INFINITE-DIMENSIONAL REPRESENTATIONS OF REAL REDUCTIVE GROUPS

DAVID A. VOGAN, JR.

1. First introduction: $GL(2)$

The purpose of these notes is to introduce the infinite-dimensional representations of real reductive Lie groups: to say what they are, what is known about them, what is *not* known about them, and why one might care.

I'll begin with an extremely general definition, then look at some examples in $GL(2,\mathbb{R})$.

Definition 1.1. *Suppose G is a topological group. A (continuous)* representation *of G is a pair (π, V) consisting of a complete locally convex topological vector space V, and a continuous homomorphism $\pi\colon G \to \mathrm{Aut}(V)$. "Continuity" means that the action map*

$$G \times V \to V, \qquad (g,v) \mapsto \pi(g)v$$

is continuous. "Locally convex" means that the space has lots of continuous linear functionals, which is technically fundamental. "Complete" allows us to take limits in V, and so define things like integrals and derivatives.

The representation (π, V) is irreducible *if V has exactly two closed invariant subspaces (which are necessarily 0 and V).*

The representation (π, V) is unitary *if V is a Hilbert space, and the operators $\pi(g)$ are unitary.*

I will assume that you already know lots of interesting examples of finite-dimensional representations, and pass directly to the infinite-dimensional setting. Because we will look at representations on many different kinds of topological space (sometimes preferring to use smooth functions rather than L^2 functions, for example) it is useful to generalize the definition of unitary a little. We will say that π is *pre-unitary* if there is a positive definite G-invariant (continuous) Hermitian form $\langle , \rangle_\pi$ on V.

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Example 1.2. *Suppose $G = GL(2,\mathbb{R})$, the 4-dimensional group of invertible 2×2 real matrices. Then G acts transitively on the space Z of positive definite quadratic forms on $\mathbb{R}^2$. Inside Z there is a natural basepoint z_0, the standard inner product on $\mathbb{R}^2$. The stabilizer of z_0 is the one-dimensional group $O(2)$ of real orthogonal matrices, and therefore*

$$Z \simeq GL(2,\mathbb{R})/O(2),$$

a three-dimensional homogeneous space. The space Z appears for example in the study of rational quadratic forms in two variables. The fundamental idea of representation theory is to study such (nonlinear) objects by means of (linear) spaces of functions on them. So for example we can consider

$$\begin{aligned} V^\infty &= \textit{complex-valued smooth functions on } Z\\ V^\infty_m &= \textit{complex-valued smooth densities on } Z\\ V^\infty_{m,c} &= \textit{compactly supported smooth densities}\\ V^{-\infty} &= \textit{generalized functions on } Z = \textit{dual of } V^\infty_{m.c}\\ L^2 &= \textit{complex-valued square integrable functions on } Z \end{aligned}$$

(with respect to a G-invariant measure on Z). Each of these spaces, and countless others like them, is a complete locally convex topological vector space. The group G acts on functions on Z by

$$\pi(g)f(z) = f(g^{-1}\cdot z),$$

and in a similar fashion on densities and on distributions. In this way we get (continuous) representations (written (π^∞, V^∞), $(\pi^\infty_m, V^\infty_m)$, and so on) of G on each of these vector spaces.

Just as a reminder that these abstract matters are not quite trivial, I will mention a non-example:

$$L^\infty = \text{bounded measurable functions on } Z$$

This is a perfectly good complete locally convex topological vector space, even a Banach space. But if f is a discontinuous bounded function on Z, then even small translations of f may produce large changes in the values of f; so the action of G is not a (continuous) Banach space representation.

The representation π^2 on L^2 is unitary. None of the other representations is unitary, because none of the other spaces is a Hilbert space. Notice however that there is a G-equivariant continuous inclusion of $V^\infty_{m.c}$ into L^2 with a dense image, using the G-invariant measure on Z. It follows immediately that $\pi^\infty_{m,c}$ is pre-unitary. Using duality, it follows also that L^2 is included in $V^{-\infty}$ as a dense subspace.

Exercise 1.3. *Prove that each of the representations in Example 1.2 is reducible, by exhibiting as explicitly as possible a proper closed G-invariant subspace.*

One lesson to be extracted from this example is that there are many different topological representations (like $V^\infty_{m.c} \hookrightarrow L^2 \hookrightarrow V^{-\infty}$) that look to an algebraic eye to be almost the same. Exactly which representation we should consider depends on exactly what kind of questions we are asking. On the Hilbert space L^2 we have a powerful and general spectral theory, allowing us to diagonalize many operators. On the other two spaces the action of the group can be differentiated to get a Lie algebra representation, which is algebraically a much more elementary object. (The reason is that a Lie algebra representation is a *linear* map.)

Here is a rather different family of representations.

Example 1.4. *Suppose $G = GL(2,\mathbb{R})$. Then G acts transitively on the space $X(\mathbb{R})$ of complete flags in $\mathbb{R}^2$; that is, chains of subspaces*

$$\{0\} = V_0 \subset V_1 \subset V_2 = \mathbb{R}^2, \qquad \dim V_j = j.$$

Of course this is the same as the one-dimensional projective space $\mathbb{RP}^1$ of lines in $\mathbb{R}^2$; it may be identified with the unit circle in $\mathbb{R}^2$ modulo ± 1.

Inside $X(\mathbb{R})$ there is a natural basepoint

$$x_0 = (\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2),$$

with $\mathbb{R}^1$ the span of the first coordinate vector (the x axis). The stabilizer of x_0 is the Borel subgroup

$$B(\mathbb{R}) = \left\{ \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \mid a_1, b, a_2 \in \mathbb{R}, a_1 a_2 \neq 0 \right\}$$

of upper-triangular matrices, and therefore

$$X(\mathbb{R}) \simeq GL(2,\mathbb{R})/B(\mathbb{R}).$$

On the space $X(\mathbb{R})$ there are two natural G-equivariant real line bundles: $\mathcal{L}_1$, whose fiber at the flag (V_0, V_1, V_2) is V_1, and $\mathcal{L}_2$, whose fiber is V_2/V_1. The bundle $\mathcal{L}_1$ is often called the tautological bundle. *The isotropy group $B(\mathbb{R})$ acts on the fiber at x_0 of $\mathcal{L}_j$, defining two characters*

$$\chi_j \colon B(\mathbb{R}) \to \mathbb{R}^\times, \qquad \chi_j\left(\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}\right) = a_j.$$

If n_1 and n_2 are integers, then one can define the tensor power line bundle

$$\mathcal{L}_{(n_1,n_2)} = {\mathcal{L}_1}^{\otimes n_1} \otimes {\mathcal{L}_2}^{n_2}.$$

It is easy to see that smooth sections of $\mathcal{L}_{(n_1,n_2)}$ correspond to smooth functions

$$f\colon G \to \mathbb{R}, \qquad f(gb) = f(g)\chi_1(b)^{-n_1}\chi_2(b)^{-n_2} \qquad (b \in B(\mathbb{R})).$$

If we introduce absolute values, then the exponents n_i in this formula can be replaced by any complex numbers. We begin (for any $\nu \in \mathbb{C}^2$) by defining a character

$$\chi_\nu\colon B(\mathbb{R}) \to \mathbb{C}^\times, \qquad \chi_\nu(b) = |\chi_1(b)|^{\nu_1}|\chi_2(b)|^{\nu_2}.$$

Then we get a complex line bundle $\mathcal{L}_{(\nu_1,\nu_2)}$ on $X(\mathbb{R})$ whose smooth sections correspond to smooth functions

$$f\colon G \to \mathbb{C}, \qquad f(gb) = \chi_\nu(b^{-1})f(g) \quad (b \in B(\mathbb{R})).$$

Clearly G acts by left translation on this space of smooth sections. We write

$$W^\infty_{(\nu_1,\nu_2)} = \textit{smooth sections of } \mathcal{L}_{(\nu_1+\frac{1}{2},\nu_2-\frac{1}{2})}.$$

The shift by $(\frac{1}{2},-\frac{1}{2})$ is a traditional convenience. Here is why it is introduced. The bundle $\mathcal{L}_{(1,-1)}$ can be identified with the bundle of densities on $X(\mathbb{R})$ (by picking a Lebesgue measure on $T_{x_0}(X(\mathbb{R})$; the character $\chi_{(1,-1)}$ is how the isotropy group $B(\mathbb{R})$ acts on such measures). This means that if ϕ_1 and ϕ_2 are smooth sections of $\mathcal{L}_{(\frac{1}{2},-\frac{1}{2})}$, then $\phi_1\overline{\phi}_2$ (a smooth section of $\mathcal{L}_{(1,-1)}$) may be identified with a complex-valued smooth density on $X(\mathbb{R})$. Integrating this density over the compact manifold $X(\mathbb{R})$ gives a complex number, and so defines a G-invariant pre-Hilbert space structure on $W^\infty_{(0,0)}$.

Exactly the same argument applies whenever ν is purely imaginary: we can define

$$\langle\phi_1,\phi_2\rangle = \int_{X(\mathbb{R})} \phi_1\overline{\phi_2} \qquad \phi_i \in W^\infty_{(\nu_1,\nu_2)}, \nu_j \in i\mathbb{R},$$

a pre-Hilbert space structure. Write

$$L^2_{(\nu_1,\nu_2)} = \textit{Hilbert space completion of } W^\infty_{(\nu_1,\nu_2)} \ (\nu_j \in i\mathbb{R}).$$

This may be identified with certain measurable sections of $\mathcal{L}_{(\nu_1+\frac{1}{2},\nu_2-\frac{1}{2})}$, and so with measurable functions on G transforming by $\chi_{(\nu_1-\frac{1}{2},\nu_2+\frac{1}{2})}$ under $B(\mathbb{R})$ on the right.

Exercise 1.5. *Prove that the representation $W^\infty_{(-\frac{1}{2},\frac{1}{2})}$ is reducible, by exhibiting a one-dimensional G-invariant subspace.*

This is the representation on smooth functions on $X(\mathbb{R})$.

Exercise 1.6. *Prove that the representation $W^\infty_{(\frac{1}{2},-\frac{1}{2})}$ is reducible.*

This is the representation on smooth densities on $X(\mathbb{R})$.

Exercise 1.7. *Suppose that $F_{(n_1,n_2)}$ is the finite-dimensional algebraic representation of $GL(2,\mathbb{C})$ of highest weight (n_1,n_2). Prove that if n_1 and n_2 are even, then $F_{(n_1,n_2)}$ has a vector transforming by the character $\chi_{(n_1,n_2)}$ of $B(\mathbb{R})$.*

Exercise 1.8. *Suppose that $n_1 \geq n_2$ are non-negative even integers. Prove that the representation $\rho^\infty_{(-(n_1+\frac{1}{2}),-(n_2-\frac{1}{2}))}$ contains $F^*_{n_1,n_2}$ as a subrepresentation. Prove that $\rho^\infty_{((n_1+\frac{1}{2}),(n_2-\frac{1}{2}))}$ is reducible. Prove that $\rho^\infty_{(\nu_1,\nu_2)}$ is reducible whenever $\nu_1-\nu_2$ is an odd integer.*

Exercise 1.9. *Insert into Example 1.4 two parities ϵ_1 and ϵ_2 in $\mathbb{Z}/2\mathbb{Z}$. Wherever $|\chi_i(b)|^{\nu_i}$ appears, replace it by $|\chi_i(b)|^{\nu_i}\operatorname{sgn}(\chi_i(b))^{\epsilon_i}$. What happens to the example, and to the exercises that follow?*

Notice that the pair (ν,ϵ) (with $\nu \in \mathbb{C}$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$) defines a general one-dimensional character of the multiplicative group $\mathbb{R}^\times$.

Exercise 1.10. *Replace the 2 (in $GL(2,\mathbb{R})$) by any positive integer n in Example 1.4 and in the exercises that follow.*

Proposition 1.11. *In the setting of Example 1.4, suppose that $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$.*

1. *The representation ρ^∞_ν on W^∞_ν is irreducible unless $\nu_1-\nu_2$ is an odd integer.*
2. *If $\nu_1-\nu_2$ is a positive odd integer m, then W^∞_ν has a unique m-dimensional quotient representation E_ν. The corresponding subrepresentation*
$$D_\nu = \ker(W^\infty_\nu \to E_\nu) \qquad (\nu_1-\nu_2 \textit{ odd positive})$$
is irreducible.
3. *The representation $\rho^\infty_{(\nu_1,\nu_2)}$ is equivalent to $\rho^\infty_{(\nu_2,\nu_1)}$ unless $\nu_1-\nu_2$ is an odd integer.*
4. *If $\nu_1-\nu_2$ is a positive odd integer m, then there is a short exact sequence*
$$0 \to E_\nu \to W^\infty_{(\nu_2,\nu_1)} \to D_\nu \to 0,$$
exhibiting E_ν as the unique proper subrepresentation of $W^\infty_{\nu_2,\nu_1}$. In particular, $\rho^\infty_{(\nu_1,\nu_2)}$ and $\rho^\infty_{(\nu_2,\nu_1)}$ have equivalent composition series.
5. *The collection of representations*
$$\{W^\infty_\nu \mid \nu_1-\nu_2 \notin (2\mathbb{Z}+1)\} \cup \{E_\nu \mid \nu_1-\nu_2 \textit{ odd positive integer}\}$$
in a certain sense exhausts the irreducible representations of G having a non-zero vector fixed by $O(2)$.

The phrase "in a certain sense" requires a serious explanation. There are many representations (like L^2_ν for $\nu \in (i\mathbb{R})^2$) that are not *equivalent* to any W^∞_ν, but rather are topological variations of them. We will return to Harish-Chandra's resolution of this issue in Section 7.

Here is one of the morals of the story so far.

> **Representations of a real reductive group $G(\mathbb{R})$ are almost parametrized by characters (homomorphisms to $\mathbb{C}^\times$) of a Cartan subgroup $H(\mathbb{R})$.**

A little more precisely, we expect for each $H(\mathbb{R})$ to be able to construct a family of representations $\{\pi(\chi) \mid \chi \in \widehat{H}(\mathbb{R})\}$ indexed (more or less) by characters of $H(\mathbb{R})$. A submoral is that the indexing may behave better if we shift it a little bit (as with the shift $(\frac{1}{2}, -\frac{1}{2})$ in Example 1.4). We expect most of the representations $\pi(\chi)$ to be irreducible. They should depend only on the conjugacy class of $(H(\mathbb{R}), \chi)$; in particular, they should (for fixed $H(\mathbb{R})$) be constant on orbits of the real Weyl group.

Because unitary representations are so fundamental, we interrupt the algebraic development to describe those.

Proposition 1.12. *In the setting of Example 1.4, suppose that $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$.*

1. *The representation ρ^∞_ν is pre-unitary whenever ν is purely imaginary, and whenever*
$$\nu = (s + it, -s + it), \qquad s \in (-\frac{1}{2}, \frac{1}{2}),\ t \in \mathbb{R}.$$
None of the other irreducible representations ρ^∞_ν is pre-unitary.
2. *The one-dimensional representations $E_{(\frac{1}{2}+it, -\frac{1}{2}+it)}$ (for $t \in \mathbb{R}$) are unitary; none of the other finite-dimensional representations E_ν is unitary.*
3. *For m a positive odd integer and $t \in \mathbb{R}$, the representation $D_{(m/2+it, -m/2+it)}$ is pre-unitary. None of the other representations D_ν is pre-unitary.*

Example 1.13. *In order to test the first moral of the story, we look for representations attached to the other conjugacy class of Cartan subgroup in $GL(2, \mathbb{R})$. We get this Cartan by identifying $\mathbb{R}^2$ with $\mathbb{C}$; then the multiplicative action of $\mathbb{C}^\times$ on $\mathbb{C}$ defines an inclusion*
$$T(\mathbb{R}) \simeq \mathbb{C}^\times \hookrightarrow GL(2, \mathbb{R}), \qquad x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Just as in the previous example, we build all the characters of $T(\mathbb{R})$ from two special characters that are the eigenvalues of the matrix:

$$\xi_1(z) = z, \quad \xi_2(z) = \overline{z}, \qquad z \in T(\mathbb{R}) \simeq \mathbb{C}^\times.$$

It turns out that the characters of $T(\mathbb{R})$ are

$$\xi_{(\nu_1,\nu_2)}(z) = z^{\nu_1}\overline{z}^{\nu_2} \qquad \nu_i \in \mathbb{C}, \quad \nu_1 - \nu_2 \in \mathbb{Z}.$$

This may look a bit more plausible if it is rewritten as

$$\xi_{(\nu_1,\nu_2)}(r\, e^{i\theta}) = r^{\nu_1+\nu_2}\, e^{i(\nu_1-\nu_2)\theta} \qquad \nu_i \in \mathbb{C}, \quad \nu_1 - \nu_2 \in \mathbb{Z}.$$

Using Proposition 1.11, we can now define a representation

$$D_{(\nu_1,\nu_2)} = D_{(\nu_2,\nu_1)} = W_\nu^\infty / E_\nu, \qquad \nu \in \mathbb{C}^2, \quad \nu_1 - \nu_2 \textit{ odd positive.}$$

Exercise 1.14. *Show that the Weyl group of $T(\mathbb{R})$ in $G(\mathbb{R})$ has order two, and acts by the Galois action of complex conjugation on $\mathbb{C}^\times$. Therefore this action carries $\xi_{(\nu_1,\nu_2)}$ to $\xi_{(\nu_2,\nu_1)}$.*

Exercise 1.15. *(Assuming Exercise 1.9.) Suppose $\nu \in \mathbb{C}^2$ and $\epsilon \in (\mathbb{Z}/2\mathbb{Z})^2$. Assume $\nu_1 - \nu_2$ is a non-negative integer m, of parity opposite to $\epsilon_1 - \epsilon_2$. Prove that $V_{\nu,\epsilon}^\infty$ has an m-dimensional quotient representation*

$$E_{\nu,\epsilon} = W_{\nu,\epsilon}^\infty / D_{\nu,\epsilon}.$$

It turns out that $D_{\nu,\epsilon}$ is independent of ϵ (always subject to the parity assumptions of the exercise); so this exercise completes a definition of an irreducible representation D_ν attached to each character ν of $T(\mathbb{R})$ with $\nu_1 - \nu_2$ a non-negative integer. We define

$$D_{(\nu_1,\nu_2)} = D_{(\nu_2,\nu_1)}$$

when $\nu_1 - \nu_2$ is a non-positive integer.

Of course this is an extraordinarily unsatisfactory definition: although it is more or less constructive, the construction makes almost no reference to $T(\mathbb{R})$. There is also a peculiar feature in the case $\nu_1 - \nu_2 = 0$: in that case D_ν is defined to be *equal* to the irreducible representation $W_{\nu,(\text{even},\text{odd})}^\infty$.

I will conclude this introduction with some hints about classical harmonic analysis problems: about how one might hope to relate irreducible representations (typically related to flag varieties, like the representations W_ν^∞ of Example 1.4) to interesting reducible representations (like the function spaces of Example 1.2). We will follow [8], to which we refer for details and much more information.

