Weyl group representations and Harish-Chandra cells

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Outline

Introduction

Joseph cells

Harish-Chandra cells

Springer correspondence

Lusztig's results

Consequences of Lusztig's results for HC modules

Slides at `http://www-math.mit.edu/~dav/paper.html`

Language for today

$G(\mathbb{R})$ real reductive $\supset K(\mathbb{R}) = G(\mathbb{R})^\theta$

$G \supset K = G^\theta$ complexifications, $\mathfrak{g} = \mathrm{Lie}(G)$

Cartan and Borel $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$

$\lambda \in \mathfrak{h}^*$ dominant regular $\rightsquigarrow$ infinitesimal character.

$\mathcal{M}(\mathfrak{g}, K)_\lambda =$category of $(\mathfrak{g}, K)$-modules of infl char λ

$\mathrm{Irr}(\mathfrak{g}, K)_\lambda =$ (finite) set of irreducible representations

$K\mathcal{M}(\mathfrak{g}, K)_\lambda =$ Grothendieck group.

$K\mathcal{M}(\mathfrak{g}, K)_\lambda =$ fin-rk free $\mathbb{Z}$-module, basis $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$W(\lambda) =$ integral Weyl group $\subset W$.

What's a Harish-Chandra cell?

Have preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$: following Joseph,

$$Y \underset{LR}{\leq} X \iff \exists F, \quad Y \text{ comp factor of } F \otimes X.$$

Here F is fin-diml rep of G^{ad}.

Equivalence relation $Y \underset{LR}{\sim} X$ means $Y \underset{LR}{\leq} X \underset{LR}{\leq} Y$.

Strict order $Y \underset{LR}{<} X$ means $Y \underset{LR}{\leq} X$ but $X \underset{LR}{\not\leq} Y$.

A Harish-Chandra cell is an $\underset{LR}{\sim}$ equiv class in $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

Harish-Chandra cells partition $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

How is that a Weyl group representation?

Integral Weyl group $W(\lambda)$ acts on $K\mathcal{M}(\mathfrak{g}, K)_\lambda = \mathbb{Z}\,\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

Action defined using tensor products with fin-diml G^{ad} reps.

Can therefore use $W(\lambda)$ rep to describe $\underset{LR}{\leq}$:

$$Y \underset{LR}{\leq} X \iff \exists w \in W(\lambda), \quad [Y] \text{ appears in } w \cdot X.$$

$C(X) = \underset{LR}{\sim}$ equiv class of X = HC cell $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$\overline{C}(X) = \underset{LR}{\leq}$ interval below X = HC cone $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$\partial C(X) = \overline{C}(X) - C(X)$.

Theorem.

1. $W(\lambda)$ acts on $\mathbb{Z}\overline{C}(X) = \left[\sum_{Y \underset{LR}{\leq} X} \mathbb{Z}Y\right] \supset \mathbb{Z}\partial C(X)$.
2. $W(\lambda)$ acts on $\mathbb{Z}C(X) \simeq \mathbb{Z}\overline{C}(X)/\mathbb{Z}\partial C(X)$.

$\mathbb{Z}C(X)$ is called a HC cell repn of $W(\lambda)$.

What's the plan?

Cells $\rightsquigarrow$ connect Weyl group reps and inf diml $\mathfrak{g}$ reps.

What Joseph did for highest weight modules.

What Joseph's results give for HC modules.

Lusztig's calculation of Joseph's cell representations.

conjectural extension of Lusztig's results to HC cells.

`https://1drv.ms/u/s!AuIZlbpNWacjgVxg4fFkfQkHZfLn`

link to examples of Joseph and HC cells.

What's true about Joseph cells?

$w \in W_\lambda \rightsquigarrow$ irr $L(w)$, highest weight $w\lambda - \rho$.

