The orbit method and primitive ideals for semisimple Lie algebras

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1. Introduction

One of the central ideas in algebraic geometry is the relationship between ideals in the polynomial ring $\mathbf{C}[x_1,\ldots x_n]$, and (algebraic) varieties in $\mathbf{C}^n$. To each ideal I, one associates the variety

(1.1) $$\underline{V}(I) = \{x \in \mathbf{C}^n \mid f(x) = 0, \text{ all } x \in I\}.$$

Recall that an algebraic variety is called <u>irreducible</u> if it is not the union of two proper subvarieties. One version of Hilbert's Nullstellensatz is then

Theorem 1.2. The map $I \to \underline{V}(I)$ of (1.1) defines a bijection from the set of prime ideals in $\mathbf{C}[x_1,\ldots,x_n]$ onto the set of irreducible algebraic varieties in $\mathbf{C}^n$.

For completeness, we recall the definition of a prime ideal. (All our rings will have unit elements.)

Definition 1.3. Suppose R is a commutative ring. An ideal I in R is called <u>prime</u> if any of the following equivalent conditions is satisfied:

a) If a and b are elements of R, and $ab \in I$, then either $a \in I$ or $b \in I$.

a′) The quotient ring R/I is an integral domain (that is, it has no zero divisors).

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b) If J_1 and J_2 are ideals in R, and J_1J_2 is contained in I, then either $J_1 \subseteq I$ or $J_2 \subseteq I$.

c) there is a field F and a ring homomorphism

$$\phi : R \to F$$

with kernel I.

The purpose of these notes is to outline an important non-commutative analogue of Theorem 1.2, suggested by Dixmier (Conjecture 1.26). (The main point is to replace the polynomial ring by the enveloping algebra of a Lie algebra.) For the moment it remains in part conjectural; we will make the conjecture a little more precise than has been done previously, and suggest some connections with representation theory. To begin, it is helpful to have a more abstract formulation of Theorem 1.2.

Definition 1.4. Suppose V is a vector space over a field k. Write

a) $\quad T^nV = V \otimes \ldots \otimes V \qquad$ (n factors; $n > 0$)

b) $\quad T^0V = k$

c) $\quad TV = \oplus\, T^nV$,

the <u>tensor algebra</u> of V, endowed with the usual multiplication. The <u>symmetric algebra</u> of V,

d) $\quad SV = \oplus\, S^nV$

is defined to be the quotient of TV by the (homogeneous) ideal generated by elements

e) $\quad v \otimes w - w \otimes v \qquad (v, w \in V)$.

Definition 1.5. Suppose V is a vector space. The <u>ring of regular functions</u> $R(V)$ is the algebra of functions on V generated by constant functions and linear functionals. That is (at least if k is infinite), it is the algebra of polynomials in any basis of V^*. The ring of <u>strongly regular functions</u> on V^*, $R_S(V^*)$, is the algebra of functions on V^* generated by constant functions and evaluation at points of V. (For v in V, evaluation at v is the function on V^* defined by

$$f_v(x) = x(v).)$$

There is an obvious map from $T(V^*)$ onto $R(V)$, defined on the generators by sending x in V^* to the function x on V. This map sends $x_1 \otimes x_2 - x_2 \otimes x_1$ to $x_1x_2 - x_2x_1$, which is the zero function; so it lifts to $S(V^*)$ by Definition 1.4(e). It is then easy to show that it defines an isomorphism

$$S(V^*) = R(V). \tag{1.6}$$

Similarly,

$$S(V) = R_S(V^*) \qquad R(V^*) = S(V^{**}) \tag{1.7}$$

If I is an ideal in $S(V)$, we now define

$$\underline{V}(I) = \{x \in V^* \mid f(x) = 0, \text{ all } f \in I\}, \tag{1.8}$$

using the identification (1.7) of $S(V)$ as strongly regular functions on V^*. Now we can reformulate Theorem 1.2 as

Theorem 1.9. Suppose V is a finite dimensional complex vector space. Then the map $I \to \underline{V}(I)$ of (1.8) defines a bijection from the set of prime ideals in $S(V)$ onto the set of irreducible algebraic varieties in V^*.