Always we are considering $G = GL(2,\mathbb{R})$. Recall first from Example 1.2 the (large, very reducible) representation space

$$\text{(1.16a)} \qquad V^\infty = C^\infty(Z) \simeq \{f \in C^\infty(G) \mid f(gk) = f(g),\ k \in O(2)\},$$

and the (small, usually irreducible) representation spaces

$$W^\infty_\nu \simeq \{\phi \in C^\infty(G) \mid \phi(gb) = \chi_{\nu+\rho}(b^{-1})\phi(g),\ b \in B(\mathbb{R})\}. \tag{1.16b}$$

Here we write $\rho = (\frac{1}{2}, -\frac{1}{2})$, and

$$\chi_\eta(b) = |\chi_1(b)|^{\eta_1}|\chi_2(b)|^{\eta_2} \qquad (b \in B(\mathbb{R}),\ \eta \in \mathbb{C}^2); \tag{1.16c}$$

the Borel subgroup $B(\mathbb{R})$ and its characters χ_i are as defined in Example 1.4. On the spaces V^∞ and W^∞_ν the group G acts by left translation. In order to clarify the connection with classical mathematics, we note

Lemma 1.17. *Suppose $q \in Z$ is a positive definite quadratic form on $\mathbb{R}^2$ (Example 1.2). Write $q_{ij} (i,j \in \{1,2\})$ for the four entries of q (regarded as a 2×2 symmetric matrix). Define*

$$d(q) = \det(q)^{1/2} = (q_{11}q_{22} - q_{12}^2)^{1/2},$$

a positive real number, and

$$z(q) = \frac{q_{11} - q_{22}}{q_{11} + q_{22}} - i\frac{2q_{12}}{q_{11} + q_{22}},$$

a complex number. Then $z(q)$ belongs to the open unit disk D in the complex plane. The map (d, z) is a diffeomorphism of Z onto $\mathbb{R}^+ \times D$. The action of G on Z corresponds to the action on $\mathbb{R}^+$ by multiplication by $\det^{-1}$, and the action on D by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{[(a+d) + (b-c)i]z + [(a-d) + (-b-c)i}{[(a-d) + (b+c)i]z + [(a+d) + (c-b)i]}$$

for matrices of positive determinant.

The proof (and most likely the correction of the statement) are left as an exercise.

Exercise 1.18. *If $t > 0$ and $q \in Z$ is a positive definite quadratic form, then tq is also a positive definite quadratic form. Show that $d(tq) = td(q)$, and that $z(tq) = z(q)$. Show that the action of G commutes with this action of $\mathbb{R}^+$.*

A function $f \in V^\infty$ on positive forms is called homogeneous of degree λ_1 *if*

$$f(tq) = t^{\lambda_1} f(q).$$

This definition makes sense for any $\lambda_1 \in \mathbb{C}$. Show that that the subspace of functions homogeneous of degree λ_1 is a nonzero closed G-invariant subspace, isomorphic to $C^\infty(D)$ (in fact as a representation of $SL(2, \mathbb{R})$).

Because the group $O(2)$ is compact, integration over $O(2)$ is an easy way to produce functions in V^∞. The *Poisson transform* is the map

$$P\colon W_\nu^\infty \to V^\infty, \qquad (P\phi)(g) = \int_{O(2)} \phi(gk)dk. \tag{1.19}$$

Exercise 1.20. *Show that P is a G-equivariant continuous linear map. Show that W_ν^∞ has a one-dimensional subspace of $O(2)$-invariant vectors, and that P is non-zero on this subspace. Conclude (using Proposition 1.11) that P is one-to-one unless $\nu_1 - \nu_2$ is a positive odd integer.*

Show that the range of P is contained in functions homogeneous of degree $\nu_1 + \nu_2$.

Exercise 1.21. *The classical Poisson transform goes from functions on the unit circle to functions on the unit disk, and it's given by a formula that looks significantly more complicated than (1.19). Find some relationship between the two. (Hint: since $G = SO(2) \cdot B(\mathbb{R})$, functions in W_ν^∞ are determined by their restriction to $SO(2)$. Show that this restriction defines an isomorphism*

$$W_\nu^\infty \to \textit{smooth even functions on SO(2).}$$

Even functions on $SO(2)$ may in turn be identified with functions on the circle. In this way the domain of P is identified with functions on the circle.

The range of P, on the other hand, consists of homogeneous functions of degree $\nu_1 + \nu_2$. According to Exercise 1.18, the range may therefore be identified with smooth functions on the disk. That is, we can think of the Poisson transform as a map

$$P_{\nu_1 - \nu_2}\colon C^\infty(\textit{circle}) \to C^\infty(\textit{disk}),$$

depending on the parameter $\nu_1 - \nu_2$. Your final mission, should you decide to accept it, is to write an entirely classical-looking formula for this map, making no reference to $GL(2, \mathbb{R})$. If everything goes well, this should look like the classical Poisson formula when $\nu_1 - \nu_2 = -1$. As a hint, here is an approximation of a final formula (for ϕ_0 a smooth function on the circle, regarded as a periodic function of θ, and f the image function on the unit disk):

$$f(r\, e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2} \right)^{\frac{1 - (\nu_1 - \nu_2)}{2}} \phi_0(\theta) d\theta$$

In order to prove a formula like this, notice that both the Poisson mapping P and this mapping respect the rotation actions. It is therefore enough to consider the case $t = 0$.

So what is special about the functions on Z that are in the image of the Poisson mapping P? We have already seen that they are homogeneous (Exercise 1.18), which means that they are eigenfunctions of a certain first order differential operator (like $t\frac{\partial}{\partial t}$) on Z, commuting with the action of G. There is also a *second* order operator on Z commuting with the action of G: it is the Laplace operator Δ_D for the hyperbolic metric on the unit disk D ([8], page 31; this is the operator that Helgason calls L).

Proposition 1.22 ([8], page 331, Exercise 4)**.** *Consider the action of $SL(2,\mathbb{R})$ on functions on the unit disk D, arising by restriction of the $GL(2,\mathbb{R})$ action on Z. This makes the enveloping algebra of $SL(2,\mathbb{R})$ act by differential operators on D. The Casimir operator Ω for $SL(2,\mathbb{R})$ acts by the Laplace operator Δ_D.*

Corollary 1.23 ([8], page 36, Theorem 4.3)**.** *The Casimir operator Ω for $SL(2,\mathbb{R})$ acts on the representation $W^\infty_{\nu_1,\nu_2}$ by the scalar $\lambda(\nu) = [(\nu_1-\nu_2)^2-1]/4$. Since the Poisson mapping P commutes with the action of $SL(2,\mathbb{R})$, the image of P is contained in the $\lambda(\nu)$ eigenspace of Δ_D.*

In coordinates $z = x+iy$ on the unit disk D, the Laplace operator is

$$\Delta_D = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \tag{1.24}$$

An eigenvalue of Δ_D with eigenvalue 0 (corresponding to the case $\nu_1-\nu_2 = \pm 1$) is therefore the same as a harmonic function on D (for the Euclidean Laplace operator). For other eigenvalues this is not the case. That may be part of the reason that the Poisson kernel is treated only for the case $\nu_1-\nu_2=-1$ in most texts. Of course linear fractional transformations of the disk are central to any study of the Poisson kernel. These transformations preserve the hyperbolic Riemannian metric and Δ_D, but not the Euclidean metric and Euclidean Laplace operator.

Theorem 1.25. *Let Δ_1 be the first-order differential operator on Z arising by differentiation of the dilation action, and Δ_2 the action of the Casimir operator for $SL(2,\mathbb{R})$ (corresponding to the Laplace operator on D). If $\lambda = (\lambda_1,\lambda_2)\in\mathbb{C}^2$, define*

$$V^\infty_\lambda = \{f\in C^\infty(Z) \mid \Delta_i f = \lambda_i f \quad (i=1,2)\}.$$

Then V^∞_λ is a closed G-invariant subspace of V^∞.

Fix $\nu \in \mathbb{C}^2$, and define

$$\lambda_1(\nu) = \nu_1 + \nu_2, \qquad \lambda_2(\nu) = [(\nu_1 - \nu_2)^2 - 1]/4.$$

Then the Poisson transform P of (1.19) *carries W_ν^∞ into V_λ^∞.*

Notice that every λ is of the form $\lambda(\nu)$ for some ν. In fact ν is uniquely determined up to interchanging its coordinates. Since we know that P is always nonzero on the $O(2)$-fixed line in W_ν^∞ (Exercise 1.20), it follows that each eigenspace V_λ^∞ is nonzero. If $\nu_1 - \nu_2$ is not an odd negative integer, which we can always arrange by perhaps interchanging the coordinates, it even follows that $V_\lambda^\infty(\nu)$ contains a copy of the (infinite-dimensional) representation W_ν^∞.

Of course it is natural to ask whether *all* eigenfunctions of the Δ_i are in the image of P. We refer to [8] for details about this question, but here is a sketch. Recall that W_ν^∞ may be identified with smooth functions on the circle. The formula in Exercise 1.20 writes P as an integral over the circle of this smooth function (of θ) against a kernel that is analytic in θ (and in the parameter $z = re^{it} \in D$). It follows that the integral is defined not only for *smooth* functions on the circle, but also for continuous functions, or even for distributions, or even for *hyperfunctions*. On all of these spaces it yields (analytic) functions in $V_{\lambda(\nu)}^\infty$. As long as $\nu_1 - \nu_2$ is not a positive odd integer, this extension of the Poisson transform (from smooth functions to hyperfunctions on the circle) is a surjective isomorphism ([7]; see [8], page 36, Theorem 4.3). This theorem of Helgason has been the foundation of enormous generalizations, beginning with [10].

What about going *from* functions in V_λ^∞ back to W_ν^∞? Since the Poisson map from the second to the first is not surjective, we should not expect to have an everywhere-defined inverse. Again we must refer to [8] for details, but here is one weak result.

Proposition 1.26. *Suppose that $\nu \in \mathbb{C}^2$, $\mathrm{Re}(\nu_1 - \nu_2) < 0$, that $\phi \in W_\nu^\infty$, and $f = P\phi \in V_{\lambda(\nu)}^\infty$. Define $a^t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in G$. Then there is a constant $c(\nu)$ (independent of ϕ) so that*

$$\lim_{t \to +\infty} e^{-t(\nu_2 - \nu_1 - 1)} f(ga^t) = c(\nu)\phi(g).$$

In case $\nu = (-1/2, 1/2)$ the exponential in the limit is 1, $c(\nu) = 1$, and the Proposition is the classical fact that the Poisson kernel solves the boundary value problem for the Laplacian on the disk.

2. Second introduction: compact groups

Hiding in the background in many of the results about $GL(2,\mathbb{R})$ discussed in Section 1 was the theory of Fourier series on the circle. In order to describe Harish-Chandra's theory for reductive groups, we need to recall some generalizations of the theory of Fourier series to other compact groups. Since compact connected Lie groups are also examples of reductive groups, this will afford also the opportunity to express familiar facts about compact groups in the language we will use for general reductive groups.

In order to maintain an air of concreteness,[1] we will discuss only the case of the unitary group; but everything in this section extends in a straightforward fashion to any compact connected Lie group G. Doing this is an additional exercise for the reader.

$$\begin{aligned} G = U(n) &= \text{linear maps of } \mathbb{C}^n \text{ preserving inner product} \\ &= n \times n \text{ complex matrices } g \text{ such that } {}^t\overline{g} = g^{-1}. \end{aligned} \tag{2.1a}$$

Study of the unitary group depends on the maximal torus

$$\begin{aligned} H = U(1)^n &= \text{diagonal matrices in } U(n) \\ &= \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \qquad \theta_j \in \mathbb{R}. \end{aligned} \tag{2.1b}$$

A one-dimensional (complex) representation of the group H means a continuous action of H on a one-dimensional (complex) vector space. Any automorphism of a one-dimensional space is multiplication by a non-zero scalar; so a one-dimensional representation is the same as a continuous group homomorphism

$$\chi\colon H \to \mathbb{C}^\times \tag{2.1c}$$

Suppose $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. One example of a homomorphism χ is

$$\chi_m\colon H \to \mathbb{C}^\times, \qquad \chi_m(\operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})) = e^{i\sum m_j\theta_j}. \tag{2.1d}$$

Equivalently,

$$\chi_m(h) = \prod_{j=1}^{n} (j\text{th entry of } h)^{m_j}. \tag{2.1e}$$

(The motivation for this second formulation is to show that, despite the various $e^{i\theta}$ terms floating around, the representation χ_m can be

[1] Only in the study of higher mathematics is the smell of cement regarded as a consummation devoutly to be wished.

thought of as essentially algebraic rather than transcendental.) These are *all* the continuous characters of H, so we write

$$\widehat{H} = X^*(H) = \{\chi_m \mid m \in \mathbb{Z}^n\} \simeq \mathbb{Z}^n, \tag{2.1f}$$

the *lattice of characters* of H. For the moment one can think of $\widehat{H}$ and $X^*(H)$ as just two alternative notations for the same thing. (Here is a sneak peek ahead. For a Cartan subgroup H in a general real reductive group G, we are going to write $\widehat{H}$ for the (abelian group) of all continuous characters of H, and $X^*(H)$ for the sublattice of characters that extend algebraically to the complex points.)

Proposition 2.2. *Suppose (π, V) is a finite-dimensional (complex) representation of H. For $\chi \in \widehat{H}$, define*

$$V(\chi) = \{v \in V \mid \pi(h)v = \chi(h)v, \ \text{all } h \in H\}.$$

Then

$$V = \sum_{\chi \in \widehat{H}} V(\chi).$$

Exercise 2.3. *Prove this. What happens if H is replaced by the group $T(\mathbb{R})$ of Example 1.13?*

We define the *set of weights of H in V* as

$$\Delta(V, H) = \{\chi \in \widehat{H} \mid V(\chi) \neq 0\}, \tag{2.4}$$

a finite subset of $\widehat{H}$. Sometimes it is useful to think of $\Delta(V, H)$ as a multiset; that is, as a set in which each element has some positive integer multiplicity. Here the multiplicity of χ is equal to $\dim V(\chi)$, so that the cardinality of $\Delta(V, H)$ (as a multiset) is equal to $\dim V$.

Here also is some inconsistent notation: the *set of roots of H in $\mathfrak{g}_{\mathbb{C}}$* is

$$\Delta(\mathfrak{g}_{\mathbb{C}}, H) = \{\text{nonzero weights of } \mathrm{Ad}(H) \text{ on } \mathfrak{g}_{\mathbb{C}}\} \subset X^*(H). \tag{2.5}$$

Here

$$\mathfrak{g} = \mathrm{Lie}(G) = n \times n \text{ skew hermitian matrices}$$

is the Lie algebra of $G = U(n)$, and

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = n \times n \text{ complex matrices}$$

is its complexification. (The inconsistency with $\Delta(V, H)$ is that the zero weights of the adjoint action of H are not counted as roots.) The set of roots is

$$\Delta(\mathfrak{g}_{\mathbb{C}}, H) = \{\chi_{e_i - e_j}, 1 \leq i \neq j \leq n\}; \tag{2.6}$$

here $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}^n$, so

$$\chi_{e_i - e_j}(h) = h_i h_j^{-1} \qquad h \in H,$$

with h_i the ith diagonal entry of h. Finally, we fix a set of positive roots

$$\Delta^+ = \Delta^+(\mathfrak{g}_{\mathbb{C}}, H) = \{\chi_{e_i - e_j} \mid 1 \le i < j \le n\}. \tag{2.7}$$

Using the set of positive roots, we define a partial order on $\widehat{H}$ by

$$\chi_m \preceq \chi_{m'} \Leftrightarrow \chi_{m'} - \chi_m = \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \qquad n_\alpha \in \mathbb{N}. \tag{2.8}$$

Exercise 2.9. *Show that $\chi_m \preceq \chi_{m'}$ if and only if*

$$\sum_{i=1}^{p} m_i \le \sum_{i=1}^{p} m_i' \qquad (1 \le p < n),$$

and also

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i'.$$

Can you generalize this result to any compact connected group G?

Definition 2.10. *In the setting of* (2.1b) *the* Weyl group *of H in G is*

$$W = W(G, H) =_{def} N_G(H)/Z_G(H),$$

the quotient of the normalizer of H by the centralizer (which is H) of H. The Weyl group acts on H and on $\widehat{H}$. It is isomorphic to the symmetric group S_n, acting on H by permutation of the coordinates. It is therefore generated by the root reflections

$$s_{ij} = \textit{transposition of } i \textit{ and } j \qquad (1 \le i \ne j \le n).$$

*A weight $\chi_m \in \widehat{H}$ is called G-*dominant *(or Δ^+-*dominant *if we wish to emphasize the choice of positive roots) if m is decreasing; that is, if*

$$m_i \ge m_j, \quad \textit{all } i < j.$$

This condition is equivalent to

$$\chi_m \succeq w \cdot \chi_m, \quad \textit{all } w \in W.$$

The dominant weights are a fundamental domain for the action of W on $\widehat{H}$.

Definition 2.11. *Suppose (τ, V) is a representation of $G = U(n)$. A* highest weight *of τ is an element of $\Delta(V, H)$ that is maximal for the partial order $\preceq$ of* (2.8)*. An* extremal weight *of τ is an element of $\Delta(V, H)$ that is maximal for the partial order corresponding to* some *set of positive roots.*

Theorem 2.12 (Cartan-Weyl; see for example [11], Theorem 5.110, and [9], Proposition 21.3)**.** *Suppose $G = U(n)$ and H is the diagonal maximal torus of* (2.1b)*.*

(1) *Every irreducible representation of G is finite-dimensional.*
(2) *Every irreducible representation τ of G has a unique highest weight (Definition 2.10) $\mu_{hw}(\tau) \in \widehat{H}$.*
(3) *If τ and τ' are irreducible representations of G with $\mu_{hw}(\tau) = \mu_{hw}(\tau')$, then τ is equivalent to τ'.*
(4) *The highest weight of any irreducible representation of G is dominant (Definition 2.10).*
(5) *Every dominant weight $\mu \in \widehat{H}$ is the highest weight of a unique irreducible representation $\tau_{hw}(\mu)$ for G. Every weight μ' is an extremal weight of a unique irreducible representation $\tau_{ex}(\mu')$ of G. If μ is dominant, then $\tau_{hw}(\mu) = \tau_{ex}(\mu)$.*
(6) *The set of extremal weights of τ is equal to the W-orbit of the highest weight.*
(7) *Every weight of τ belongs to the lattice coset*
$$\mu_{hw}(\tau) + \mathbb{Z}\Delta(\mathfrak{g}_{\mathbb{C}}, H) \subset \widehat{H}.$$
(8) *The set of weights of τ is equal to the intersection of the lattice coset $\mu_{hw}(\tau) + \mathbb{Z}\Delta(\mathfrak{g}_{\mathbb{C}}, H)$ with the rational convex hull (in $\widehat{H} \otimes_{\mathbb{Z}} \mathbb{Q}$) of the extremal weights $W \cdot \mu_{hw}(\tau)$.*

The phrase "convex hull" does not appear in the references cited here. The following exercise concerns the connection between the notion of *saturated* set of weights appearing in [9] and the formulation in the theorem.

Exercise 2.13. *Suppose χ_m is a dominant weight and $\chi_{m'}$ is any other character of H. Prove (using the theorem) that $\chi_{m'}$ is a weight of $\tau_{hw}(\chi_m)$ if and only if*
$$\sum_{j \in J} m'_j \le \sum_{i=1}^{|J|} m_i \qquad (J \subsetneq \{1, \ldots, n\}),$$

and also

$$\sum_{i=1}^{n} m_i' = \sum_{i=1}^{n} m_i.$$

Corollary 2.14. *Suppose $G = U(n)$ and H is the diagonal maximal torus of* (2.1b)*. Then the correspondence*

$$\tau \leftrightarrow \{\textit{extremal weights of } \tau\}$$

establishes a bijection

$$\widehat{G} \leftrightarrow \widehat{H}/W(G,H)$$

between irreducible representations of G and Weyl group orbits of characters of H.

This is a good time to say again that exactly the same statement is true whenever G is a compact connected Lie group and H is a maximal torus.

Here are a few statements peculiar to $U(n)$.