Theorem (Joseph)

1. Relations $\underset{L}{\sim}$ and $\underset{LR}{\sim}$ partition $W(\lambda)$ into left cells and two-sided cells $C_L(w) \subset C_{LR}(w) \subset W(\lambda)$ $(w \in W(\lambda))$.
2. Free $\mathbb{Z}$-module $\mathbb{Z}C_L(w) =$ left cell rep of $W(\lambda)$.
3. $\mathbb{Z}C_{LR}(w(\lambda)) =$ 2-sided cell rep of $W(\lambda) \times W(\lambda)$.
4. $\sum_{C_{LR}} \mathbb{Z}C_{LR} \quad \simeq_{W(\lambda)\times W(\lambda)} \quad \mathbb{Z}W(\lambda)$, regular rep of $W(\lambda)$.
5. Two-sided cells C_{LR} partition $\widehat{W}(\lambda)$ into subsets $\Sigma(C_{LR})$ called families: $\mathbb{Z}C_{LR} \simeq \sum_{\sigma\in\Sigma(C_{LR})} \sigma \otimes \sigma^*$.
6. As rep of the first $W(\lambda)$, $\mathbb{Z}C_{LR} \simeq \sum_{C_L \subset C_{LR}} \mathbb{Z}C_L$.

Joseph's Goldie rank $W(\lambda)$ reps

$w \in W_\lambda \rightsquigarrow$ irr $L(w)$, highest weight $w\lambda - \rho$.

Theorem (Joseph)

1. $\mathrm{Ann}(L(w)) = \mathrm{Ann}(L(w')) \iff w,\ w'$ in same left cell C_L.
2. $\mathcal{AV}(L(w)) = \mathcal{AV}(L(w'))$ if $w,\ w'$ in same right cell C_R.
3. Left cell rep $\mathbb{Z}C_L$ has "lowest degree" irr $\sigma_0(C_L)$, mult one.
4. $\sigma_0(C_L) = \sigma_0(C_L') \iff C_L,\ C_L'$ in same C_{LR}.
5. Two-sided cells C_{LR} in bijection with $W(\lambda)$ reps $\sigma_0(C_{LR})$.
6. #(left cells in C_{LR}) $= \dim(\sigma_0(C_{LR}))$.
7. $\#\mathrm{Prim}_\lambda\, U(\mathfrak{g})$ = sum of dims of reps σ_0.

The reps $\sigma_0(C_{LR})$ are Joseph's Goldie rank representations.

Turn out also to be Lusztig's special $W(\lambda)$ representations.

The moral of this story:

1. Cell reps of $W(\lambda)$ are critical to $\mathrm{Prim}\, U(\mathfrak{g})$.
2. Cell reps are helpful to $\mathcal{AV}$(highest wt modules).

More about (2) on the next slide. . .

What's special about Goldie rank reps?

$w \in W_\lambda \rightsquigarrow$ irr $L(w)$, highest weight $w\lambda - \rho$.

C_{LR} two-sided cell $\rightsquigarrow \Sigma(C_{LR}) \subset \widehat{W}(\lambda)$.

Say σ_0 is the $\left(\substack{\text{Goldie rank}\\ \text{special}}\right)$ $W(\lambda)$ rep in $\Sigma(C_{LR})$.

Write $b(\sigma_0) =$ smallest integer b so $\sigma_0 \subset S^b(\mathfrak{h})$.

Theorem (Joseph)

1. $b(\sigma_0) < b(\sigma')$ for any other $\sigma' \in \Sigma(C_{LR})$.
2. $W \cdot [\sigma_0 \subset S^{b(\sigma_0)}]$ is an irr rep $\sigma_1 \in \widehat{W}$, $b(\sigma_0) = b(\sigma_1)$.
3. σ_1 is a Springer rep $\leftrightsquigarrow$ nilpotent orbit $\mathcal{O}(\sigma_1) \subset \mathfrak{g}^*$.
4. For $w \in C_{LR}$, $\mathcal{AV}(L(w)) \subset \overline{\mathcal{O}}(\sigma_1)$.
5. $\mathcal{AV}(L(w)) =$ some irr comps of $\overline{\mathcal{O}}(\sigma_1) \cap (\mathfrak{g}/\mathfrak{b})^*$.

What's true about Harish-Chandra cells?

Applying Joseph's clever definitions, find easily. . .

Theorem.