What we want to do is replace $S(V)$ by a slightly non-commutative analogue, and look for similar theorems. To formulate the setting more precisely, we introduce a natural measure of non-commutativity.

Definition 1.10. Suppose R is a ring, and x is in R. Define

$$ad(x): R \to R, \quad ad(x)a = xa - ax \qquad (a \in R)$$

Obviously $ad(x)$ is a derivation of R:

$$ad(x)(ab) = [ad(x)a]b + a[ad(x)b]. \tag{1.11}$$

If R is an algebra over a field K (or, more generally, if K is any subring of R consisting of elements commuting with x), then $ad(x)$ is

K-linear.

Definition 1.12. Suppose R is a K-algebra. We say that ad(x) is <u>locally finite</u> if each element a of R belongs to a finite dimensional vector subspace V of R, such that

$$ad(x)V \subseteq V.$$

It is this local finiteness of ad(x) which will serve as a substitute for commutativity. Notice that x is central in R exactly when the subspace V of the definition can always be chosen to be one dimensional.

Proposition 1.13. Suppose R is an algebra over the field K. Assume that R is generated by a set S of ad-locally finite elements S (Definition 1.12). Then R is a quotient of the universal enveloping algebra of an ad-locally finite Lie subalgebra, which we may also take to be generated by S. Conversely, suppose $\underline{g}$ is a Lie algebra generated by an ad-locally finite set S. Then S is an ad locally-finite set of generators for the universal enveloping algebra U($\underline{g}$).

In what is probably too general a form, the problem we consider is this. Let $\underline{g}$ be a Lie algebra generated by ad-locally finite elements. Then we seek to relate ideal theory in U($\underline{g}$) to geometry in $\underline{g}^*$. To make this more precise, we must first specify which ideals we consider. All ideals will be two-sided unless otherwise stated.

Definition 1.14. Suppose R is a (possibly non-commutative) ring. A (two-sided) ideal I is called <u>prime</u> if either of the following equivalent conditions is satisfied:

a) If a and b are elements of R, and $axb \in I$ for all $x \in R$, then either $a \in I$ or $b \in I$.

b) If J_1 and J_2 are ideals in R, and $J_1 J_2 \subseteq I$, then either $J_1 \subseteq I$ or $J_2 \subseteq I$.

Definition 1.15. Suppose R is a ring. An ideal I is called <u>completely prime</u> if any of the following equivalent conditions is satisfied:

a) If a and b are elements of R, and $ab \in I$, then either $a \in I$ or $b \in I$.

a′) The quotient ring R/I is an integral domain.

b) If J_1 is a left ideal, J_2 is a right ideal, and $J_1 J_2 \subseteq I$, then either $J_1 \subseteq I$ or $J_2 \subseteq I$.

If R is noetherian, these are equivalent also to

c) There is a division algebra D and a ring homomorphism

$$\Phi : R \rightarrow D$$

with kernel I.

Obviously every completely prime ideal is prime, and the two concepts coincide for commutative R. We will be concerned chiefly with completely prime ideals, since they have a better geometric theory, and are more closely connected with unitary group representations (Proposition 7.12).

Example 1.16. Take $\underline{g} = \mathbf{C}^2$, with the Lie algebra structure

$$[e_1, e_2] = e_2$$

(This is a short way of saying

$$[ae_1+be_2, ce_1+de_2] = (ad-bc)e_2.)$$

The prime ideals in $S(\underline{g})$ correspond by Theorem 1.9 to the points of $\mathbf{C}^2$, the irreducible algebraic curves in $\mathbf{C}^2$, and $\mathbf{C}^2$ itself: in particular, there are many of dimension 1. Every prime ideal I in $U(\underline{g})$ turns out to be completely prime, and we have only the following possibilities for I:

$I = I(z)$, $z \in \mathbf{C}$; or

$I = I^1$; or

$I = \{0\}$.