Proposition 2.15. *Suppose $G = U(n)$ and $m \in \mathbb{Z}^n$ is dominant (that is, decreasing). Write $M = \sum_i m_i$. If $m_n \geq 0$, then the representation $\tau(\xi_m)$ of highest weight χ_m occurs as an irreducible constituent of the Mth tensor power $(\mathbb{C}^n)^{\otimes M}$.*

The one-dimensional representation det *of G has weight* $(1, \ldots, 1)$. *Consequently*

$$\tau(\chi_m) \otimes \det{}^N \simeq \tau(\chi_{m+(N,\ldots,N)}).$$

In particular, $\tau(\chi_m)$ occurs as an irreducible constituent of

$$(\mathbb{C}^n)^{\otimes(M-nm_n)} \otimes (\det{}^{m_n}).$$

After we define matrix coefficients in Definition 3.1 below, we will be able to deduce

Corollary 2.16. *Suppose $G = U(n)$. Write e_{ij} for the function on G whose value at the unitary matrix g is equal to the (i,j) entry of g. Then the matrix coefficients of finite-dimensional representations of G are precisely the polynomials in the functions e_{ij} and* $\det^{-1}$.

Unfortunately Theorem 2.12 and the Corollary 2.14 do *not* provide examples of the parametrization we want for general reductive groups (of irreducible representations by characters of Cartan subgroups). The difficulty is that there is nothing like the shift (appearing in Example

1.4) by $(1/2, -1/2)$. In the present setting, we have described an irreducible representation of G in terms of its set of weights. The description that generalizes better is in terms of its *character*.

Definition 2.17. *Suppose $G = U(n)$, $H = U(1)^n$ is the diagonal maximal torus, and $\Delta^+(\mathfrak{g}_{\mathbb{C}}, H)$ is the set of positive roots chosen in (2.7). Define*

$$2\rho = \sum_{\alpha \in \Delta^+} \alpha \in X^*(H) = \widehat{H},$$

the sum of the positive roots. (We are not yet defining ρ; the symbol 2ρ is so far a single unit.) In the identification of $\widehat{H}$ with $\mathbb{Z}^n$, it is easy to calculate

$$2\rho \leftrightarrow (n-1, n-3, \cdots, -(n-3), -(n-1)).$$

That is,

$$2\rho(h) = \prod_{j=1}^{n} (h_j)^{n-2j+1},$$

with h_j the jth diagonal entry of H.

The ρ double cover of H *is the group*

$$H_\rho = \{(h, z) \in H \times \mathbb{C}^\times \mid 2\rho(h) = z^2\}.$$

We write $\pi\colon H_\rho \to H$ for the projection on the first factor, and

$$\rho\colon H_\rho \to \mathbb{C}^\times, \rho(h, z) = z$$

for the projection on the second factor. Finally, define

$$\epsilon = (1, -1) \in H_\rho.$$

A representation τ of H_ρ is called genuine *if $\tau(\epsilon) = -\operatorname{Id}$. It is called* ordinary *if $\tau(\epsilon) = \operatorname{Id}$. Similarly, a function f on H_ρ is called* genuine *if $f(\epsilon\widetilde{h}) = -f(\widetilde{h})$; it is called* ordinary *if $f(\epsilon\widetilde{h}) = f(\widetilde{h})$.*

Proposition 2.18. *In the setting of Definition 2.17, there is a short exact sequence*

$$1 \to \{1, \epsilon\} \to H_\rho \xrightarrow{\pi} H \to 1$$

The function ρ is a genuine character of H_ρ.

The characters of H are in one-to-one correspondence (by composition with π with the ordinary characters of H_ρ. The characters of H are also in one-to-one correspondence (by composition with π and adding ρ) with the genuine characters of H_ρ. Consequently

$$\widehat{H_\rho} \simeq \widehat{H} \sqcup (\widehat{H} + \rho),$$

with the symbol $\sqcup$ indicating disjoint union.

There is a natural action of the Weyl group $W(G,H)$ on H_ρ, fixing ϵ and compatible via π with the action on H.

The proof is an exercise. The first short exact sequence explains the terminology "double cover."

Corollary 2.19. *In the setting of Definition 2.17, the covering group H_ρ does not depend on the choice of positive root system. That is, if $(\Delta^+)'$ is any other choice of positive roots, with $2\rho'$ the corresponding character, then there is a natural isomorphism*

$$H_\rho \to H_{\rho'}$$

over the identity map on H, carrying ϵ to ϵ'.

Finding a formula for this isomorphism is an exercise.

Exercise 2.20. *(The first part applies only to $G = U(n)$.) Show that the character 2ρ of H has a square root if and only if n is odd. Show that for even n, H_ρ is a connected torus*

$$H_\rho \simeq (i\mathbb{R})^n/(2\pi i L),$$

where

$$L = \{m \in \mathbb{Z}^n \mid \sum (n-2j+1)m_j \in 2\mathbb{Z}\}.$$

Show that for odd n,

$$H_\rho \simeq H \times \{1, \epsilon\}.$$

Can you find parallel statements that apply to a general compact connected Lie group G?

The definitions of $\preceq$ and dominant extend immediately to characters of H_ρ. For characters of H_ρ we need also another definition (which makes sense for characters of H, but is less important there).

Definition 2.21. *Suppose $G = U(n)$, $H = U(1)^n$ is the diagonal maximal torus, and H_ρ is the ρ double cover (Definition 2.17). A character $\xi \in \widehat{H_\rho}$ is called* regular *if for every coroot $\alpha^\vee$, the integer $\langle \xi, \alpha^\vee \rangle$ is not zero. It is equivalent to require that ξ is not fixed by any reflection $s_\alpha \in W(G,H)$, for $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, H)$. This in turn is equivalent to*

$$w \cdot \xi = \xi \Rightarrow w = 1 \qquad (w \in W(G,H)).$$

The sign character of W *is the homomorphism*

$$\mathrm{sgn}\colon W(G,H) \to \{\pm 1\}, \qquad \mathrm{sgn}(w) = \det(w),$$

where on the right we regard w as a linear transformation of the vector space $\mathfrak{h} = \mathrm{Lie}(H)$. This homomorphism is characterized by

$$\mathrm{sgn}(s_\alpha) = -1, \qquad (\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, H)).$$

For any character $\xi \in \widehat{H_\rho}$, we can define the Weyl numerator function *in $C^\infty(\widehat{H}_\rho)$ as*

$$N(\xi)(\widetilde{h}) = \sum_{w\in W(G,H)} \operatorname{sgn}(w)(w\cdot\xi)(\widetilde{h}).$$

This function is genuine or ordinary (Definition 2.17) according as the character ξ is genuine or ordinary.

The Weyl denominator function *is the genuine function*

$$\Delta(\widetilde{h}) = N(\rho)(\widetilde{h}) = \sum_{w\in W(G,H)} \operatorname{sgn}(w)(w\cdot\rho)(\widetilde{h}).$$

We emphasize that the following theorem applies as stated to any compact connected Lie group G.

Theorem 2.22 (Weyl Character Formula; see [11], Theorem 5.75)**.** *Suppose $G = U(n)$ and $H = U(1)^n$ is the diagonal maximal torus. Use the notation of Definition 2.21.*

(1) *The Weyl numerator functions satisfy $N(w\cdot\xi) = \operatorname{sgn}(w)N(\xi)$.*
(2) *Each Weyl numerator function satisfies*

$$N(\xi)(w\cdot\widetilde{h}) = \operatorname{sgn}(w)N(\xi)(\widetilde{h}) \qquad (\widetilde{h} \in H_\rho).$$

*Such a function is called W-*skew invariant.
(3) *The function $N(\xi)$ is nonzero if and only if ξ is regular.*
(4) *The functions $\{N(\xi) \mid \xi$ dominant regular$\}$ form an orthogonal basis for the W-skew invariant functions on H_ρ.*
(5) *Suppose τ is an irreducible representation of G, with character*

$$\Theta_\tau = \operatorname{tr}\tau \in C^\infty(G).$$

Then there is a unique dominant regular genuine character

$$\mu_{char}(\tau) \in \widehat{H_\rho}$$

so that

$$(\Theta_\tau|_H)\cdot N(\rho) = N(\mu_{char}(\tau)).$$

(6) *We have $\mu_{char}(\tau) = \mu_{hw}(\tau) + \rho$ (see Theorem 2.12).*
(7) *Adding ρ is a bijection from dominant characters of H to dominant regular genuine characters of H_ρ. In particular, μ_{char} is a bijection from $\widehat{G}$ to dominant regular genuine characters of H.*

Corollary 2.23. *Suppose $G = U(n)$ and $H = U(1)^n$ is the diagonal maximal torus.*

(1) *For each genuine character μ of $\widehat{H_\rho}$, there is a unique representation $\tau_{char}(\mu)$ characterized by the property*

$$(\Theta_{\tau_{char}(\mu)}|_H) \cdot N(\rho) = \pm N(\mu).$$

(2) *The representation $\tau_{char}(\mu)$ depends only on the Weyl group orbit $W \cdot \mu$.*
(3) *The representation $\tau_{char}(\mu)$ is irreducible if and only if μ is regular.*
(4) *The representation $\tau_{char}(\mu)$ is zero if and only if μ is not regular.*
(5) *The correspondence $\mu \mapsto \tau_{char}(\mu)$ establishes a bijection*

$$\widehat{G} \leftrightarrow (\textit{regular genuine characters of } H_\rho)/W(G,H).$$

Proposition 2.24. *Suppose G is a compact connected Lie group with maximal torus H. Then restriction to $\mathfrak{h} = \mathrm{Lie}(H)$ defines a one-to-one correspondence from G-invariant negative Riemannian structures on G to W-invariant negative inner products on $\mathfrak{h}$. These correspond in turn by duality to W-invariant positive inner products on*

$$i\mathfrak{h}^* \simeq X^*(H) \otimes_{\mathbb{Z}} \mathbb{R}.$$

If $G = U(n)$ and $H = U(1)^n$ is the diagonal maximal torus, then the identification $X^*(H) \simeq \mathbb{Z}^n$ identifies $i\mathfrak{h}^*$ with $\mathbb{R}^n$.

Fix such a W-invariant positive inner product $\langle , \rangle$ on $i\mathfrak{h}^*$; in the case of $U(n)$, we can choose the standard inner product on $\mathbb{R}^n$. Let $\Omega \in U(\mathfrak{g}_{\mathbb{C}})$ be the corresponding Casimir operator, so that Ω acts on $C^\infty(G)$ by the Laplace-Beltrami operator L_Ω for the (negative) Riemannian structure on G.

Suppose (τ, V) is an irreducible representation of G of highest weight $\mu_{hw}(\tau) \in \widehat{H} \subset i\mathfrak{h}^*$. Then the Casimir operator Ω acts in τ by the scalar

$$\begin{aligned}
\tau(\Omega) &= \langle \mu_{hw}(\tau) + 2\rho, \mu_{hw}(\tau)\rangle \\
&= \langle \mu_{hw}(\tau) + \rho, \mu_{hw}(\tau) + \rho\rangle - \langle \rho, \rho\rangle \\
&= \langle \mu_{char}(\tau), \mu_{char}(\tau)\rangle - \langle \rho, \rho\rangle
\end{aligned}$$

(notation as in Theorems 2.12, 2.22).

The corresponding Laplace-Beltrami operator L_Ω acts on the matrix coefficients of τ (Definition 3.1) by the same scalar $\tau(\Omega)$.

The choice of a negative Riemannian structure on G is natural for two reasons. First, it corresponds to a positive inner product on the lattice $X^*(H)$. Second, it means that the Laplace-Beltrami operator on $C^\infty(G)$ has non-negative spectrum.

3. Harmonic analysis on compact groups

In this section we will recall the Peter-Weyl and Paley-Wiener theorems for a compact group G, and look at some generalizations to abstract representations of G. The central idea is that the compactness of G provides a decomposition theory for general representations that is comparable in power to the spectral theory for operators on a Hilbert space. It is this decomposition theory for compact group representations that Harish-Chandra used as his basic tool for studying representations of noncompact groups.

Definition 3.1. *Suppose (π, V) is a representation of the topological group G (Definition 1.1). Suppose $v \in V$ and $\xi \in V^*$ is a continuous linear functional on V. Define*

$$f_{v,\xi}(g) = \xi(\pi(g^{-1})v,$$

a continuous function on G. A matrix coefficient *of π is a finite linear combination of functions $f_{\xi,v}$.*

Exercise 3.2. *Show that, for fixed ξ, the map $v \mapsto f_{v,\xi}$ is continuous from V to the space $C(G)$ of continuous functions on G, and that it intertwines the representation π on V with the left translation action on continuous functions.*

Exercise 3.3. *Suppose T is a finite-rank continuous linear operator on V (that is, that T has finite-dimensional image). Find a reasonable definition of the trace of T, and define*

$$f_T(g) = \mathrm{tr}(\pi(g^{-1})T).$$

Prove that f_T is a matrix coefficient of π, and that every matrix coefficient has this form.

The map

$$\mathrm{End}_{\mathrm{fin}}(V) \to C(G), \qquad T \mapsto f_T(g) = \mathrm{tr}(\pi(g^{-1})T) \tag{3.4}$$

(from finite-rank operators on V to continous functions on G) is called the *inverse Fourier transform.* We may also write it as $T^\vee(g)$.

Exercise 3.5. *Suppose π_1 and π_2 are representations of G, and that $\dim \pi_1 < \infty$. Explain how to define the tensor product representation $\pi_1 \otimes \pi_2$—specifically, how to put a nice topology on the tensor product vector space. Prove that the matrix coefficients of $\pi_1 \otimes \pi_2$ are precisely finite sums of products*

(matrix coefficient of π_1)(matrix coefficient of π_2).

Exercise 3.6. *Suppose* (π, V) *is a representation of* G. *Write* $\overline{V}$ *for the vector space with the same additive structure as* V, *but with the new scalar multiplication by* z *equal to the old scalar multiplication by* $\overline{z}$. *Write* $\overline{\pi}$ *for the same set of operators as* π, *now regarded as linear maps on* $\overline{V}$. *Prove that* $\overline{\pi}$ *is a representation of* G, *and that the matrix coefficients of* $\overline{\pi}$ *are the complex conjugates of the matrix coefficients of* π.

Give an example in which $\overline{\pi}$ *is not equivalent to* π.

Definition 3.7. *Suppose* G *is a topological group. Write* $\mathcal{H}_c(G)$ *for the collection of compactly supported complex-valued Radon measures on* G. *A measure* D *is a continuous linear functional*

$$\phi \mapsto \int_G \phi(g) dD(g) \in \mathbb{C} \qquad (\phi \in C(G))$$

on the space $C(G)$ *of continous functions on* G. *Similarly, if* E *is a complete locally convex topological vector space, then we can regard* D *as a continuous linear map from* E*-valued continuous functions on* G *to* E*:*

$$C(G, E) \to E, \qquad \Phi \mapsto \int_G \Phi(g) dD(g) \in E.$$

If (π, V) *is a representation of* G, *we can regard* π *as a continous map from* G *to* $E = \mathrm{Hom}(V, V) = \mathrm{End}(V)$, *the algebra of continuous endomorphisms of* V. *The corresponding map*

$$\pi \colon \mathcal{H}_c(G) \to \mathrm{End}(V), \qquad \pi(D) = \int_G \pi(g) dD(g)$$

is called the operator-valued Fourier transform, *and sometimes written* $\widehat{D}(\pi)$.

Suppose D_1 *and* D_2 *are compactly supported Radon measures. Then* $D_1 \boxtimes D_2$ *is a compactly supported Radon measure on* $G \times G$*:*

$$\int_{G\times G} \psi(g_1, g_2) d(D_1 \boxtimes D_2) = \int_G \left(\int_G \psi(g_1, g_2) dD_1(g_1) \right) dD_2(g_2)$$

The convolution product *is the Radon measure on* G *given by*

$$\int_G \phi(g) d(D_1 \star D_2) = \int_{G\times G} \phi(g_1 g_2) d(D_1 \boxtimes D_2).$$

Some elements of $\mathcal{H}_c(G)$ are the point masses δ_g at elements of G. Clearly $\pi(\delta_g) = \pi(g)$, and it is easy to check that $\delta_g \star \delta_{g'} = \delta_{gg'}$. The convolution product makes $\mathcal{H}_c(G)$ into an associative algebra, containing the group algebra of G as a subalgebra. The reason for the definition of convolution is

Proposition 3.8. *Suppose π is a representation of the topological group G. Then the operator-valued Fourier transform is an algebra homomorphism from the convolution product on $\mathcal{H}_c(G)$ to* $\mathrm{End}(V)$*.*

We leave the proof as an exercise.

For general representations of general topological groups, we therefore have an inverse Fourier transform

$$\mathrm{End}_{\mathrm{finite}}(V) \xrightarrow{\text{inverse Fourier}} C(G),$$

and a Fourier transform

$$\mathcal{H}_c(G) \xrightarrow{\text{operator Fourier}} \mathrm{End}(V).$$

These maps cannot in general be composed: the continuous functions in the range of the first typically have noncompact support, and so cannot be related to compactly supported measures; and the operators in the range of the second map typically have infinite rank.

For the balance of this section we assume that G is compact. Then G has a Haar measure dg, which allows us to define a natural inclusion

$$C(G) \hookrightarrow \mathcal{H}_c(G), \qquad \phi \mapsto \phi(g)dg. \tag{3.9}$$

This inclusion allows us to define the operator-valued Fourier transform of any continuous function:

$$\widehat{\phi}(\pi) = \int_G \phi(g)\pi(g)dg \in \mathrm{End}(V) \qquad (\phi \in C(G)).$$

It turns out that $C(G)$ is a subalgebra under convolution: the formula is

$$(\phi \star \psi)(x) = \int_G \phi(xg)\psi(g^{-1})dg.$$

Since G is compact, every irreducible representation (τ, V) of G is finite-dimensional, and so all the operators are of finite rank. The inverse Fourier transform is therefore defined on all operators:

$$T^\vee(g) = \mathrm{tr}(\mu(g^{-1})T) \qquad (T \in \mathrm{End}(V)). \tag{3.10}$$

Theorem 3.11 (Peter-Weyl; see for example [11], pages 243–248)**.** *Suppose G is a compact group, and (τ, V) is an irreducible representation of G. Define*

$$d(\tau) = \mathrm{Vol}(G)/\dim(V),$$

a positive real number. For any $T \in \mathrm{End}(V)$,

$$\widehat{T^\vee}(\tau') = \begin{cases} d(\tau)T', & \tau \simeq \tau', \\ 0, & \tau \not\simeq \tau'. \end{cases}$$

(Here we use the isomorphism of τ *with* τ' *to identify* T *with an operator* T' *on* V'*.) A little more explicitly,*

$$\int_G \mathrm{tr}(\tau'(g^{-1}T)\tau'(g)dg = \begin{cases} \frac{\mathrm{Vol}(G)}{\dim V}T' & \text{if } \tau \simeq \tau' \\ 0 & \text{if } \tau \not\simeq \tau'. \end{cases}$$

Corollary 3.12. *Suppose* G *is a compact group, and* τ *is an irreducible representation of* G*. Then the space* $C(G)(\tau)$ *of matrix coefficients of* τ *is a convolution subalgebra of* $\mathcal{H}_c(G)$*, isomorphic by the operator-valued Fourier transform (at* τ*) to* $\mathrm{End}(V)$*. The function*

$$e_\tau = \mathrm{tr}(\tau(g^{-1}) \cdot \dim(V)/\operatorname{Vol}(G)$$

is the identity element of this subalgebra, and therefore a (convolution) idempotent in $\mathcal{H}_c(G)$*.*

The subspace $C(G)(\tau)$ *of* $C(G)$ *(or of* $\mathcal{H}_c(G)$*) is isomorphic to a direct sum of copies of* τ *under the left translation action; it is the largest subspace with this property.*

Definition 3.13. *Suppose* $D \in \mathcal{H}_c(G)$ *is a complex-valued Radon measure on the compact group* G*, and* τ *is an irreducible representation of* G*. The* τ Fourier coefficient of D *is the continuous function*

$$D_\tau(x) = (e_\tau \star D) = d(\tau)\int_G \mathrm{tr}(\tau(gx^{-1})dD(g) \in C(G)_\tau,$$

with $d(\tau)$ *as in Theorem 3.11. The* Fourier series of D *is the formal sum*

$$\sum_{\tau\in\widehat{G}} D_\tau.$$

Suppose (π, V) *is an arbitrary representation of* G *(continuous, and on a complete topological vector space, as always). The* τ-isotypic subspace $V(\tau)$ *is the largest invariant subspace of* V *that is a direct sum of copies of* τ*. Equivalently,*

$$V_\tau = \textit{image of } \pi(e_\tau) = \textit{kernel of } (\pi(e_\tau) - \mathrm{Id}).$$

If $v \in V$*, the* τ Fourier coefficient of v *is*

$$v_\tau = \pi(e_\tau)v \in V(\tau).$$

The Fourier series of v *is the formal series*

$$\sum_{\tau\in\widehat{G}} v_\tau.$$

It is certainly *not* the case that the Fourier series defined here must converge; our final goal in this section is to give some very general setting in which they do.