1. $Y \underset{LR}{\leq} X \implies \mathcal{AV}(Y) \subset \mathcal{AV}(X)$.
2. $Y \underset{LR}{<} X \implies \mathcal{AV}(Y) \subsetneq \mathcal{AV}(X)$.
3. $Y \underset{LR}{\sim} X \implies \mathcal{AV}(Y) = \mathcal{AV}(X)$.

Since $C(X) = \underset{LR}{\sim}$ equiv class of X = HC cell $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$,

Deduce associated varieties are constant on HC cells.

Applying Joseph's deep thms about Goldie rk polys, find

Theorem. Suppose X is an irr $(\mathfrak{g}, K)$-module belonging to Harish-Chandra cell $C = C(X)$.

1. All $W(\lambda)$ reps on $\mathbb{Z}C(X) \subset$ one family $\Sigma(C_{LR}) \subset \widehat{W}(\lambda)$.
2. $\mathbb{Z}C(X)$ contains $\left(\begin{smallmatrix}\text{Goldie rank}\\ \text{special}\end{smallmatrix}\right)$ $W(\lambda)$ rep $\sigma_0 \in \Sigma(C_{LR})$.

Associated varieties for HC cells

$\mathrm{Irr}(\mathfrak{g}, K)_\lambda$ is partitioned into Harish-Chandra cells C;

$\mathbb{Z}C$ carries a HC cell representation of $W(\lambda)$;

$\mathbb{Z}C$ is built from one two-sided cell $\Sigma \subset \widehat{W}(\lambda)$;

$\mathbb{Z}C \supset$ unique $\binom{\text{Goldie rank}}{\text{special}}$ rep $\sigma_0(C)$;

$\sigma_0(C) \rightsquigarrow$ Springer W rep $\sigma_1(C) \rightsquigarrow$ nilp orbit $\mathcal{O}(C) \subset \mathfrak{g}^*$.

Kostant-Rallis: $\mathcal{O}(C) \cap (\mathfrak{g}/\mathfrak{k})^* =$ finite union of K orbits.

Analogous to Joseph's irr comps of $\overline{\mathcal{O}} \cap (\mathfrak{g}/\mathfrak{b})^*$.

Analog of Joseph's result for right cells in $W(\lambda)$ is

Theorem. For any irr HC module $X \in C$,

$\mathcal{AV}(X) =$ union of closures of K-orbits in $\mathcal{O}(C) \cap (\mathfrak{g}/\mathfrak{k})^*$.

Assoc var (which union of closures) depends only on C.

What next?

Jeffrey Adams + DV $\rightsquigarrow$ algorithm for $\mathcal{AV}(X)$.

Nothing parallel known for highest weight modules.

Get complex nilp orbit from $\binom{\text{Goldie rank}}{\text{special}}$ $W(\lambda)$ rep.

Other $W(\lambda)$ reps in $\mathbb{Z}C(X)$ carry more info about X.

Seek to understand these other reps!

$\mathbb{Z}C(X)$ made of $W(\lambda)$ reps in one family $\Sigma(C_{LR})$.

First topic is Lusztig's description of families.

Springer's description of $\widehat{W}$

Already used: Springer identified each nilpotent coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ with a Weyl group rep $\sigma(\mathcal{O})$.

Now need to extend that: fix $\eta \in \mathcal{O}$, so $\mathcal{O} \simeq G/G^\eta$.

Eqvt fundamental group is $\pi_1^G(\mathcal{O}) = A(\mathcal{O}) =_{\text{def}} G^\eta/G_0^\eta$.

(Subgps of $A(\mathcal{O})$) $\leftrightsquigarrow$ (covers of $\mathcal{O}$ with G action)

Theorem (Springer)

1. Each irreducible $\xi \in \widehat{A}(\mathcal{O}) \rightsquigarrow \sigma(\mathcal{O}, \xi)$, rep of W.
2. $\sigma(\mathcal{O}, \xi)$ is irreducible or zero; $\sigma(\mathcal{O}, 1) \neq 0$.
3. "The" Springer rep for $\mathcal{O}$ is $\sigma(\mathcal{O}) = \sigma(\mathcal{O}, 1)$.
4. Nonzero $\sigma(\mathcal{O}, \xi)$ are all distinct ($\mathcal{O}$ and ξ vary).
5. Every irr rep of W is $\sigma(\mathcal{O}, \xi)$ for unique $(\mathcal{O}, \xi)$

