \le 1 \quad (\text{all } \alpha \text{ simple})\}.$$

The FPP conjecture

$FPP =_{\text{def}} \{\nu \in \mathfrak{h}^*(\mathbb{R})_{\text{nat}} \mid 0 \le \langle \nu, \alpha^\vee \rangle \le 1 \quad (\text{all } \alpha \text{ simple})\}.$

Theorem of Barbasch-Ciubotaru-Pantano is evidence/motivation for

FPP Conjecture Suppose $G(\mathbb{R})$ semisimple, and J is an irr unitary rep of real infl char $\gamma \in \mathfrak{h}^*(\mathbb{R})_{\text{nat}}$. If J is not cohomologically induced in the good range from a unitary J_L on the Levi subgroup L of a proper θ-stable parabolic, then $\gamma \in FPP$.