Exercise 3.14. *Prove the equivalence of the three definitions of* $V(\tau)$.

Before turning to the convergence criteria we will use, we record a classical one.

Theorem 3.15. *Suppose* G *is compact and* $(\pi, \mathcal{H})$ *is a unitary representation of* G. *Then the spaces* $\mathcal{H}(\tau)$ *are mutually orthogonal and their sum is dense;* $\pi(e_\tau)$ *is the orthogonal projection. Consequently*

$$\mathcal{H} = \widehat{\bigoplus}_{\tau \in \widehat{G}} \mathcal{H}(\tau)$$

(Hilbert space direct sum). The Fourier series of Definition 3.13 converges in the Hilbert space norm to v.

We have

$$\mathcal{H}_c(G) \supset L^2(G) \supset C(G),$$

and the middle representation (under left translation) is unitary. Consequently the Fourier series of a square-integrable function on G converges in L^2 to the function.

Even on the circle, the Fourier series of a continuous function need not converge to the function pointwise. To get a good convergence theorem, the example of the circle suggests that we should consider nicer functions than continuous functions. We therefore assume now that G is a compact *Lie* group, and consider

(3.16a) $$C^\omega(G) \subset C^\infty(G) \subset C(G) \subset \mathcal{H}_c(G) \subset C^{-\infty}(G) \subset C^{-\omega}(G).$$

Here $C^\omega(G)$ is the space of real analytic functions on G; C^∞ is the space of smooth functions; $C^{-\infty}(G)$ is the continuous dual space of C^∞, the space of *distributions* on G; and $C^{-\omega}$ is the continuous dual of C^ω, the space of *hyperfunctions* on G. (For more details about analytic functions and hyperfunctions, see for example [8].) The left (and right) translation representations of G are well-defined on all of these spaces. Any finite-dimensional representation of a Lie group is automatically (smooth and) real analytic; so the matrix coefficient spaces $C(G)_\mu$ are contained in C^ω. In particular, the definition of Fourier coefficients extends from $\mathcal{H}_c(G)$ to $C^{-\omega}(G)$, and the Fourier coefficients all belong to $C^\omega(G)$.

What makes a (Fourier) series converge well is that the terms tend to zero rapidly. In order to make precise sense of "rapidly," we use the precise structure on $\widehat{G}$ established in Section 2. So fix a (connected) maximal torus $H \subset G$. Define

(3.16b) $$W(G,H) = N_G(H)/H \supset W(G_0, H).$$

It is easy to check that

$$W(G,H)/W(G_0,H) \simeq G/G_0, \tag{3.16c}$$

a finite group. Fix a set of positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, H)$, and use it to define dominant weights in $\widehat{H}$ as in Definition 2.10. Fix also a $W(G,H)$-invariant positive definite inner product

$$\langle , \rangle \colon i\mathfrak{h}^* \times i\mathfrak{h}^* \to \mathbb{R} \tag{3.16d}$$

as in Proposition 2.24; write Ω for the corresponding Casimir operator and L_Ω for the Laplace-Beltrami operator on G. If τ is any irreducible representation of G, define the *norm of* τ

$$\|\tau\| = \tau(\Omega) + \langle \rho, \rho \rangle = \langle \mu_{char}(\tau_0), \mu_{char}(\tau_0) \rangle.$$

Here τ_0 is any irreducible constituent of $\tau|_{G_0}$; the length of the corresponding character weight is independent of the choice of τ_0.

Theorem 3.17 (Paley-Wiener theorem for compact Lie groups; see [15], page 401)**.** *In the setting* (3.16)*, let* $\|f\|$ *be the* L^p *norm on* $C^\infty(G)$*, for some* $p \in [1,\infty]$*. Recall from Definition 3.13 the definition of the Fourier coefficients* f_τ *for a function (or distribution, or hyperfunction)* f *on* G*.*

(1) *Suppose* $f \in C^\infty(G)$ *and* N *is a non-negative integer. Then there is a positive constant* $C_{N,f}$ *so that*

$$\|f_\tau\| \le C_{N,f}(1 + \|\tau\|)^{-N}.$$

Conversely, if $\{f_\tau \in C(G)(\tau) \mid \tau \in \widehat{G}\}$ *is a collection of functions satisfying the estimates above, then the Fourier series*

$$\sum_{\tau \in \widehat{G}} f_\tau$$

converges absolutely (in any of the seminorms defining the C^∞ *topology) to some* $f \in C^\infty(G)$*.*

(2) *Suppose* $D \in C^{-infty}(G)$ *is a distribution. Then there is a non-negative integer* M *and a positive constant* $C_{M,D}$ *so that*

$$\|D_\tau\| \le C_{M,f}(1 + \|\tau\|)^{M}.$$

Conversely, if these inequalities are satisfied, then the Fourier series $\sum_\tau D_\tau$ *converges absolutely to a distribution* D*.*

(3) *Suppose* $f \in C^\omega(G)$ *is a real analytic function. Then there is a* $\delta > 0$ *and a positive constant* $C_{\delta,f}$ *so that*

$$\|f_\tau\| \le C_{\delta,f}(1-\delta)^{\|\tau\|}.$$

Conversely, if these inequalities are satisfied, then the Fourier series $\sum_\tau f_\tau$ *converges absolutely to a real analytic function* f.

(4) *Suppose* $D \in C^{-\omega}(G)$ *is a hyperfunction. Then for every* $\epsilon > 0$ *there is a positive constant* $C_{\epsilon,D}$ *so that*

$$\|D_\tau\| \leq C_{\epsilon,f}(1+\epsilon)^{\|\tau\|}.$$

Conversely, if these inequalities are satisfied, then the Fourier series $\sum_\tau D_\tau$ *converges absolutely to a hyperfunction* D.

One interesting aspect of the theorem is that the formulation can use so many different norms on $C^\infty(G)$. A consequence is that, on the space $C(G)(\tau)$ of matrix coefficients of τ, any of the norms is bounded by a polynomial in $\|\tau\|$ times any other (with a polynomial independent of τ).

The theorem is stated in [15] only for smooth functions, but the other versions can be proved in a very similar fashion. The question of the history of this theorem is an interesting one, to which I cannot offer a reasonable answer, despite a delightful hour of instruction from David Jerison. The theorem has an obvious extension with G replaced by a real analytic Riemannian manifold M, and the spaces $C(G)(\tau)$ by eigenspaces of the Laplace-Beltrami operator. That extension is proved in two papers of Seeley from 1965 and 1969. According to Jerison, the theorem (as far as it concerns smooth and analytic functions) was very likely known to Weyl; and in fact Hadamard had the necessary tools to prove it.

The earliest explicit reference that I can find is [5], Lemma 31 (proved in [6], Lemma 3). This essentially proves a much more abstract and general version, Theorem 3.20 below; we turn now to the formulation of that result.

Definition 3.18. *Suppose* G *is a Lie group, and* (π, V) *is a continuous representation of* G. *A vector* $v \in V$ *is called* smooth *if the map*

$$G \to V, \qquad g \mapsto \pi(g)v$$

is infinitely differentiable. The vector space of smooth vectors is written V^∞.

It is not difficult to show that V^∞ is dense in V, and that this subspace is preserved by $\pi(G)$. Differentiating the G action defines an algebraic representation of $\operatorname{Lie} G$, and so of the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$, on V^∞. In fact there is a natural complete locally convex topology on V^∞: if $\{|\cdot|_\alpha\ (\alpha \in A)\}$ is a family of seminorms defining the topology of V, then

$$|v|_{u,\alpha} =_{\text{def}} |\pi(u)v|_\alpha \qquad (u \in U(\mathfrak{g}_\mathbb{C}), \alpha \in A) \tag{3.19}$$

is a collection of seminorms defining the topology of V^∞. With this definition, the action of $U(\mathfrak{g})$ on V^∞ is by continuous operators.

Theorem 3.20 (Harish-Chandra; see [5], Lemma 31)**.** *Suppose (π, V) is a representation of the compact group G, and $v \in V^\infty$ is a smooth vector. If $|\cdot|_\alpha$ is any continuous seminorm on V, and N is a non-negative integer, then there is a constant C_N so that the Fourier components of v (Definition 3.13) satisfy*

$$|v_\tau|_\alpha \le c_{N,v,\alpha}(1 + \|\tau\|)^{-N}$$

In particular, the Fourier series of v converges absolutely to v.

4. Real algebraic groups

We begin this section with the relationship between compact Lie groups and complex reductive algebraic groups; then, with these wonderful complex objects in hand, describe how to get from them the real Lie groups that we wish to study. For a beautiful and complete introduction to algebraic groups, the reader should consult [14]. Here we just offer a few pieces that we need.

Definition 4.1. *A* complex algebraic group *is a complex affine algebraic variety G that is also a group, in such a way that the group operations*

$$m\colon G \times G \to G, \qquad m(x, y) = xy$$

and

$$i\colon G \to G, \qquad i(x) = x^{-1}$$

are morphisms of algebraic varieties. This means first of all that G is endowed with a complex commutative algebra $\mathbb{C}[G]$ of functions on G, called regular functions*. The set of functions vanishing at a point of G is then a maximal ideal in $\mathbb{C}[G]$, and we require that this correspondence is a bijection* $\operatorname{Max}\mathbb{C}[G] \simeq G$*. A morphism $\phi\colon X \to Y$ of affine algebraic varieties is by definition the same thing as an algebra homomorphism $\phi^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$; so the group operations correspond to algebra homomorphisms*

$$m^*\colon \mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G], \qquad i^*\colon \mathbb{C}[G] \to \mathbb{C}[G].$$

The identity element $e \in G$ corresponds to an algebra homomorphism $e^*\colon \mathbb{C}[G] \to \mathbb{C}$. The group axioms can be expressed as properties of the algebra homomorphisms m^*, i^*, and e^* ([14]*, 2.1.2).*

Exercise 4.2. *A complex algebraic variety X has a "classical topology" as a closed subset of some complex vector space. The set of smooth points X_{smooth} is then a dense open subset (in the classical topology);*

it is a complex manifold of dimension equal to the dimension of the algebraic variety X.

Assuming these facts, explain how a complex algebraic group "is equal to" a complex Lie group of the same dimension.

Example 4.3. *Let $G = GL(n)$, the group of invertible $n \times n$ complex matrices. The vector space $M(n)$ of* all *$n \times n$ matrices is an affine algebraic variety, with regular functions the polynomial algebra in n^2 variables*

$$\mathbb{C}[e_{ij}] \qquad (1 \le i, j \le n).$$

The group $GL(n)$ is the principal (Zariski) open subset $\{\det(x) \neq 0\}$. Therefore $GL(n)$ is an affine algebraic variety, with regular functions obtained by localizing the polynomial algebra in all matrix entries at the polynomial function det*:*

$$\mathbb{C}[GL(n)] = \mathbb{C}[e_{ij}, \det^{-1}] \qquad (1 \le i, j \le n).$$

Comparing Example 4.3 with Corollary 2.16, we see that the underlying bones and muscles on which the algebraic group $GL(n)$ is constructed (the regular functions) are very close to the bones and muscles of harmonic analysis on the compact group $U(n)$ (the matrix coefficients). Our goal in this section is to generalize this connection so that it includes all compact Lie groups and all complex reductive algebraic groups.

To construct a complex algebraic group, we must construct the (complex commutative) algebra $\mathbb{C}[G]$ and the algebra homomorphisms m^* and i^*, and e^* of Definition 4.1. Springer explains in [14], Section 2.5, how to recover an algebraic group from its representations. We will begin instead with the representations of a compact Lie group, and follow the same construction to build a (larger) complex algebraic group.

We begin therefore with a compact Lie group K (possibly disconnected). Define

$$R =_{\text{def}} C(K)_{\text{fin}} = \sum_{\tau \in \widehat{K}} C(K)_\tau. \tag{4.4a}$$

Here the first formula means all the continuous functions on K that generate finite-dimensional subspaces under left translation. Since finite-dimensional representations of K are completely reducible (since K is compact), we get the second identification of R, as the sum over irreducible representations τ of the τ-isotypic subspaces. According to Corollary 3.12, R is also precisely the space of matrix coefficients of finite-dimensional representations of K.

Clearly R is a complex vector space of functions on K. According to Exercise 3.5, R is closed under multiplication of functions; so it is a commutative algebra. (The constant function 1 is a matrix coefficient of the trivial representation of K, so R has a unit.) Evaluation at $e \in K$ defines an algebra homomorphism

$$e^*\colon R \to \mathbb{C}. \tag{4.4b}$$

In the notation of Definition 3.1, we have

$$f_{\xi,v}(g^{-1}) = \xi(\pi(g)v) = (\pi^*(g^{-1})\xi)(v) = f_{v,\xi}(g).$$

That is, the composition with the inversion map i of a matrix coefficient (of π) is again a matrix coefficient (of the contragredient representation on V^*). We therefore have an algebra homomorphism

$$i^*\colon R \to R, \qquad i^*(f_{\xi,v}) = f_{v,\xi}. \tag{4.4c}$$

Finally, we need to see that the mapping m^* (corresponding to multiplication of group elements) can be defined on matrix coefficients. What this requires is knowing how to multiply matrices. To see this, fix again a matrix coefficient $f_{\xi,v}$. Choose a basis $\{v_1, \ldots, v_n\}$ for V, and let $\{\xi_1, \ldots, \xi_n\}$ be the dual basis of V^*. Then

$$w = \sum_{j=1}^n \xi_j(w) v_j \qquad (w \in V).$$

Now we can compute

$$\begin{aligned} f_{\xi,v}(xy) &= \xi(\pi(y^{-1}x^{-1})v) \\ &= \xi(\pi(y^{-1})\pi(x^{-1})v) \\ &= \xi\left(\pi(y^{-1})\left(\sum_{j=1}^n \xi_j(\pi(x^{-1})v)v_j\right)\right) \\ &= \sum_{j=1}^n \xi_j(\pi(x^{-1}v))\xi(\pi(y^{-1}v_j)) \\ &= \sum_{j=1}^n f_{v,\xi_j}(x) f_{v_j,\xi}(y). \end{aligned}$$

. That is,

$$m^*(f_{v,\xi}) = \sum_{j=1}^n f_{v,\xi_j} \otimes f_{v_j,\xi}. \tag{4.4d}$$

Theorem 4.5. *Suppose K is a compact Lie group. Define the complex commutative algebra R of matrix coefficients as in* (4.4a)*, and define algebra morphisms i^* and m^* as in* (4.4)*. Then R is the ring of regular functions on a complex reductive algebraic group*

$$K(\mathbb{C}) = \operatorname{Max} R,$$

(the set of maximal ideals in R) which we call the complexification *of K. Every complex reductive algebraic group arises in this way from an appropriate compact Lie group K.*

Evaluation of matrix coefficients at $x \in K$ exhibits x as a maximal ideal of R; so we get

$$K \hookrightarrow K(\mathbb{C}).$$

The complexification of $U(n)$ is $GL(n, \mathbb{C})$. If K is a closed subgroup of $U(n)$ of real dimension N, then $K(\mathbb{C})$ is a Zariski-closed algebraic subgroup of $GL(n, \mathbb{C})$ of complex dimension N.

Restriction of representations from $K(\mathbb{C})$ to K defines a natural identification

$$\begin{aligned}&(\textit{finite-dimensional algebraic representations of } K(\mathbb{C}))\\ &\qquad \xrightarrow{\sim} (\textit{finite-dimensional continuous representations of } K).\end{aligned}$$

The algebraic group $K(\mathbb{C})$ is a kind of maximal domain of holomorphic extension for the representations of K. The points of $K(\mathbb{C})$ are maximal ideals in the algebra of matrix coefficients; and these maximal ideals arise by analytic continuation of the ideals of vanishing at a point of K.

To understand and use this correspondence completely, we need to know how to recover K from $K(\mathbb{C})$. We will see that K is the group of real points of a real form of $K(\mathbb{C})$. Here is one way to look at the definition. We start with a very simple case.

Definition 4.6. *Suppose V is a complex vector space. A* real form *of V is a real vector space $V_0 \subset V$, with the property that the natural map*

$$j_0 \colon V_0 \otimes_{\mathbb{R}} \mathbb{C} \to V, \qquad j_0(x \otimes z) = z \cdot x (x \in V_0, z \in \mathbb{C})$$

is an isomorphism. Equivalently, a real form is a conjugate-linear map

$$\sigma \colon V \to V, \qquad \sigma^2 = \operatorname{Id}.$$

Here "conjugate-linear" means that

$$\sigma(av + bw) = \overline{a}\sigma(v) + \overline{b}\sigma(w) \qquad (a, b \in \mathbb{C}, \quad v, w \in V).$$

Here is how to see the two versions of the definition are equivalent. Given V_0, the defining condition means that every element $v \in V$ has a unique expression

$$v = x + iy, \qquad x, y \in V_0.$$

Using this expression, we can define

$$\sigma(v) = x - iy,$$

and check that σ is a conjugate-linear map of order two. Conversely, given σ, we can define

$$V_0 = V^\sigma,$$

which is clearly a real vector space. We leave as an exercise the verification that these two constructions are mutual inverses.

Definition 4.7. *Suppose X is a complex affine algebraic variety, with (complex) algebra of regular functions $A = \mathbb{C}[X]$. A* real form *of X is a real commutative subalgebra $A_0 \subset A$ with the property that A_0 is a real form of A as a vector space. Equivalently, a real form is a conjugate-linear ring homomorphism*

$$\sigma^*\colon A \to A, \qquad (\sigma^*)^2 = \mathrm{Id}\,.$$

Equivalently, a real form is map on sets

$$\sigma\colon X \to X, \qquad \sigma^2 = \mathrm{Id},$$

subject to the requirement that composition with σ carries a regular function to the complex conjugate of a regular function:

$$\sigma^*(f)(x) =_{def} \overline{f(\sigma(x))} \in \mathbb{C}[X], \qquad f \in \mathbb{C}[X].$$

Two real forms are called weakly equivalent *if they are conjugate by an automorphism of X.*

Given a real form, we write

$$\mathbb{R}[X] = A_0 = (\mathbb{C}[X])^{\sigma^*},$$

the (real) algebra of regular functions on the real form.