$\widehat{W}$ partitioned by nilp orbits into Springer sets

$$S(\mathcal{O}) = \left\{\text{nonzero } \sigma(\mathcal{O}, \xi) \mid \xi \in \widehat{A}(\mathcal{O})\right\}.$$

Lusztig's description of families

For any finite group F, Lusztig in 1979 defined

$$\mathcal{M}(F) = \{\,(x,\xi) \mid x \in F,\ \xi \in \widehat{F^x}\,\}/(\text{conjugation by } F)$$

Fix $(\Sigma \subset \widehat{W})$ family $\leftrightsquigarrow$ $(C_{LR} \subset W)$ two-sided cell.

Recall Joseph: $\Sigma \ni \binom{\text{Goldie rank}}{\text{special}} \quad \sigma_0 \underset{\text{Springer}}{\longleftrightarrow} \mathcal{O} = \mathcal{O}(\Sigma)$ nilpotent.

$$S(\mathcal{O}) = \{\sigma(\mathcal{O},\xi) \mid \xi \in \widehat{A}(\mathcal{O})\} \quad \text{Springer set.}$$

Theorem (Lusztig 1984)

1. List W-reps in Σ attached by Springer to $\mathcal{O}$:

   $$\Sigma \cap S(\mathcal{O}) = \{\sigma(\mathcal{O},\xi_1), \sigma(\mathcal{O},\xi_2) \ldots, \sigma(\mathcal{O},\xi_r)\}.$$

2. Define $\overline{A}(\mathcal{O}) = A(\mathcal{O})/[\cap_j \ker \xi_j]$.
3. Have inclusion $\Sigma \hookrightarrow \mathcal{M}(\overline{A})$, $\quad \sigma \mapsto (x(\sigma), \xi(\sigma))$ so

   $$\sigma_0 \mapsto (1,1), \quad \sigma(\mathcal{O},\xi_j) \mapsto (1,\xi_j).$$

Source: bijection $\mathrm{Unip}(G(\mathbb{F}_q)) \leftrightarrow$ all pairs $(\mathcal{O}, m) \quad (m \in \mathcal{M}(\overline{A}(\mathcal{O})))\ldots$

…and natural inclusion of $\widehat{W} \subset \mathrm{Unip}(G(\mathbb{F}_q))$.

Lusztig's description of left cells: formalism

Recall that finite group F gives

$$\mathcal{M}(F) = \{ (x,\xi) \mid x \in F,\ \xi \in \widehat{F^x} \}/(\text{conj by } F)$$

$$\simeq \text{irr conj-eqvt coherent sheaves } \mathcal{E}(x,\xi) \text{ on } F.$$

Sheaf $\mathcal{E}(x,\xi)$ supported on F-conjugacy class of x.

Subgp $S \subset F \rightsquigarrow$ const sheaf $\mathcal{S}$ on S is S-eqvt for conjugation.

Push forward to F-eqvt sheaf supp on F-conjugates of S:

$$i_*(\mathcal{S}) = \sum_{s,\xi} m_S(s,\xi)\mathcal{E}(s,\xi), \qquad m_S(s,\xi) = \dim \xi^{S^s}.$$

Sum runs over S conj classes $s \in S$. Can write this as

$$i_*(\mathcal{S}) = \sum_s \mathcal{E}(s, \mathrm{Ind}_{S^s}^{F^s}(\text{triv})).$$

This construct nonnegative integer combination of elements of $\mathcal{M}(F)$ from any subgroup $S \subset F$.

$$m_S(x,1) = \#(S\text{-conj classes in } F\text{-conj class of } x).$$

Fin grps question: do numbers $m_S(x,1)$ determine S up to conj in F?

Some examples

Fix $\mathcal{O} = \mathcal{O}(C_{LR})$ (special) nilpotent orbit.

If G has only classical simple factors, then $\overline{A}(\mathcal{O})$ is an elementary abelian 2-group.

That is, $\overline{A}(\mathcal{O}) = V$, d-diml vector space over $\mathbb{F}_2$.