Finally, the set of real points of the real form *is*

$$\begin{aligned} X(\mathbb{R}) &= \{\textit{algebra homomorphisms } \mathbb{R}[X] \to \mathbb{R}\} \\ &= \{\textit{fixed points of } \sigma \textit{ on } X(\mathbb{C})\}. \end{aligned}$$

The notion of equivalence is not standard (or very interesting) for general algebraic varieties; we include it only in order to set up the case of algebraic groups. The term "weak" is present only to distinguish this notion from a stronger one in the case of algebraic groups; the stronger notion does not make sense for general varieties.

Notice that the mapping σ from X to X is *not* a morphism of algebraic varieties, because the underlying ring homomorphism is not complex linear.

It is immediate from the definitions that the functions in $\mathbb{R}[X]$ take real values on $X(\mathbb{R})$; so we have a homomorphism of real algebras

$$\mathbb{R}[X] \to (\mathbb{R}\text{-valued functions on } X(\mathbb{R})).$$

But, in contrast to the complex case, this homomorphism may be far from injective: we cannot identify $\mathbb{R}[X]$ as an algebra of functions on the set of real points. For example, it may happen that $X(\mathbb{R})$ is empty even though the algebra real form $\mathbb{R}[X]$ is not zero. We will see that this subtlety does not arise for connected real algebraic *groups*.

Definition 4.8. *Suppose G is a complex algebraic group (Definition 4.1). A* real form of G *is a real form of commutative algebras*

$$\mathbb{R}[G] \subset \mathbb{C}[G],$$

preserved by the maps m^* and i^*, and such that the e^* (evaluation at the identity) takes real values on $\mathbb{R}[G]$. Equivalently, a real form is a group homomorphism

$$\sigma \colon G \to G, \qquad \sigma^2 = \mathrm{Id}$$

with the property that the mapping on functions

$$(\sigma^*(f))(x) = \overline{f(\sigma(x))}$$

preserves $\mathbb{C}[G]$.

Two real forms σ_1 and σ_2 are called weakly equivalent *if they are conjugate by an algebraic automorphism α of G:*

$$\sigma_2(x) = \alpha\left(\sigma_1\left(\alpha^{-1}(x)\right)\right) \qquad (\textit{some } \alpha \in \mathrm{Aut}(G)).$$

They are called strongly equivalent *if they are conjugate by an inner automorphism of G:*

$$\sigma_2(x) = a\sigma_1(a^{-1}xa)a^{-1} \qquad (\textit{some } a \in G).$$

They are called inner *(another equivalence relation on real forms) if they are in the same coset of the inner automorphisms:*

$$\sigma_2(x) = b\sigma_1(x)b^{-1} \qquad (\textit{some } b \in G).$$

The group of real points *of the real form is*

$$G(\mathbb{R}) =_{def} G^{\sigma}.$$

Exercise 4.9. *Show that two strongly equivalent real forms are necessarily inner. Give examples to show that two real forms that are inner to each other need not be weakly equivalent, and that weakly equivalent real forms need not be inner.*

Exercise 4.10. *(Use the ideas from Exercise 4.2.) Suppose σ is a real form of the complex algebraic group G. Define $\mathfrak{g}$ to be the Zariski tangent space to G at the identity element, a complex vector space. Show that $d\sigma$ is a real form of the complex vector space $\mathfrak{g}$. Deduce that $G(\mathbb{R})$ is a closed subgroup of G as a real Lie group, and that*

$$\dim_{\mathbb{R}}(G(\mathbb{R})) = \dim_{\mathbb{C}}(G).$$

Prove that if G is connected, then restriction to $G(\mathbb{R})$ identifies $\mathbb{C}[G]$ with an algebra of real-analytic functions on $G(\mathbb{R})$.

Exercise 4.11. *Give an example of a complex algebraic group G and a real form so that the homomorphism*

$$\mathbb{R}[G] \to \textit{functions on } G(\mathbb{R})$$

is not one-to-one.

In the general theory of algebraic varieties, the question of whether a variety is defined over a certain field F (that is, whether there is an F-form) is a natural one. Typically this is a fairly uncommon occurrence; and if an F-form exists, it is typically unique. The situation for algebraic groups is very different. Connected reductive groups (the ones we are most concerned with) admit F-forms for *any* field F. There can be many quite different F-forms, depending on the arithmetic of F. Here are the basic definitions for keeping track of these matters. We write

$$\mathrm{Gal} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \mathrm{bar}\} \tag{4.12}$$

for the Galois group.

Definition 4.13. *Suppose X is a complex algebraic variety. Define*

$$\mathrm{Aut}(X) = \{\mathbb{C}\textit{-linear automorphisms of } \mathbb{C}[X]\},$$

the group of automorphisms of X as a complex algebraic variety. Define

$$\mathrm{Aut}(X)_{\mathrm{bar}} = \{\textit{conjugate-linear automorphisms of } \mathbb{C}[X]\}.$$

Define the Galois-extended automorphism group

$$\mathrm{Aut}(X)^{\mathrm{Gal}} = \mathrm{Aut}(X) \sqcup \mathrm{Aut}(X)_{\mathrm{bar}},$$

a group of real-linear automorphisms of $\mathbb{C}[X]$. There is an exact sequence

$$1 \to \mathrm{Aut}(X) \to \mathrm{Aut}(X)^{\mathrm{Gal}} \to \mathrm{Gal}\,.$$

Proposition 4.14. *In the setting of Definition 4.13, the following notions are equivalent:*

1. *a real form of X;*
2. *an element of order two in the subset* $\mathrm{Aut}(X)_{\mathrm{bar}}$ *of* $\mathrm{Aut}(X)^{\mathrm{Gal}}$*;*
3. *a splitting* $\mathrm{Gal} \to \mathrm{Aut}(X)^{\mathrm{Gal}}$ *of the sequence of Definition 4.13.*

Suppose σ_0 is a particular real form of X. Then

$$\mathrm{Aut}(X)_{\mathrm{bar}} = \mathrm{Aut}(X)\sigma_0,$$

providing an identification

$$\mathrm{Aut}(X)^{\mathrm{Gal}} \simeq \mathrm{Aut}(X) \rtimes \{1, \sigma_0\},$$

and also an action of Gal *on* $\mathrm{Aut}(X)$*:*

$$\overline{a} = \sigma_0 a \sigma_0^{-1} \qquad (a \in \mathrm{Aut}(X)).$$

With respect to these structures, a real form of X is also equivalent to a one-cocycle of Gal *with coefficients in* $\mathrm{Aut}(X)$*; that is, to an element* $a_{\mathrm{bar}} \in \mathrm{Aut}(X)$ *satisfying* $a_{\mathrm{bar}}\overline{a_{\mathrm{bar}}} = 1$*. (The equivalence sends the real form σ to $a_{\mathrm{bar}} = \sigma\sigma_0^{-1}$.) Two one-cocycles a_{bar} and b_{bar} are cohomologous—that is, $b_{\mathrm{bar}} = \alpha a_{\mathrm{bar}} \overline{\alpha}^{-1}$ for some $\alpha \in \mathrm{Aut}(X)$—if and only if the corresponding real forms are weakly equivalent (Definition 4.8).*

Notice that the first half of this proposition makes use of the Galois-extended group $\mathrm{Aut}(X)^{\mathrm{Gal}}$, and classical group-theoretic notions. The restatement in the second half is phrased entirely in terms of $\mathrm{Aut}(X)$, and uses Galois cohomology. There is no serious mathematical difference; the choice between the two points of view is a matter of taste. I will almost always prefer to talk about extended groups, but the reader may regard the problem of finding cohomological formulations of everything as an extended exercise.

Corollary 4.15. *In the setting of Definition 4.8, the following notions are equivalent:*

1. *a real form of G;*
2. *an element of order two in the subset* $\mathrm{Aut}(G)_{\mathrm{bar}}$ *of* $\mathrm{Aut}(G)^{\mathrm{Gal}}$*;*
3. *a splitting* $\mathrm{Gal} \to \mathrm{Aut}(G)^{\mathrm{Gal}}$ *of the sequence of Definition 4.13.*

Suppose σ and σ' are two real forms in $\mathrm{Aut}(G)_{\mathrm{bar}}$.

1. *The real forms are weakly equivalent (Definition 4.8) if and only if σ and σ' are conjugate by* $\mathrm{Aut}(G)$.
2. *The real forms are strongly equivalent (Definition 4.8) if and only if σ and σ' are conjugate by* $\mathrm{Int}(G) \subset \mathrm{Aut}(G)$.
3. *The real forms are inner to each other if they belong to the same coset of* $\mathrm{Int}(G)$.

Theorem 4.16. *Suppose K is a compact Lie group, and $K(\mathbb{C})$ the corresponding complex reductive algebraic group (Theorem 4.5). Then there is a unique real form σ of $K(\mathbb{C})$ with the property that the group of real points is K.*

Proof. Write R for the algebra of matrix coefficient functions on K, which by definition is the same as the ring of regular functions on $K(\mathbb{C})$. Define a conjugate-linear automorphism σ^* of R by

$$(\sigma^*(f))(k) = \overline{f(k)} \qquad (k \in K). \tag{4.17}$$

It is easy to see from the definitions that this is the only possible definition of an algebra map $\sigma*$ corresponding to a real form of $K(\mathbb{C})$ with group of real points containing K. The first issue is to check that σ^* is well-defined; that is, that the function on the right side is again a matrix coefficient for K. So suppose $f = f_{\pi,\xi,v}$ is a matrix coefficient for a finite-dimensional representation (π, V) of K (Definition 3.1). Write V^{conj} for the vector space with the same additive structure as V, but with scalar multiplication modified by complex conjugation. The space V^{conj} has exactly the same linear operators as V, so we can interpret π as a representation π^{conj} on V^{conj}. The linear functionals on V^{conj} are just complex conjugates of linear functionals on V. So $\overline{\xi}$ is a linear functional on V^{conj}. It is now easy to check that, as functions on K,

$$\sigma^*(f_{\pi,\xi,v}) = \overline{f_{\pi,\xi,v}} = f_{\pi^{\text{conj}},\overline{\xi},v}.$$

This proves that σ^* is well-defined.

That σ^* is a conjugate-linear algebra automorphism respecting m^* and i^* is very easy; so σ^* is a real form of $K(\mathbb{C})$. It is clear from (4.17) and the definitions that

$$K \subset K(\mathbb{R}). \tag{4.18}$$

Exercise 4.19. *In the setting above, fix a K-invariant Hilbert space structure on V (as is always possible since K is compact). For $g \in K(\mathbb{C})$, show that $\sigma(g) = g$ if and only if $\pi(g)$ is a unitary operator. Deduce that the group of real points $K(\mathbb{R}) = K(\mathbb{C})^\sigma$ is a compact group containing K (a subgroup of the product over π of the groups $U(V)$).*

Prove that every finite-dimensional representation of K has a canonical extension to $K(\mathbb{R})$.

Use the Peter-Weyl theorem to deduce that $K = K(\mathbb{R})$.

This exercise completes the proof of Theorem 4.16. □

In order to analyze other real forms of complex connected reductive algebraic groups, we begin with a long detour into the structure of algebraic automorphisms. For that we need to recall the fundamentals of the structure theory; details may be found in [14].

Suppose G is a complex connected reductive algebraic group, with Lie algebra $\mathfrak{g}$. Fix a Borel subgroup $B \subset G$, and a maximal torus $H \subset B$. The torus H is isomorphic to a product of copies of $\mathbb{C}^\times$ (but not canonically). The coordinate-free way to keep track of H is using

$$X^* = \mathrm{Hom}_{\mathrm{alg}}(H, \mathbb{C}^\times)), \tag{4.20a}$$

the lattice of characters; this is a finitely generated torsion-free abelian group. It is therefore a product of copies of $\mathbb{Z}$ (but not canonically). We also make use of

$$X_* = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^\times), H), \tag{4.20b}$$

the lattice of cocharacters of H. Composing a one-parameter subgroup $\xi\colon \mathbb{C}^\times \to H$ with a character $\lambda\colon H \to \mathbb{C}^\times$ gives an algebraic group morphism from $\mathbb{C}^\times$ to itself. Such a morphism is given by raising to an integer power n; so we write

$$\langle \lambda, \xi \rangle = n \quad \text{whenever} \quad \lambda(\xi(z)) = z^n \qquad (z \in \mathbb{C}^\times). \tag{4.20c}$$

This pairing $\langle \cdot, \cdot \rangle$ identifies

$$X_* = \mathrm{Hom}(X^*, \mathbb{Z}), \qquad X^* = \mathrm{Hom}(X_*, \mathbb{Z}). \tag{4.20d}$$

Put

$$\Delta = \{\text{roots of } H \text{ in } \mathfrak{g}\} \subset X^*, \tag{4.20e}$$

$$\Delta^\vee = \{\text{coroots of } H \text{ in } \mathfrak{g}\} \subset X_*. \tag{4.20f}$$

The choice of Borel subgroup B defines a set of positive roots

$$\Delta^+ = \{\text{roots of } H \text{ in } \mathfrak{b}\} \subset X^*. \tag{4.20g}$$

Write

$$\Pi = \{\text{simple roots for } \Delta^+\}, \qquad \Pi^\vee = \{\text{simple coroots}\}. \tag{4.20h}$$

The quadruple $\mathcal{R} = (X^*, \Delta, X_*, \Delta^\vee)$ is called the *root datum of* G; more precisely, of the pair (G, H). Here we think of X^* and X_* as lattices in duality by the pairing $\langle \cdot, \cdot \rangle$, Δ as a finite subset of X^*, and $\Delta^\vee$ a finite subset of X_* with a specified bijection $\alpha \leftrightarrow \alpha^\vee$ between Δ and $\Delta^\vee$. An *isomorphism of root data* is required to preserve all of that structure.

The quadruple $\mathcal{R}_b = (X^*, \Pi, X_*, \Pi^\vee)$ is called the *based root datum of* G.

Root data are characterized by remarkably simple axioms.

Definition 4.21. *An* abstract root datum *is a quadruple*

$$\mathcal{R} = (X^*, \Delta, X_*, \Delta^\vee),$$

subject to the following requirements.

(1) *The abelian groups X^* and X_* are dual lattices (that is, finitely generated torsion-free abelian groups). We write the pairing between them as*

$$\langle \cdot, \cdot \rangle \colon X^* \times X_* \to \mathbb{Z}.$$

(2) *The set Δ is a finite subset of X^*, and $\Delta^\vee$ a finite subset of X_*. There is a specified bijection $\alpha \leftrightarrow \alpha^\vee$ between Δ and $\Delta^\vee$.*

We interrupt the axioms to introduce a little more notation. For each $\alpha \in \Delta$, define endomorphisms of X^* and of X_* by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \qquad (\lambda \in X^*),$$

$$s_{\alpha^\vee}(\xi) = \xi - \langle \alpha, \xi \rangle \alpha^\vee \qquad (\xi \in X_*).$$

These endomorphisms are transposes of each other. The next axiom will ensure that they are invertible, so that we can define the Weyl group of the root datum

$$W = \textit{group generated by the } s_\alpha \subset \mathrm{Aut}(X^*).$$

The inverse transpose map identifies W with

$$W^\vee = \textit{group generated by the } s_{\alpha^\vee} \subset \mathrm{Aut}(X_*).$$

We will follow standard practice and confuse these two groups, allowing the context to indicate which one is intended.

(3) *For each $\alpha \in \Delta$, $\langle \alpha, \alpha^\vee \rangle = 2$ (and so $s_\alpha^2 = \mathrm{Id}$, $s_{\alpha^\vee}^2 = \mathrm{Id}$).*
(4) *For each $\alpha \in \Delta$, $s_\alpha(\Delta) = \Delta$, and $s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee$.*

Finally, the root datum is called reduced *if in addition*

(5) *For $\alpha \in \Delta$, $2\alpha \notin \Delta$.*

It is equivalent to require that $2\alpha^\vee \notin \Delta^\vee$.

An isomorphism of root data

$$g \colon (X^*, \Delta, X_*, \Delta^\vee) \to ((X')^*, \Delta', (X')_*, (\Delta')^\vee)$$

is an isomorphism of abelian groups

$$g^* \colon (X')^* \to X^*$$

so that (writing $g_* \colon X_* \to (X')_*$ for the transpose isomorphism) we have

$$g^*(\Delta') = \Delta, \qquad (\alpha')^\vee = g_*(g^*(\alpha')^\vee) \quad (\alpha' \in \Delta').$$

Theorem 4.22 (see [14], Theorem 9.6.2)**.** *Suppose G is a complex connected reductive algebraic group, and H is a maximal torus in G. The root datum defined in* (4.20) *is a reduced abstract root datum.*

Suppose G' is a second complex connected reductive algebraic group, and H' is a maximal torus in G. Suppose that g is an isomorphism from the root datum of G to the root datum of G' (Definition 4.21). Then there is an isomorphism of algebraic groups

$$\gamma\colon G \to G', \qquad \gamma(H) = H',$$

so that γ induces the isomorphism g. Furthermore γ is unique up to (pre-) composition with an inner automorphism coming from H, or up to (post-) composition with an inner automorphism coming from H'.

Suppose $(X^*, \Delta, X_*, \Delta^\vee)$ is a reduced abstract root datum (Definition 4.21). Then there is a complex connected reductive algebraic group G and a maximal torus H giving rise to this root datum by means of (4.20).

This theorem says very precisely the way in which a complex reductive algebraic group is determined by its root datum. (It is important to keep in mind that a maximal torus is uniquely determined up to conjugacy in G.)

One proof of this theorem is based on proving the more precise Theorem 4.29 below. That in turn is based on writing a presentation of G by generators and relations; details may be found in [14], Chapter 9.

Exercise 4.23. *A* weak isogeny *is a morphism of connected reductive algebraic groups*

$$\gamma\colon G \to G'$$

with the properties

1. $\ker\gamma$ *is abelian;*
2. $\operatorname{im}\gamma$ *is a closed normal subgroup of G', and*
3. $G'/(\operatorname{im}\gamma)$ *is abelian.*

(For an isogeny*, the two abelian groups are also required to be finite; this forces the second group to be trivial.)*

A weak isogeny of root data

$$g\colon (X^*, \Delta, X_*, \Delta^\vee) \to ((X')^*, \Delta', (X')_*, (\Delta')^\vee)$$

is a morphism of abelian groups

$$g^*\colon (X')^* \to X^*$$

so that (writing $g_*\colon X_* \to (X')_*$ for the transpose morphism) we have

$$g^*(\Delta') = \Delta, \qquad (\alpha')^\vee = g_*(g^*(\alpha')^\vee) \quad (\alpha' \in \Delta').$$

Generalize Theorem 4.22 to prove that weak isogenies of reductive groups correspond to weak isogenies of root data.

Exercise 4.24. *Find a definition of "morphism of root data" so that something like Theorem 4.22 is true with isomorphisms replaced by morphisms (of root data and of algebraic groups).*

This is not an exercise but rather a possible career path. Dynkin has done serious work in [4], but the problem is incredibly subtle.

In order to identify and work with individual automorphisms of G, we need to refine Theorem 4.22 even further. Attached to each root α is the three-dimensional (closed algebraic) subgroup G_α generated by the root spaces for $\pm\alpha$. The intersections

$$B_\alpha = B \cap G_\alpha \supset H_\alpha = H \cap G_\alpha \tag{4.25a}$$

are a Borel subgroup and a maximal torus in G_α. One of the fundamental results in the structure theory is that there are algebraic group morphisms

$$\phi_{\alpha^\vee} \colon SL(2) \to G_\alpha \subset G \tag{4.25b}$$

with the property that

$$\phi_{\alpha^\vee}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \alpha^\vee(z) \in H_\alpha \subset H \qquad (z \in \mathbb{C}^\times). \tag{4.25c}$$

Such a morphism is called a *root* $SL(2)$ for the root α. Using the morphism $\phi_{\alpha^\vee}$, we can define a preferred basis vector

$$X_\alpha = d\phi_{\alpha^\vee}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_\alpha \tag{4.26}$$

of the α root space. Conversely, if we first fix the basis vector X_α of the root space, then we can find a unique group morphism $\phi_{\alpha^\vee}$ satisfying (4.25) and (4.26). We record this fact as a lemma.

Lemma 4.27. *The group morphism $\phi_{\alpha^\vee}$ is determined by the requirements* (4.25) *up to (pre-) composition with the adjoint action of an element of the diagonal torus of $SL(2)$, or up to (post-) composition with the adjoint action of an element of H_α. The choice of $\phi_{\alpha^\vee}$ is equivalent to the choice of the basis vector X_α of the root space $\mathfrak{g}_\alpha$.*

Definition 4.28 ([3], Exposé XXIII)**.** *Suppose G is a complex connected reductive algebraic group. A* pinning *(in* [3] épinglage*) of G is a triple*

$$\mathcal{P} = (B, H, \{\phi_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee\}),$$

with B a Borel subgroup; $H \subset G$ a maximal torus; $\Pi^\vee$ the set of simple coroots for H corresponding to B (see (4.20)*); and $\phi_{\alpha^\vee}$ a root $SL(2)$*

(for each simple root). According to Lemma 4.27, a pinning is the same thing as a triple

$$(B, H, \{X_\alpha \mid \alpha \in \Pi\}),$$

with X_α a nonzero element of the root space $\mathfrak{g}_\alpha$.

Theorem 4.29 ([3], Exposé XXIII, Theorem 4.1)**.** *Every complex connected reductive algebraic group admits a pinning, which is unique up to inner automorphism.*

Suppose $(G, \mathcal{P})$ and $(G', \mathcal{P}')$ are complex connected reductive algebraic groups endowed with pinnings. Write $\mathcal{R}_b$ and $\mathcal{R}_b'$ for the based root data corresponding to the pairs (B, H) and (B', H'). Suppose g is an isomorphism from $\mathcal{R}_b$ to $\mathcal{R}_b'$. Then there is a unique isomorphism $\gamma\colon G \to G'$ that carries $\mathcal{P}$ to $\mathcal{P}'$ and induces g on the root datum.

As explained after Theorem 4.22, one proof of this theorem is based on using the pinnings to write presentations of G and of G'; see [14], Section 9.4.

Corollary 4.30. *Suppose G is a complex connected reductive algebraic group, and*

$$\mathcal{P} = (B, H, \{\phi_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee\})$$

is a pinning of G. Write

$$\begin{aligned} \mathrm{Aut}(G) &= \textit{algebraic automorphisms of } G \\ \mathrm{Int}(G) &= \textit{inner automorphisms of } G \simeq G/Z(G) \\ \mathrm{Out}(G) &= \mathrm{Aut}(G)/\operatorname{Int}(G) \end{aligned}$$

so that there is a short exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

Finally, define $\mathrm{Aut}(G, \mathcal{P})$ to be the subgroup of automorphisms preserving the pinning.

Then $\operatorname{Aut} G$ is a semidirect product

$$\mathrm{Aut}(G) = \mathrm{Int}(G) \rtimes \mathrm{Aut}(G, \mathcal{P})$$

of the normal subgroup of inner automorphisms by the subgroup of automorphisms preserving the pinning.

Write $\mathcal{R}$ (resp. $\mathcal{R}_b$) for the root datum (resp. based root datum) attached to (G, H) (resp. to (G, B, H).) Then $\mathrm{Aut}(\mathcal{R})$ is a semidirect product

$$\mathrm{Aut}(\mathcal{R}) = W(\mathcal{R}) \rtimes \mathrm{Aut}(\mathcal{R}_b)$$

of the Weyl group (Definition 4.21) with the subgroup of automorphisms preserving the positive roots.

We therefore have natural isomorphisms

$$\mathrm{Out}(G) \simeq \mathrm{Aut}(G, \mathcal{P}) \simeq \mathrm{Aut}(\mathcal{R}_b) \simeq \mathrm{Out}(\mathcal{R}).$$

(The last notation is explained in (4.31) *below.)*

The corollary is a straightforward consequence of Theorem 4.29. It is natural to think of elements of the Weyl group as inner automorphisms of a root datum $\mathcal{R}$, and so to write

$$\mathrm{Int}(\mathcal{R}) =_{\mathrm{def}} W(\mathcal{R}). \tag{4.31a}$$

Then the short exact sequence

$$1 \to \mathrm{Int}(\mathcal{R}) \to \mathrm{Aut}(\mathcal{R}) \to \mathrm{Out}(\mathcal{R}) \to 1 \tag{4.31b}$$

defines $\mathrm{Out}(\mathcal{R})$. The fundamental fact that $W(\mathcal{R})$ acts in a simply transitive way on positive root systems (see for example [14], Proposition 8.2.4) shows that

$$\mathrm{Aut}(\mathcal{R}) = W(\mathcal{R}) \rtimes \mathrm{Aut}(\mathcal{R}_b), \tag{4.31c}$$

and therefore that

$$\mathrm{Out}(\mathcal{R}) \simeq \mathrm{Aut}(\mathcal{R}_b). \tag{4.31d}$$

Corollary 4.32. *Suppose G is a complex connected reductive algebraic group with based root datum $\mathcal{R}_b$. Then* $\mathrm{Aut}(G)$ *has a natural structure of complex Lie group. The identity component is the complex connected semisimple algebraic group*

$$\mathrm{Int}(G) = G/Z(G),$$

consisting of inner automorphisms of G. The group of connected components is $\mathrm{Aut}(\mathcal{R}_b)$*, a subgroup of* $\mathrm{Aut}(X^*) \simeq GL(n, \mathbb{Z})$*. In fact there is a semidirect product*

$$\mathrm{Aut}(G) \simeq \mathrm{Int}(G) \rtimes \mathrm{Aut}(\mathcal{R}_b).$$

Essentially all of the analysis to this point can be applied with fairly small changes to reductive algebraic groups over any field F. Theorems 4.22 and 4.29 are true essentially as stated with $\mathbb{C}$ replaced by any algebraically closed field. Something like Proposition 4.14 is true for any perfect field. With our analysis of $\mathrm{Aut}(G)$ in hand, it is now natural to study the larger group $\mathrm{Aut}(G)^{\mathrm{Gal}}$, and so the rational forms of G. Proposition 4.14 suggests that it can be useful to do this using a "base" rational form. When one wants to do this over a more or less arbitrary base field, it is natural to look for a rational form that makes sense over an arbitrary field. The only such form is the split form. Chevalley's great achievement in [2] is to construct (for each semisimple root system) a corresponding (split) algebraic group "over

$\mathbb{Z}$." In [3], Demazure extends this to give a split form for any root datum "over $\mathbb{Z}$," and therefore over any field F. One can then use ideas like Galois cohomology to study other F-forms as "twists" of the split form. This is the right way to proceed over general fields.

In the case of the real numbers, there is another way to proceed. Every complex reductive group has (in addition to the split real form) a *compact* real form. We are going to view the compact form as fundamental, and other real forms as "twists" of the compact form. Of course there is a price to be paid (in the theory of automorphic forms, for example) in treating the real field differently from p-adic fields. But the benefits are enormous. Theorem 4.5 explains that subtle analytic questions about harmonic analysis on the compact real form are precisely equivalent to easy algebraic statements about the complexification. Harish-Chandra's theory provides parallel results for any real reductive group, relating problems of harmonic analysis on the real group to harmonic analysis on the "nearby" compact real form, and so to algebra.

So here is a starting point for the analysis of rational forms over general fields. (We leave to the reader's imagination the underlying definitions.)

Theorem 4.33. *Suppose that $\overline{F}$ is an algebraically closed field of characteristic zero, and that τ is any automorphism of $\overline{F}$. Suppose $(G,\mathcal{P})$ and $(G',\mathcal{P}')$ are connected reductive algebraic groups over $\overline{F}$ endowed with pinnings. Write $\mathcal{R}_b$ and $\mathcal{R}_b'$ for the based root data corresponding to the pairs (B,H) and (B',H'). Suppose g is an isomorphism from $\mathcal{R}_b$ to $\mathcal{R}_b'$. Then there is a unique isomorphism*

$$\gamma = \gamma(g,\tau)\colon G \to G'$$

characterized by the following requirements:

(1) $\gamma(B,H) = (B',H')$;
(2) $\gamma[\xi(z)] = (g_*\xi)(\tau(z)) \qquad (\xi \in X_*, \quad z \in \overline{F})$; *and*
(3) $\gamma[\phi_{\alpha^\vee}(x)] = \phi_{g_*\alpha^\vee}(\tau(x)) \qquad (\alpha^\vee \in \Pi^\vee, \quad x \in SL(2,\overline{F}))$.

Here τ acts on the matrix group $SL(2,F)$ by acting on each entry.

The group homomorphism γ is a τ-algebraic morphism of algebraic varieties, in the sense that it corresponds to a ring homomomorphism

$$\gamma^*\colon \overline{F}[G'] \to \overline{F}[G]$$

carrying scalar multiplication by z to scalar multiplication by $\tau(z)$.

Here is the variant that is special to the real numbers.

Theorem 4.34. *Suppose $(G, \mathcal{P})$ and $(G', \mathcal{P}')$ are connected complex reductive algebraic groups over $\overline{F}$ endowed with pinnings. Write $\mathcal{R}_b$ and $\mathcal{R}'_b$ for the based root data corresponding to the pairs (B, H) and (B', H'). Suppose g is an isomorphism from $\mathcal{R}_b$ to $\mathcal{R}'_b$. Then there is a unique isomorphism*

$$\gamma_0 = \gamma_0(g, \mathrm{bar})\colon G \to G'$$

characterized by the following requirements:

(1) $\gamma_0(B, H) = (B', H')$;
(2) $\gamma_0[\xi(z)] = (g_*\xi)(\overline{z}^{-1}) \quad (\xi \in X_*, z \in \mathbb{C})$; *and*
(3) $\gamma_0[\phi_{\alpha^\vee}(x)] = \phi_{g_*\alpha^\vee}({}^t\overline{x}^{-1}) \quad (\alpha^\vee \in \Pi^\vee, x \in SL(2, \mathbb{C})$.

The isomorphism γ_0 is conjugate-algebraic, in the sense that it carries regular functions on G' to complex conjugates of regular functions on G.

Both versions of the theorem can be proved in exactly the same way as Theorem 4.29, by inspection of the generators and relations for G and G' in Section 9.4 of [14].

Definition 4.35. *Suppose G is a complex connected reductive algebraic group, and*

$$\mathcal{P} = (B, H, \{\phi_{\alpha^\vee} \mid \alpha \in \Pi\})$$

is a pinning of G. The compact real form of G attached to $\mathcal{P}$ *is the conjugate-algebraic isomorphism $\sigma_c = \sigma_c(\mathcal{P})$ attached by Theorem 4.34 to $\mathcal{P}$ and the identity map on the corresponding based root datum.*

The split real form of G attached to $\mathcal{P}$ *is the conjugate-algebraic isomorphism σ_s attached by Theorem 4.33 to $\mathcal{P}$ and the identity map on the based root datum.*

Since pinnings are unique up to inner automorphism, the compact and split real forms of G are unique up to inner automorphism.

Example 4.36. *Suppose $G = GL(n, \mathbb{C})$, B is the Borel subgroup of upper triangular matrices, and H is the group of diagonal matrices. There are obvious choices for simple root $SL(2)$ subgroups (which you should write down); write $\mathcal{P}_{std}$ for the corresponding pinning. The corresponding compact real form is*

$$\sigma_c(g) = {}^t\overline{g}^{-1},$$

which is the real form $U(n)$.

The corresponding split real form is

$$\sigma_s(g) = \overline{g},$$

which is the real form $GL(n, \mathbb{R})$.

Exercise 4.37. *Suppose G is a complex connected reductive algebraic group, and*

$$\mathcal{P} = (B, H, \{\phi_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee\})$$

The opposite pinning *is*

$$\mathcal{P}^{\text{op}} = (B^{\text{op}}, H, \{\psi_{\beta^\vee} \mid \beta^\vee \in -\Pi^\vee\}).$$

Here $B^{\text{op}} \supset H$ is the Borel subgroup corresponding to the positive root system $-R^+$, with simple roots $-\Pi$; and

$$\psi_{\beta^\vee}(x) = \phi_{-\beta^\vee}({}^t x^{-1}) \qquad (x \in SL(2)).$$

Write

$$\mathcal{R}_b = (X^*, \Pi, X_*, \Pi^\vee)$$

for the based root datum attached to (B, H), and

$$\mathcal{R}_b^{\text{op}} = (X^*, -\Pi, X_*, -\Pi^\vee)$$

for the opposite based root datum attached to (B^{op}, H). Let g be the isomorphism of based root data from $\mathcal{R}_b$ to $\mathcal{R}_b^{\text{op}}$ given by $g^* = -\operatorname{Id}$ on X^* (Definition 4.21). The Chevalley involution *attached to $\mathcal{P}$ is the element $\gamma \in \operatorname{Aut}(G)$ corresponding to g (Theorem 4.29). Prove that $\gamma^2 = 1$; that γ acts on H by inversion; that γ commutes with σ_c and with σ_s; and that*

$$\sigma_s = \gamma\sigma_c.$$

Theorem 4.38. *Suppose G is a complex connected reductive algebraic group, and $\sigma \in \operatorname{Aut}(G)_{\text{bar}}$ is a real form (Corollary 4.15); write $G(\mathbb{R})$ for the group of real points. Then there is a compact real form σ_c (Definition 4.35) that commutes with σ. The involution σ_c is unique up to conjugation by an inner automorphism in* $\operatorname{Int}(G(\mathbb{R}))$.

The composition

$$\theta = \sigma \circ \sigma_c \in \operatorname{Aut}(G)$$

called a Cartan involution *attached to the real form σ is an (algebraic) automorphism of order 2; it is also unique up to conjugation by* $\operatorname{Int}(G(\mathbb{R}))$. *Write*

$$K = G^\theta$$

for the group of fixed points of θ, a complex reductive algebraic subgroup of G.

The restriction to K of σ_c defines a compact real form $K(\mathbb{R})$ of K. It is the group of common fixed points

$$G^{\sigma,\sigma_c} = G^{\sigma,\theta} = G^{\sigma_c,\theta} = K(\mathbb{R}).$$

In particular, K is the complexification of the compact group $K(\mathbb{R})$ (Theorem 4.5).

Conversely, suppose $\theta \in \mathrm{Aut}(G)$ is an automorphism with $\theta^2 = 1$; write $K = G^\theta$ for the (complex reductive algebraic) group of fixed points. Then there is a compact real form σ_c of G that commutes with σ. The real form σ is unique up to conjugation by an inner automorphism in $\mathrm{Int}(K)$.

Corollary 4.39. *Suppose G is a complex connected reductive algebraic group. The construction of Theorem 4.38 establishes a bijection*

$$\begin{aligned}&\{\textit{real forms of } G\}/(\textit{conjugation by } \mathrm{Int}(G))\\ &\qquad\qquad \leftrightarrow \{\textit{involutions in } \mathrm{Aut}(G)\}/(\textit{conjugation by } \mathrm{Int}(G)).\end{aligned}$$

The conjugacy classes on the left in the Corollary are by definition strong equivalence classes of real forms. On the right, each class belongs to a single coset of $\mathrm{Int}(G)$ in $\mathrm{Aut}(G)$. Because the conjugacy class of compact real forms is also contained in a single coset of $\mathrm{Int}(G)$, we deduce

Corollary 4.40. *Suppose G is a complex connected reductive algebraic group. Then the bijection of Corollary 4.39 establishes (by passage to the group $\mathrm{Out}(G)$ of components of $\mathrm{Aut}(G)$) a bijection*

$$\{\textit{inner classes of real forms}\} \leftrightarrow \{\textit{elements of order 2 in } \mathrm{Out}(G)\}.$$

According to Corollary 4.30, these are exactly the automorphisms of order 2 of the based root datum of G.

5. Interlude on linear algebra

We are going to need a little structure theory for the real reductive groups that we have just described. The central fact is the following theorem of linear algebra.

Theorem 5.1 (Polar decomposition)**.** *Suppose $G = GL(n, \mathbb{C})$, and*

$$\sigma_c(g) = {}^t\overline{g}^{-1} \qquad (g \in GL(n, \mathbb{C})$$

is the compact real form (with real points $K = U(n)$. Then σ_c acts on the Lie algebra $\mathfrak{g}$ of G (consisting of $n \times n$ complex matrices) by

$$\sigma_c(X) = -{}^t\overline{X} = -X^* \qquad (X \in \mathfrak{g}).$$

Write

$$\begin{aligned}\mathfrak{p}(n) &= -1 \textit{ eigenspace of } \sigma_c \textit{ on } \mathfrak{g}\\ &= n \times n \textit{ Hermitian matrices.}\end{aligned}$$

Then the map

$$U(n) \times \mathfrak{p}(n) \to GL(n, \mathbb{C}) \qquad (k, X) \mapsto k \exp(X)$$

is an analytic diffeomorphism of $U(n) \times \mathfrak{p}$ *onto* $GL(n, \mathbb{C})$*, called the* polar decomposition*.*

Suppose H *is any closed subgroup of* $GL(n, \mathbb{C})$ *preserved by* σ_c*, and having finitely many connected components. Then* H *inherits the polar decomposition: if* $\mathfrak{h}$ *is the real Lie algebra of* H*, then*

$$(H \cap U(n)) \times (\mathfrak{h} \cap \mathfrak{p}(n)) \to H \qquad (k, X) \mapsto k \exp(X)$$

is an analytic diffeomorphism of $(H \cap U(n)) \times \mathfrak{h} \cap \mathfrak{p}(n))$ *onto* H*.*

One reason for the terminology is the case $n = 1$, when this is the decomposition $z = re^{i\theta}$ of a non-zero complex number. The positive number r is $\exp(X)$ for unique real number X, and $e^{i\theta}$ is the unitary matrix.

In order to be able to apply this to general reductive groups, we need

Theorem 5.2 ([14], Theorem 2.3.7, and [11], Theorem 6.31)**.** *Suppose* G *is a complex algebraic group. Then* G *may be realized as a (Zariski-closed algebraic) subgroup of* $GL(n, \mathbb{C})$ *for some* n*.*

Suppose in addition that G *is connected reductive algebraic, and that* σ_c^G *is a compact real form of* G *(Definition 4.35). Then* σ_c *can be extended to a compact real form* σ_c *of* $GL(n, \mathbb{C})$*.*

It is the first assertion (about embedding G in $GL(n)$) that is proved in [14]. The tricky point is the extension of σ_c^G to $U(n)$. (Springer proves that any rational form of G can be arranged to extend to the split form of $GL(n)$; but in our real-oriented world, we want instead the corresponding statement about compact forms.) The main difficulty is to show that the real form $G(\mathbb{R}, \sigma_c^G)$ is really a compact group. With this in hand, it is easy to show that the compact group must preserve an inner product on the representation $\mathbb{C}^n$. The existence of the extension (which is inverse Hermitian transpose with respect to the invariant inner product) follows.