Hence Lusztig's finite set is

$$\mathcal{M}(\overline{A}(\mathcal{O})) = V \times V^* = T^*(V),$$

$2d$-diml symplectic vector space over $\mathbb{F}_2$.

If $S \subset V$ is any subspace, then

$$m_S(x, \xi) = \begin{cases} 1 & \xi|_S = 0 \\ 0 & \xi|_S \neq 0 \end{cases}$$

This is the characteristic function of

$$S \times (V/S)^* = T^*_S(V) \subset T^*(V)$$

a d-dimensional Lagrangian subspace:

$$|\mathcal{M}(\overline{A}(\mathcal{O}))| = 2^{2d}, \qquad |T^*_S(V)| = 2^d.$$

Lusztig's description of left cells

Fix $(\Sigma \subset \widehat{W})$ family $\leftrightsquigarrow$ $(C_{LR} \subset W)$ two-sided cell.

$\mathcal{O} = \mathcal{O}(C_{LR})$ corresponding (special) nilpotent orbit.

Goal: describe Weyl rep $\mathbb{Z}C_L$, each left cell $C_L \subset C_{LR}$.

Know: $\mathbb{Z}C_L$ is nonnegative integer comb of elements of Σ.

Describe the combination using Lusztig's $\Sigma \hookrightarrow \mathcal{M}(\overline{A})$.

Theorem (Lusztig)

1. $\exists$ subgp $\Gamma = \Gamma(C_L) \subset \overline{A}$ so $\mathbb{Z}C_L \simeq \sum_{(x,\xi)} m_\Gamma(x,\xi)\sigma(x,\xi)$
2. $m_\Gamma(1,1) = 1$, so special rep σ appears once in $\mathbb{Z}C_L$.
3. $\exists$ Lusztig cells with $\Gamma = \overline{A}$, so $\mathbb{Z}C_L \simeq \sum_x \sigma(x,1)$.
4. $\exists$ Springer cells with $\Gamma = \{e\}$, so $\mathbb{Z}C_L \simeq \sum_{\xi \in \overline{A}^\frown} \dim(\xi)\sigma(1,\xi)$, all Springer reps for $\mathcal{O}$ in Σ.

Back to the example board

G classical, two-sided cell $(C_{LR} \subset W) \leftrightsquigarrow (\Sigma \subset \widehat{W})$ family.

Write $\mathcal{O} = \mathcal{O}(C_{LR})$, $\overline{A}(\mathcal{O}) = V$ (order 2^d).

Lusztig: cardinality of the family in $\widehat{W}$ is

$$\#\Sigma = \binom{2d+1}{d} = \binom{2d}{d} + \binom{2d}{d-1}.$$

That is the size of $\mathrm{im}(\Sigma) \subset T^*V$, $\quad \#T^*V = 2^{2d}$.

Each $C_L \leftrightsquigarrow T^*_{\Gamma(C_L)}V$, $\quad$ subspace $\Gamma(C_L) \subset V$.

$T^*_{\Gamma(C_L)} \subset \mathrm{im}(\Sigma)$, so $\mathbb{Z}C_L =$ sum of 2^d distinct irrs in family Σ.

Springer cell has $\Gamma = \{0\} \subset V$, 2^d W reps $\sigma(\mathcal{O}, \xi) \leftrightarrow V^* = (0, \xi)$.

Lusztig cell has $\Gamma = \overline{A}(\mathcal{O}) = V$, W reps attached to $(x, 0)$.

$$2^d \le \binom{2d+1}{d} \le 2^{2d}, \text{ equality } \iff d = 0.$$

Consequences of Lusztig for HC cells

HC world $(\mathfrak{g}, K)$: $\mathrm{Irr}(\mathfrak{g}, K)_\lambda \supset C =$ HC cell $\rightsquigarrow W(\lambda)$ rep $\mathbb{Z}C$.

By Joseph etc: $\mathbb{Z}C \supset$ unique $\sigma(C)$ special in $\widehat{W(\lambda)}$.