Corollary 5.3 ([11], Theorem 6.31)**.** *Suppose* G *is a complex connected reductive algebraic group, with compact real form* σ_c^G*. Embed* G *in* $GL(n, \mathbb{C}$*, and fix a compact real form* σ_c *of* $GL(n, \mathbb{C}$ *extending* σ_c^G *(Theorem 5.2). After change of basis in* $\mathbb{C}^n$*,*

$$\sigma_c(g) = {}^t\overline{g}^{-1}.$$

Write $U = G(\mathbb{R}, \sigma_c^G)$ *for the corresponding group of real points, and* $\mathfrak{p} \subset \mathfrak{g}$ *for the* -1 *eigenspace of* σ_c^G*. Then*

1. $U = G \cap U(n)$ *consists of the unitary matrices in* G*, a compact group;*
2. $\mathfrak{p}$ *consists of the Hermitian matrices in* $\mathfrak{g}$*; and*

(3) *the map*

$$U \times \mathfrak{p} \to G, \qquad (k, X) \mapsto k \exp(X)$$

is an analytic diffeomorphism of $U \times \mathfrak{p}$ *onto* G.

Finally suppose that σ *is a real form of* G *that commutes with* σ_c^G*; write* $\theta = \sigma \circ \sigma_c^G$ *for the corresponding (algebraic) Cartan involution of* G*. Write*

$$G(\mathbb{R}) = G^\sigma, \qquad K = G^\theta.$$

Finally write $\mathfrak{s}$ *for the* -1 *eigenspace of* θ *on* $\mathfrak{g}$*, so that*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}.$$

(1) *The (differentiated) action of* σ *on* $\mathfrak{g}$ *defines a real form of this complex Lie algebra; the fixed points are*

$$\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G(\mathbb{R})).$$

The subspaces $\mathfrak{k}$ *and* $\mathfrak{s}$ *are defined over* $\mathbb{R}$.
(2) *The (differentiated) action of* θ *preserves* $\mathfrak{g}(\mathbb{R})$*; the* $+1$ *and* -1 *eigenspaces are* $\mathfrak{k}(\mathbb{R})$ *and* $\mathfrak{s}(\mathbb{R})$.
(3) *The group* $K(\mathbb{R}) = G(\mathbb{R})^\theta = G(\mathbb{R}) \cap U(n)$ *consists of the unitary matrices in* $G(\mathbb{R})$*; it is compact.*
(4) *The space* $\sim(\mathbb{R})$ *consists of the Hermitian matrices in* $\mathfrak{g}(\mathbb{R})$.
(5) *The map*

$$K(\mathbb{R}) \times \mathfrak{s}(\mathfrak{R}) \to G, \qquad (k, X) \mapsto k \exp(X)$$

is an analytic diffeomorphism of $K(\mathbb{R}) \times \mathfrak{s}(\mathfrak{R})$ *onto* G.

In particular, $K(\mathbb{R})$ *is maximal among compact subgroups of* $G(\mathbb{R})$*; and the homogeneous space* $G(\mathbb{R})/K(\mathbb{R})$ *is diffeomorphic (in a* $K(\mathbb{R})$*-equivariant way) to the Euclidean space* $\mathfrak{s}(\mathbb{R})$.

The corollary follows easily from the linear algebra in Theorem 5.1.

6. Interlude on angels and pinnings

We have seen in Corollary 4.39 how to parametrize equivalence classes of real forms of a complex connected reductive algebraic group G by conjugacy classes of involutions in $\mathrm{Aut}(G)$, and we have even seen (in Corollary 4.30) some hints about how one might get one's hands on such involutions, by conjugating them into some kind of standard form. We want now to turn to the problem of classifying representations of real forms. The purpose of this section is to address a fundamental formal problem. We will begin by formulating some questions in an apparently natural and reasonable way; but then we will give an example showing that the natural and reasonable definition does not behave naturally and reasonably. The last part of the section is devoted to modifying the formulations to fix this problem. *The reader should therefore beware that the definitions formulated in* (6.1) *are* ***not*** *the ones that we will ultimately use.*

In classical representation theory, the basic question looks like this:

> **Fix a real reductive group $G(\mathbb{R})$. Parametrize the equivalence classes of irreducible representations (π, V) of $G(\mathbb{R})$. Call this set of equivalence classes $\widehat{G(\mathbb{R})}$, "representations of the real form $G(\mathbb{R})$".**

If we want to think of understanding the real form by conjugating it into standard position, we are led to modify this question:

> **Fix a complex reductive group G. Parametrize the equivalence classes of pairs (σ, π), with σ a real form of G and (π, V) an irreducible representation (π, V) of $G(\mathbb{R}, \sigma)$. Call this set of equivalence classes $\widehat{G}(\mathbb{R})$, "representations of real forms of G".**

In this modified question, there is an obvious notion of "equivalence": we should say that (σ, π) is equivalent to (σ', π') if there is an element $g \in G$ with the following two properties. First, roughly speaking, we need $g \cdot \sigma = \sigma'$. This means that conjugation by g should carry the (conjugate-linear) automorphism σ to σ':

$$\sigma'(x) = g[\sigma(g^{-1}xg)]g^{-1}. \tag{6.1a}$$

This means in particular that g conjugates $G(\mathbb{R}, \sigma)$ to $G(\mathbb{R}, \sigma')$. We can therefore define a representation $g \cdot \pi$ of $G(\mathbb{R}, \sigma')$ on the space V of π by the formula

$$(g \cdot \pi)(x) = \pi(g^{-1}xg) \qquad (x \in G(\mathbb{R}, \sigma')). \tag{6.1b}$$

Then the second requirement should be that $g \cdot \pi$ is equivalent to π'.

Formally this all makes good sense. It is obvious that there is a well-defined map

$$\begin{aligned} &\text{equivalence classes of irreducibles of } G(\mathbb{R}, \sigma_0) \\ &\qquad\qquad\qquad \longrightarrow \text{equivalence classes of pairs } (\sigma, \pi) \end{aligned} \tag{6.1c}$$

by sending π to the equivalence class of (σ_0, π). It is even clear that

$$\widehat{G}(\mathbb{R}) = \coprod_{\text{equiv classes of } \sigma} (\text{image of } \widehat{G(\mathbb{R}, \sigma)}).$$

The difficulty is this: with the definitions sketched above, the mapping in (6.1c) need not be one-to-one, so that our new problem about all real forms is *not* just the disjoint union of the old problems about each real form. Here is an example. Suppose G is $SL(2)$, the group of (complex) two by two matrices of determinant one. One of the real forms of G is σ_0, which acts on matrices by complex conjugation of their entries. The corresponding real form is the group of fixed points of σ_0:

$$G(\mathbb{R}, \sigma_0) = SL(2, \mathbb{R}). \tag{6.2a}$$

Inside this real group is the maximal compact subgroup $SO(2)$, whose irreducible representations are one-dimensional and naturally indexed by the integers:

$$\tau_m \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{im\theta}. \tag{6.2b}$$

The group $SL(2, \mathbb{R})$ has discrete series representations $\pi(1)$ and $\pi(-1)$, which are characterized by the properties

(6.2c)

$$\pi(1)|_{SO(2)} = \tau_2 + \tau_4 + \tau_6 + \cdots, \quad \pi(-1)|_{SO(2)} = \tau_{-2} + \tau_{-4} + \tau_{-6} + \cdots.$$

It is clear from these characterizations that $\pi(1)$ and $\pi(-1)$ are inequivalent representations: they define distinct classes in $\widehat{SL(2, \mathbb{R})}$. What we will show is that, with the definitions above, their images in $\widehat{SL(2)}(\mathbb{R})$ are equivalent. That is, we will show that $(\sigma_0, \pi(1))$ is equivalent to $(\sigma_0, \pi(-1))$. According to the definition above, an equivalence is given by an element $g \in SL(2)$ that (first of all) conjugates σ_0 to σ_0 and (second) carries $\pi(1)$ to something equivalent to $\pi(-1)$.

So what does it mean for g to fix σ_0? According to (6.1a), the condition is

$$\sigma_0(x) = g[\sigma_0(g^{-1}xg)]g^{-1}.$$

for all $x \in SL(2)$. This is equivalent to

$$\sigma(x) = (g\sigma_0(g)^{-1})[\sigma_0(x)](g\sigma_0(g)^{-1}]$$

again for all $x \in SL(2)$. Since σ_0 is an automorphism and therefore surjective, the condition is

$$g\sigma_0(g)^{-1} \in Z(SL(2)) = \pm \mathrm{Id}. \tag{6.2d}$$

There are two cases here. If $g\sigma_0(g)^{-1} = \mathrm{Id}$, then g is fixed by σ_0, so $g \in SL(2, \mathbb{R})$. In this case $g \cdot \pi$ is the action of a group on its own representations, which is always trivial on equivalence classes: the operator $\pi(g)$ provides an intertwining operator from π to $g \cdot \pi$.

The interesting case is $g\sigma_0(g)^{-1} = -\mathrm{Id}$. In our case there is an example of such an element:

$$g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Conjugation by g normalizes $SL(2,\mathbb{R})$. This particular choice of g also normalizes $SO(2)$, acting by inversion there; so $g \cdot \tau_m = \tau_{-m}$ for any $m \in \mathbb{Z}$ (notation as in (6.2b). Using the characterizations (6.2c), we deduce immediately that

$$g \cdot \pi(1) = \pi(-1). \tag{6.2e}$$

So that is the problem. The heart of the matter is (6.2d): in the action of G on real forms, the stabilizer of a real form $\sigma \in \mathrm{Aut}(G)_{BAR}$ may be larger than $G(\mathbb{R},\sigma)$. The reason is that the mapping $G \to \mathrm{Int}(G)$ has a kernel, namely $Z(G)$. The solution is to replace $\mathrm{Aut}(G)_{BAR}$ (or rather the group $\mathrm{Aut}(G)^{\mathrm{Gal}}$ in which it lies) by a group extension, containing the extension G of $\mathrm{Aut}(G)$. Here is how.

Definition 6.3. *Suppose G is a complex connected reductive algebraic group, and*

$$\mathcal{P} = (B, H, \{\phi_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee\}$$

is a pinning. Write $\mathcal{R}_b$ for the based root datum attached to $\mathcal{P}$. Recall from Corollary 4.30 the natural isomorphisms

$$\mathrm{Aut}(G) \simeq \mathrm{Int}(G) \rtimes \mathrm{Out}(\mathcal{R}_b)$$

and

$$\mathrm{Out}(\mathcal{R}_b) \simeq \mathrm{Aut}(G,\mathcal{P}).$$

Now we make an abstract copy $\mathrm{Aut}^{\mathrm{strong}}(G,\mathcal{P})$ *of this outer automorphism group. Elements of this copy just carry a label "strong" to remind us of where they are:*

$$\mathrm{Aut}^{\mathrm{strong}}(G,\mathcal{P}) = \{\theta_1^{\mathrm{strong}} \mid \theta_1 \in \mathrm{Aut}^{\mathrm{strong}}(G,\mathcal{P})\}.$$

The strong automorphism group of G with respect to $\mathcal{P}$) *is the semidirect product*

$$\mathrm{Aut}^{\mathrm{strong}}(G) = G \rtimes \mathrm{Aut}^{\mathrm{strong}}(G,\mathcal{P}),$$

with the first factor normal and the second acting by the indicated algebraic automorphisms. This is a complex Lie group with identity component G. There is a natural exact sequence

$$1 \to Z(G) \to \mathrm{Aut}^{\mathrm{strong}}(G) \to \mathrm{Aut}(G) \to 1.$$

Let σ_c be the compact real form attached to $\mathcal{P}$ (Definition 4.35). Define an abstract copy $\{1, \sigma_c^{\mathrm{strong}}\}$ of $\mathbb{Z}/2\mathbb{Z}$, and make it act on G by σ_c. Notice that σ_c commutes with all the automorphisms of G in $\mathrm{Aut}(G,\mathcal{P})$. *The* strong Galois-extended group of G with respect to $\mathcal{P}$ *is the semidirect product*

$$\mathrm{Aut}^{\mathrm{strong}}(G)^{\mathrm{Gal}} = G \rtimes (\mathrm{Aut}(G,\mathcal{P}) \times \{1, \sigma_c^{\mathrm{strong}}\}).$$

If we make σ_c^{strong} *act trivially on the factor* $\mathrm{Aut}(G,\mathcal{P})$*, then we can write*

$$\mathrm{Aut}^{\text{strong}}(G)^{\text{Gal}} = \mathrm{Aut}(G)^{\text{strong}} \rtimes \{1, \sigma_c^{\text{strong}}\}$$

By analogy with the notation of Corollary 4.15, we write

$$\mathrm{Aut}^{\text{strong}}(G)_{\text{bar}} = \mathrm{Aut}^{\text{strong}}(G)\sigma_c^{\text{strong}}$$

for the coset inducing conjugate-linear automorphisms of G*. There is a natural exact sequence*

$$1 \to Z(G) \to \mathrm{Aut}^{\text{strong}}(G)^{\text{Gal}} \to \mathrm{Aut}(G)^{\text{Gal}} \to 1.$$

A strong real form of G *is an element* $\sigma^{\text{strong}} \in \mathrm{Aut}^{\text{strong}}(G)_{\text{bar}}$ *mapping to an element* σ *of order 2 in* $\mathrm{Aut}(G)_{\text{bar}}$*. Equivalently, the requirement is*

$$(\sigma^{\text{strong}})^2 \in Z(G).$$

Two strong real forms are strongly equivalent *if they are conjugate by* G*. (This implies strong equivalence of the corresponding real forms (Definition 4.8) but the converse is not true.*

A strong involution of G *is an element* $\theta^{\text{strong}} \in \mathrm{Aut}^{\text{strong}}(G)$ *mapping to an element* θ *of order 2 in* $\mathrm{Aut}(G)$*. Equivalently, the requirement is*

$$(\theta^{\text{strong}})^2 \in Z(G).$$

Two strong involutions are called strongly equivalent *if they are conjugate by* G*.*

Fix now an inner class of real forms of G *(Definition 4.8); equivalently (Corollary 4.40) an element* $\theta_1 \in \mathrm{Aut}(G,\mathcal{P})$ *of order (*1 *or)* 2*. Write* $\sigma_1 = \theta_1\sigma_c$ *for a real form corresponding to* θ_1 *(Theorem 4.38; this is the unique real form in the inner class preserving* $\mathcal{P}$*. (We might call it the* quasicompact inner form *since in some sense it is the most compact form in the inner class.) We define the* extended group of G *with respect to* $\mathcal{P}$ *and the inner class represented by* θ_1 *as*

$$G^\Gamma =_{def} G \rtimes \{1, \theta_1^{\text{strong}}\} \subset \mathrm{Aut}^{\text{strong}}(G).$$

Here Γ *stands for the group* $\{1, \theta_1^{\text{strong}}\}$*; it is playing a role precisely parallel to the Galois group, but we prefer to give it a different name since* θ_1^{strong} *is acting by an algebraic (not conjugate-algebraic) automorphism. Thus* G^Γ *is a complex reductive algebraic group with two connected components. Strong involutions in the inner class of* θ_1 *are elements*

$$x \in G^\Gamma \setminus G = G\theta_1^{\text{strong}}$$

such that $x^2 \in Z(G)$*.*

The Galois-extended group of G *with respect to $\mathcal{P}$ and the inner class represented by σ_1 is*

$$G^{\mathrm{Gal}} =_{def} G \rtimes \{1, \sigma_1^{\mathrm{strong}}\} \subset \mathrm{Aut}^{\mathrm{strong}}(G)^{\mathrm{Gal}}.$$

This is a Lie group with two connected components; it is not a complex Lie group, because the nonidentity component acts on the identity component by conjugate-algebraic (that is, antiholomorphic) automorphisms. Strong real forms in the inner class of σ_1 are elements

$$\gamma \in G^{\mathrm{Gal}} \setminus G = G\sigma_1^{\mathrm{strong}}$$

such that $\gamma^2 \in Z(G)$.

There is a small lie or abuse of notation at the end of this definition. If $\theta_1 = 1$, we really do intend that Γ should be a two-element group $\{1, 1^{\mathrm{strong}}\}$. Then G^Γ is not a subgroup of $\mathrm{Aut}^{\mathrm{strong}}(G)$: the map from G^Γ to $\mathrm{Aut}^{\mathrm{strong}}(G)$ has kernel $\{1, 1^{\mathrm{strong}}\}$.

The `atlas` software always works with a single inner class of real forms, and therefore with the group G^Γ (which has just two connected components) rather than with $\mathrm{Aut}^{\mathrm{strong}}(G)$ (which has $|\,\mathrm{Out}(G)|$ connected components). The underlying mathematics is set out in [1], which works instead with the extended group G^{Gal}. We should therefore record the precise correspondence between strong involutions and strong real forms, extending Theorem 4.38.

Theorem 6.4. *Suppose G is a complex connected reductive algebraic group endowed with a pinning $\mathcal{P}$; use notation as in Definition 6.3. Suppose $\sigma^{\mathrm{strong}} \in \mathrm{Aut}^{\mathrm{strong}}(G)_{\mathrm{bar}}$ is a strong real form of G. Then there is a G conjugate $(\sigma_c^{\mathrm{strong}})'$ of $\sigma_c^{\mathrm{strong}}$ that commutes with σ^{strong}; this conjugate is unique up to conjugation by $G(\mathbb{R})$. We therefore get a strong involution*

$$\theta^{\mathrm{strong}} = \sigma^{\mathrm{strong}}(\sigma_c^{\mathrm{strong}})'$$

in the same inner class as σ^{strong}, and defined up to conjugation by $G(\mathbb{R})$; it is called a strong Cartan involution *attached to σ^{strong}. We have also*

$$(\sigma^{\mathrm{strong}})^2 = (\theta^{\mathrm{strong}})^2 \in Z(G).$$

Conversely, suppose $\theta^{\mathrm{strong}} \in \mathrm{Aut}^{\mathrm{strong}}(G)$ is a strong involution; write $K = G^{\theta^{\mathrm{strong}}}$ for the (complex reductive algebraic) group of fixed points. Then there is a G conjugate $(\sigma_c^{\mathrm{strong}})'$ of $\sigma_c^{\mathrm{strong}}$ that commutes with θ^{strong}; this conjugate is unique up to conjugation by K. We therefore get a strong real form

$$\sigma^{\mathrm{strong}} = \theta^{\mathrm{strong}}(\sigma_c^{\mathrm{strong}})'$$

in the same inner class as θ^{strong}, and defined up to conjugation by K.

Corollary 6.5. *Suppose G is a complex connected reductive algebraic group. Then the correspondence of Theorem 6.4 establishes a bijection*

$$\begin{aligned}\{\textit{strong real forms of } G\}/(\textit{conjugation by } G)\\ \leftrightarrow \{\textit{strong involutions for } G\}/(\textit{conjugation by } G).\end{aligned}$$

This corollary allows for translating between the statements about group representations and real forms in [1] and the statements about Harish-Chandra modules and involutions with which the `atlas` software works.

With these definitions in hand, we can repair the difficulties explained at the beginning of this section. A better version of the basic question is this:

Definition 6.6. *Suppose G is a complex connected reductive algebraic group. A* representation of a strong real form of G *is a pair* $(\sigma^{\text{strong}}, \pi)$*, with σ^{strong} a strong real form of G (Definition 6.3) and (π, V) an irreducible quasisimple (to be defined in Definition 7.4 below) representation (π, V) of $G(\mathbb{R}, \sigma^{\text{strong}})$. We impose on this set of representations an equivalence relation to be defined below in* (6.7)*; the set of equivalence classes is called $\widehat{G}(\mathbb{R})$.*

Here is the definition of "equivalence." We say that $(\sigma^{\text{strong}}, \pi)$ is equivalent to $((\sigma^{\text{strong}})', \pi')$ if there is an element $g \in G$ with the following two properties. First, we require

$$(\sigma^{\text{strong}})' = g\sigma^{\text{strong}}g^{-1}. \tag{6.7a}$$

This means in particular that conjugation by g carries the centralizer of σ^{strong} in G (which is precisely $G(\mathbb{R}, \sigma)$) to the centralizer of $(\sigma^{\text{strong}})'$ (which is $G(\mathbb{R}, \sigma')$). We can therefore define a representation $g \cdot \pi$ of $G(\mathbb{R}, \sigma')$ on the space V of π by the formula

$$(g \cdot \pi)(x) = \pi(g^{-1}xg) \qquad (x \in G(\mathbb{R}, \sigma')). \tag{6.7b}$$

Then the second requirement is that $g \cdot \pi$ is infinitesimally equivalent to π' as a representation of $G(\mathbb{R}, \sigma)$. (We will define infinitesimal equivalence in Definition 7.9 below.)

Here is the result that makes everything good (and which *failed* in the example of (6.2)), when we were using real forms instead of strong real forms).

Lemma 6.8. *Suppose $(\sigma^{\text{strong}}, \pi)$ and $(\sigma^{\text{strong}}, \pi')$ are two representations of strong real forms of G that are equivalent in the sense of* (6.7)*. Then π and π' are (infinitesimally) equivalent as representations of $G(\mathbb{R}, \sigma)$.*

Proof. Suppose $g \in G$ implements the equivalence. According to (6.7a) this means first of all that g commutes with σ^{strong}; that is, that

$$g \in G^{\sigma^{\text{strong}}} = G(\mathbb{R}, \sigma).$$

As we have already remarked, it follows that $g \cdot \pi$ is equivalent to π in any definition of equivalence; the operator $\pi(g)$ provides the required intertwining operator to prove this. By the definition of the equivalence relation, $g \cdot \pi$ is also (infinitesimally) equivalent to π'. Putting these two equivalences together, we find that π is infinitesimally equivalent to π', as we wished to show. $\square$

For each strong real form σ_0^{strong}, the lemma provides a *one-to-one* map

$$\text{(6.9)} \quad \text{equivalence classes of irreducibles of } G(\mathbb{R}, \sigma_0^{\text{strong}}) \hookrightarrow \text{equivalence classes of pairs } (\sigma^{\text{strong}}, \pi)$$

by sending π to the equivalence class of $(\sigma_0^{\text{strong}}, \pi)$. It is now clear that

$$\text{(6.10)} \qquad \widehat{G}(\mathbb{R}) = \coprod_{\text{equiv classes of } \sigma^{\text{strong}}} \widehat{G(\mathbb{R}, \sigma^{\text{strong}})}.$$

7. Harish-Chandra modules

At last we can begin to describe Harish-Chandra's algebraic framework for studying infinite-dimensional representations of a real reductive group. Perhaps the most fundamental fact about finite-dimensional irreducible representations is Schur's lemma.

Theorem 7.1 (Schur's lemma)**.** *Suppose (π, V) is an irreducible finite-dimensional representation of G, and $T \in \operatorname{End}_G(V)$ is a linear operator on V that commutes with all the operators $\pi(g)$. Then $T = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.*

Proof. If $V = 0$, the theorem is true with any value of λ. If $V \neq 0$, then by linear algebra T must have an eigenvalue $\lambda \in \mathbb{C}$. The eigenspace

$$V_\lambda = \{v \in V \mid Tv = \lambda v\}$$

is then a non-zero closed subspace of V. It is G-invariant since T commutes with all the operators $\pi(g)$. Since V is irreducible, a non-zero closed invariant subspace must be all of V; so $V = V_\lambda$, as we wished to show. $\square$

In infinite-dimensional representations V, of course there need not be an eigenvalue of T, so we cannot imitate this proof precisely. Nevertheless there are many of "spectral theorems," providing something like eigenspaces for sufficiently nice continuous linear operators. One can therefore prove versions of Schur's lemma with additional hypotheses on T; for example, if π is an irreducible unitary representation, then the theorem is true for any bounded T commuting with π.

The point of view in Harish-Chandra's work is that Schur's lemma is a hallmark of nice representations; he makes the truth of the theorem (for some particular T) part of his definition of the right class of representations to consider. To see how that works, we need to construct some operators T.

Definition 7.2. *Suppose G is a real Lie group, with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$ (the left-invariant complex vector fields on G). Then the universal enveloping algebra $U(\mathfrak{g})$ may be identified with linear differential operators on G that commute with left translation. The group G acts on $U(\mathfrak{g})$ by algebra automorphisms* $\mathrm{Ad}(g)$. *Define*

$$\mathfrak{Z}(\mathfrak{g}) = U(\mathfrak{g})^{\mathrm{Ad}(G)},$$

the algebra of common fixed points of all the automorphisms $\mathrm{Ad}(g)$. *Evidently*

$$\mathfrak{Z}(\mathfrak{g}) \subset U(\mathfrak{g})^{\mathrm{Ad}(G_0)} = \textit{center of } U(\mathfrak{g});$$

so in any case $\mathfrak{Z}(\mathfrak{g})$ is a commutative subalgebra of $U(\mathfrak{g})$.

Lemma 7.3. *Suppose (π, V) is a representation of the Lie group G, and (π^∞, V^∞) is the corresponding representation on smooth vectors (see* (3.19)*). For $u \in U(\mathfrak{g}$ and $g \in G$, we always have*

$$\pi^\infty(g)\pi^\infty(u) = \pi^\infty(\mathrm{Ad}(g)(u))\pi^\infty(g).$$

In particular, if $z \in \mathfrak{Z}(\mathfrak{g})$, then

$$\pi^\infty(z) \in \mathrm{End}_G(V^\infty)$$

is an intertwining operator for π^∞.

Definition 7.4 (see [5], page 225)**.** *A representation (π, V) of a Lie group G is called* quasisimple *if there is an algebra homomorphism*

$$\chi_\pi \colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$$

with the property that

$$\pi^\infty(z) = \chi_\pi(z)\,\mathrm{Id}_V$$

for all $z \in \mathfrak{Z}(\mathfrak{g})$.

That is, we are asking that the conclusion of Schur's lemma should hold for each of the intertwining operators $\pi^\infty(z)$ on V^∞.

Theorem 7.5 (Segal [12]). *Suppose (π, V) is an irreducible unitary representation of a Lie group G. Then π is quasisimple (Definition 7.4).*

Harish-Chandra's representation theory concerns only quasisimple irreducible representations. According to the theorem of Segal, this includes all the irreducible unitary representations; so it is a good class to work with for harmonic analysis. Schur's lemma says that any finite-dimensional irreducible representation is quasisimple; so the class includes a lot of familiar examples. Inspection of the proof of Schur's lemma shows that a non-quasisimple irreducible representation has to be "pathological," in the sense that the operator $\pi^\infty(z)$ should have no non-trivial spectral decomposition. Such representations do exist: Wolfgang Soergel in [13] gave an example of an irreducible Banach representation of $SL(2,\mathbb{R})$ that is not quasisimple. (Soergel's construction begins with the examples, due to Enflo and Read, of bounded operators on Banach spaces having no non-trivial closed invariant subspaces.)

For the balance of this section we (change notation and) fix a complex connected reductive algebraic group G (secretly endowed with a pinning $\mathcal{P}$); and a strong real form

$$\sigma^{\text{strong}} \in \text{Aut}^{\text{strong}}(G)_{\text{bar}} \tag{7.6a}$$

(Definition 6.3). The real Lie group that we will consider is the group of real points

$$G(\mathbb{R}) = G^{\sigma^{\text{strong}}}. \tag{7.6b}$$

We fix also a strong Cartan involution

$$\theta^{\text{strong}} \in \text{Aut}^{\text{strong}}(G) \tag{7.6c}$$

corresponding to σ^{strong} in the sense of Theorem 6.4. Write

$$K = G^\theta, \qquad K(\mathbb{R}) = G(\mathbb{R})^\theta. \tag{7.6d}$$

Recall that K is a complex reductive algebraic group, with compact real form $K(\mathbb{R})$. In order to discuss the polar decomposition of $G(\mathbb{R})$, we recall from Corollary 5.3 the notation

$$\mathfrak{s} = -1 \text{ eigenspace of } \theta \text{ on } \mathfrak{g}. \tag{7.6e}$$

We can now begin the definition of Harish-Chandra modules. Suppose (π, V) is a representation of $G(\mathbb{R})$. Write

$$(\pi^\infty, V^\infty) = \text{smooth vectors in } V \tag{7.7a}$$

for the smooth representation defined in (3.19). For any irreducible representation τ of $K(\mathbb{R})$, recall from Definition 3.13 the τ-isotypic subspace $V(\tau) \subset V$. We define the space of $K(\mathbb{R})$-*finite vectors*

$$V_{K(\mathbb{R})} = \{v \in V \mid \dim\langle \pi(k)v \mid k \in K(\mathbb{R})\rangle < \infty\}. \tag{7.7b}$$

It is easy to check that

$$V_{K(\mathbb{R})} = \sum_{\tau \in \widehat{K(\mathbb{R})}} V(\tau), \tag{7.7c}$$

an algebraic direct sum. The corresponding spaces of G-smooth vectors are

$$V^\infty_{K(\mathbb{R})} = V_{K(\mathbb{R})} \cap V^\infty = \sum_{\tau \in \widehat{K(\mathbb{R})}} V^\infty(\tau). \tag{7.7d}$$

The action of G does not preserve the property of being $K(\mathbb{R})$-finite: $V_{K(\mathbb{R})}$ (and $V^\infty_{K(\mathbb{R})}$) are not G-invariant subspaces. It turns out (Lemma 7.8 below) that the Lie algebra representation π^∞ *does* preserve $V^\infty_{K(\mathbb{R})}$:

$$\pi^\infty(u)V^\infty_{K(\mathbb{R})} \subset V^\infty_{K(\mathbb{R})}. \tag{7.7e}$$

Lemma 7.8. *Suppose (π, V) is a representation of the real reductive group $G(\mathbb{R})$. On the space $V^\infty_{K(\mathbb{R})}$, we have two structures:*

(1) *a representation of the compact group $K(\mathbb{R})$ that is locally finite (that is, every vector belongs to a finite-dimensional $K(\mathbb{R})$-invariant subspace); and*
(2) *a representation of the real Lie algebra $\mathfrak{g}(\mathbb{R})$.*

These two representations are connected by two compatibility conditions:

(1) *the differential of the action of $K(\mathbb{R})$ is equal to the restriction to $\mathfrak{k}(\mathbb{R})$ of the action of $\mathfrak{g}(\mathbb{R})$; and*
(2) *for any $k \in K(\mathbb{R})$ and $X \in \mathfrak{g}(\mathbb{R})$, we have*

$$\pi(k)\pi(X)\pi(k^{-1}) = \pi(\mathrm{Ad}(k)X).$$

The only part of this lemma requiring a little thought is the fact that the action of $\mathfrak{g}$ on V^∞ preserves the subspace $V^\infty_{K(\mathbb{R})}$. The reason for this is that the Lie algebra action defines a linear map

$$a\colon \mathfrak{g} \otimes V^\infty \to V^\infty.$$

This map sends the tensor product representation $\mathrm{Ad} \otimes \pi$ of $K(\mathbb{R})$ to π (which also gives the last assertion of the lemma). Now if $v \in V^\infty_{K(\mathbb{R})}$, then v belongs to a finite-dimensional $K(\mathbb{R}$-invariant subspace V_1. It follows that $\pi(X)v$ belongs to the finite-dimensional $K(\mathbb{R})$-invariant subspace $a(\mathfrak{g} \otimes V_1)$.

Definition 7.9. *A* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-module *is a complex vector space* Y *endowed with two structures:*

(1) *a representation of the compact group* $K(\mathbb{R})$ *that is locally finite (that is, every vector belongs to a finite-dimensional* $K(\mathbb{R})$*-invariant subspace); and*
(2) *a representation of the real Lie algebra* $\mathfrak{g}(\mathbb{R})$*.*

These two actions are required to satisfy the two compatibility conditions written in Lemma7.8. A morphism of $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-modules *is a linear map respecting the two actions. With this definition the category* $\mathcal{M}(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$ *of* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-modules is an abelian category.*

An invariant Hermitian form *on a* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-module* Y *is a Hermitian form*

$$\langle , \rangle \colon Y \times Y \to \mathbb{C}$$

subject to the following two requirements:

(1) *the action of* $K(\mathbb{R})$ *preserves the form; and*
(2) *the real Lie algebra* $\mathfrak{g}(\mathbb{R})$ *acts by skew-Hermitian operators:*

$$\langle X \cdot y_1, y_2 \rangle = -\langle y_1, X \cdot y_2 \rangle \qquad (X \in \mathfrak{g}(\mathbb{R}), \quad y_j \in Y).$$

The $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-module* Y *is called* unitary *if it is endowed with a positive-definite invariant Hermitian form.*

If (π, V) *is any representation of* $G(\mathbb{R})$*, the* Harish-Chandra module of π *is the* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-module* $V^{\infty}_{K(\mathbb{R})}$ *constructed in Lemma 7.8.*

Representations (π, V) *and* (π', V') *are called* infinitesimally equivalent *if the* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-modules* $V^{\infty}_{K(\mathbb{R})}$ *and* $(V')^{\infty}_{K(\mathbb{R})}$ *are isomorphic.*

Here is the picture. According to the Cartan decomposition of Corollary 5.3, $G(\mathbb{R})$ is smoothly a product of the compact group $K(\mathbb{R})$ and a Euclidean space. It is therefore reasonable to hope that nice objects on $G(\mathbb{R})$ (like representations) can be specified by giving their restrictions to $K(\mathbb{R})$, and appropriate differential equations describing how they evolve away from $K(\mathbb{R})$.

The Harish-Chandra module of a representation is exactly such a description of the representation: we are keeping its restriction to $K(\mathbb{R})$, and the Lie algebra representation of $\mathfrak{g}(\mathbb{R})$ (as differential equations for the rest of the group action). The niceness of the Cartan decomposition allows one to hope (first) that the Harish-Chandra module of π more or less determines π; and (second) that given an abstract $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-module, we can solve the differential equations to reconstruct a corresponding representation of $G(\mathbb{R})$.

Harish-Chandra's basic theorems say that these hopes are realized.

Theorem 7.10 (Harish-Chandra [5])**.** *In the setting of* (7.6) *and* (7.7)*, suppose* (π, V) *is an irreducible quasisimple representation of* $G(\mathbb{R})$*. Then* $V^\infty_{K(\mathbb{R})}$ *is an irreducible* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-module. If* π *is unitary, then* $V^\infty_{K(\mathbb{R})}$ *is unitary (Definition 7.9).*

Conversely, if Y *is an irreducible* $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$*-module, then there is an irreducible quasisimple representation* (π, V) *such that* $V^\infty_{K(\mathbb{R})} \simeq Y$*. It is always possible to choose* V *to be a Hilbert space. If* Y *is unitary, we may choose* π *to be unitary.*

This bijection

$$\begin{aligned}&\{\textit{infinitesimal equiv classes of irreducible quasisimple reps of } G(\mathbb{R})\}\\ &\qquad\qquad \leftrightarrow \{\textit{equiv classes of irreducible } (\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))\textit{-modules}\}\end{aligned}$$

includes a bijection

$$\begin{aligned}&\{\textit{equiv classes of irreducible unitary reps of } G(\mathbb{R})\}\\ &\quad \leftrightarrow \{\textit{equiv classes of irreducible unitary } (\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))\textit{-modules}\}.\end{aligned}$$

We will take this theorem as formulating the problem we want to consider: the irreducible representations we want to study are the quasisimple ones, and the notion of equivalence is infinitesimal equivalence. To be explicit,

$$\begin{aligned}\widehat{G(\mathbb{R})} &=_{\text{def}} \{\text{infl equiv classes of irr quasisimple reps}\}\\ &\simeq \{\text{equiv classes of irr } (\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))\text{-modules}\}.\end{aligned} \tag{7.11}$$

The point of Harish-Chandra's Theorem 7.10 is to express the analytic problem of understanding group representations in an algebraic way. Once we are in the algebraic world of Definition 7.9, we should take full advantage of it, bringing to bear the linear algebra tool of complexification. If $\mathfrak{h}_0$ is a real Lie algebra and $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, then representations of $\mathfrak{h}_0$ (on a complex vector space) are exactly the same thing as complex-linear representations of $\mathfrak{h}$. In the definition of $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-modules, we can therefore replace the (somewhat subtle) real Lie algebra $\mathfrak{g}(\mathbb{R})$ by the (more elementary) complex Lie algebra $\mathfrak{g}$. Theorem 4.5 allows us to do the same thing with the group action: we can replace the locally finite continuous action of the compact group $K(\mathbb{R})$ by an algebraic action of the complex reductive algebraic group K. Here is what we get.

Definition 7.12. *Suppose* G *is a complex connected reductive algebraic group, and* $\theta^{\text{strong}} \in \text{Aut}^{\text{strong}}(G)$ *is a strong involution (Definition 6.3). Write* $K = G^\theta$ *for the (complex reductive algebraic) group* *of fixed points. A* $(\mathfrak{g}, K)$-module *is a complex vector space* Y *with two structures:*

(1) *an algebraic representation of* K*; and*
(2) *a complex-linear representation of the Lie algebra* $\mathfrak{g}$*.*

These two actions are required to satisfy the compatibility conditions

(1) *the differential of the action of* K *is equal to the restriction to* $\mathfrak{k}$ *of the action of* $\mathfrak{g}$*; and*
(2) *for any* $k \in K$ *and* $X \in \mathfrak{g}$*, we have*
$$\pi(k)\pi(X)\pi(k^{-1}) = \pi(\mathrm{Ad}(k)X).$$

A morphism of $(\mathfrak{g}, K)$-modules *is a linear map respecting the two actions. With this definition the category* $\mathcal{M}(\mathfrak{g}, K)$ *of* $(\mathfrak{g}, K)$*-modules is an abelian category.*

Proposition 7.13. *In the setting of Definitions 7.9 and 7.12, the obvious forgetful functor*
$$\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$$
is an equivalence of categories.

As indicated in the discussion before Definition 7.12, the content is in Theorem 4.5. From now on we will use this proposition without explicit mention, speaking of the $(\mathfrak{g}, K)$-module attached to a representation of $G(\mathbb{R})$. (In particular, we may write V_K^∞ instead of $V_{K(\mathbb{R})}^\infty$.)

An interesting point is that the real form σ has disappeared in Definition 7.12; all that we need is the (algebraic) Cartan involution. But one of the fundamental goals of representation theory is to understand *unitary representations*, so we do not want to lose track of invariant Hermitian forms. Complexification does not behave so well with respect to invariant Hermitian forms, because i times a skew-Hermitian operator is Hermitian (rather than skew-Hermitian). The result is that a discussion of forms requires bringing the real form back into view. Here is the definition.

Definition 7.14. *In the setting of Definition 7.12, fix also a strong real form* σ^{strong} *corresponding to* θ^{strong} *in the sense of Theorem 6.4. An* invariant Hermitian form *on a* $(\mathfrak{g}, K$*-module* Y *is a Hermitian form*
$$\langle , \rangle \colon Y \times Y \to \mathbb{C}$$
subject to the following two requirements:

(1) $\langle k \cdot y_1, y_2\rangle = \langle y_1, \sigma(k)^{-1} \cdot y_2\rangle \qquad (k \in K, \quad y_j \in Y)$*; and*
(2) $\langle X \cdot y_1, y_2\rangle = -\langle y_1, \sigma(X) \cdot y_2\rangle \qquad (X \in \mathfrak{g}, \quad y_j \in Y)$*.*

Proposition 7.15. *In the setting of Definitions 7.9 and 7.12, the equivalence of categories in Proposition 7.13 identifies invariant Hermitian forms (Definition 7.14).*

We leave the proof as an exercise for the reader; what is needed is to extend Theorem 4.5 to include invariant forms.

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