$\sigma(C) \rightsquigarrow \mathcal{O}(C) \rightsquigarrow \Sigma \subset \widehat{W}(\lambda)$ family, $\overline{A}(\mathcal{O})$, $\Sigma \hookrightarrow \mathcal{M}(\overline{A}(\mathcal{O}))$.

Theorem (McGovern, Binegar) C a HC cell in $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

1. $\mathbb{Z}C = \sum_{\tau \in \Sigma} m_C(\tau)\tau$, $\quad m_C(\tau) \in \mathbb{N}$, $\quad m_C(\sigma(C)) = 1$.
2. Assume $G(\mathbb{R})$ is a real form of $SO(n)$, $Sp(2n)$, type A, any complex group, or any exceptional group. Then $\exists \Gamma(C) \subset \overline{A}$ so $m_C(\sigma(x, \xi)) = m_\Gamma(x, \xi)$.
3. Assume $G(\mathbb{R})$ cplx, so $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_1$, $\overline{A}(\mathcal{O}) = \overline{A_1} \times \overline{A_1}$. Then we must take $\Gamma(C) = (\overline{A_1})_\Delta$. (Not one of Lusztig's Γ unless $A_1 = 1$.)
4. If G simple in (2) (excludes complex case), then $\Gamma(C)$ appears in Lusztig's description of left cells. That is, $\mathbb{Z}C$ is isomorphic to a left cell representation of $W(\lambda)$.

McGovern showed (4) fails for some forms of $Spin(n)$, $PSp(2n)$.

Conjecture. Part (2) is true for any HC cell C.

Next goal: try to understand geometric origin of $\Gamma(C)$.

Real forms of G

This page is incomplete at best! Don't trust it.

Pinning $\mathcal{P} = (G, B, H, \{X_\alpha \mid \alpha \in \Pi(B, H) \subset X^*(H)\})$.

Distinguished inv $\delta \in \mathrm{Aut}(\mathcal{P})$, extended group $G^\Gamma = G \rtimes \{1, \delta\}$.

Strong real form of G = G-conj class of $x \in G\delta$, $x^2 \in Z(G)$.

Strong form $x \rightsquigarrow$ inv aut $\theta_x = \mathrm{Ad}(x)$ Cartan for real form.

Summary: (conj classes of invs in G) $\leftrightsquigarrow$ ($\mathbb{R}$-forms of G).

Ex: $G = GL(n)$, involution $x_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \leftrightsquigarrow U(p, q)$.

Coming up: (involution respecting ??) $\leftrightsquigarrow$ (real form of ??).

Real forms of nilpotents

Pinning $\mathcal{P} = (G, B, H, \{X_\alpha\})$.

θ Cartan inv $\rightsquigarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ $\quad \mathcal{N} =$ nilp cone $\supset \mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{s}$.

Theorem (Jacobson-Morozov, Kostant, Kostant-Rallis)
$\mathcal{O} \subset \mathfrak{g}$ nilpotent orbit.

1. $\exists$ Lie triple $(T, E, F), [T, E] = 2E, [T, F] = -2F, [E, F] = T$,

$$E \in \mathcal{O}; \quad T \in \mathfrak{h} \text{ dom}; \quad T \text{ is unique} \rightsquigarrow \phi\colon SL(2) \to G$$

Define $\mathfrak{g}[j] = \{X \in \mathfrak{g} \mid [T, X] = jX\}$. JM parabolic is

$$\mathfrak{l} = \mathfrak{g}[0], \quad \mathfrak{u} = \sum_{j>0} \mathfrak{g}[j], \quad \mathfrak{q} = \mathfrak{l} + \mathfrak{u}.$$

2. $G^E = (L^E)(U^E) = G^\phi U^E$ Levi decomp.
3. BIJECTION ($\mathbb{R}$-forms of $\mathcal{O}$) $\rightsquigarrow$ (G^ϕ-conj classes
   $\{\ell \in G^\phi \mid \ell^2 \in \phi(-I)Z(G)\}$, $\ell \mapsto \mathbb{R}$-form $x = \ell \cdot \phi\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\}$).
4. Summary: (conj classes of invs in G^ϕ) $\leftrightsquigarrow$($\mathbb{R}$-forms of $\mathcal{O}$).