Here

$$I(z) = \text{ideal generated by } e_2 \text{ and } e_1 - z;$$

$$I^1 = \text{ideal generated by } e_2 \;= \bigcap_{z \in \mathbf{C}} I(z).$$

We will associate these to subvarieties of $\underline{g}^*$, as follows:

$I(z) \leftrightarrow$ the point f_z in $\underline{g}^*$ $(f_z(e_2) = 0,\ f_z(e_1) = z)$;

$I^1 \leftrightarrow$ the line $\{f_z \mid z \in \mathbf{C}\}$;

$(0) \leftrightarrow \underline{g}^*$

In particular, only one variety of dimension 1 appears.

The point of this example is that the ideal theory of $U(\underline{g})$ is much more rigid than that of $S(\underline{g})$; only certain special subvarieties of $\underline{g}^*$ should correspond to ideals in $U(\underline{g})$. To say which ones, we need to introduce a group.

Lemma 1.17. Suppose R is an algebra over C, and D is a derivation of R. (That is, D is linear and satisfies

$$D(ab) = D(a)b + aD(b) \qquad (a, b \in R).)$$

Assume that D is locally finite: that is, for each a in R, there is a finite dimensional, D-stable vector subspace of R, containing a. Then the endomorphism exp(D) of R, defined by

$$(\exp(D))(a) = \sum_n (1/n!) D^n a$$

is an algebra automorphism.

Proof. By induction on n, one shows that

$$D^n(ab) = \sum_{p+q=n} (n!/p!q!)(D^p a)(D^q b).$$

The obvious formal calculation now applies; we use the local finiteness of D to ensure (indeed to define) the absolute convergence of the series for exp(D). Q.E.D.

It is worth remarking that the lemma applies to any bilinear "product" on R, and not just to an associative algebra structure.

Definition 1.18. Suppose $\underline{g}$ is a complex Lie algebra generated by a set S of ad-locally finite elements. The <u>adjoint group</u> Ad($\underline{g}$) of

$\underline{g}$ is the group of automorphisms of $U(\underline{g})$ generated by

$$\{\exp(\mathrm{ad}(zx)) \mid z \in \mathbf{C},\ x \in S\}.$$

By the remark after Lemma 1.17, we can also regard $\mathrm{Ad}(\underline{g})$ as a group of automorphisms of $\underline{g}$. It is conceivable that $\mathrm{Ad}(\underline{g})$ depends on the choice of generating set S; but if $\underline{g}$ is finite dimensional, this cannot happen.

Lemma 1.19. In the setting of Definition 1.18, suppose I $U(\underline{g})$ is any (two-sided) ideal. Then I is stable under the automorphisms in $\mathrm{Ad}(\underline{g})$. That is, the action of $\mathrm{Ad}(\underline{g})$ lifts to any quotient ring $U(\underline{g})/I$.

Proof. By the definition of $\mathrm{Ad}(\underline{g})$, it suffices to show that I is stable under $\mathrm{ad}(x)$, for every x in S. But this is obvious from Definition 1.10. Q.E.D.

Example 1.20. In the setting of example 1.16, we have

$$\mathrm{Ad}(\underline{g}) = \{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbf{C}^{\times},\ b \in \mathbf{C}\}$$

$$\mathrm{ad}\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}(e_1) = e_1 + be_2$$

$$\mathrm{ad}\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}(e_2) = ae_2.$$

If we write f_1, f_2 for the dual basis of $\underline{g}^*$, then the (inverse transpose) action on $\underline{g}^*$ is

$$\mathrm{ad}^*\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}(xf_1+yf_2) = (x-(b/a)y)f_1+(y/a)f_2.$$

The orbits of this action are the points xf_1, and the set

$$\{xf_1+yf_2 \mid y = 0\}.$$

The only $\mathrm{Ad}(\underline{g})$-invariant irreducible algebraic curve in $\underline{g}^*$ is therefore the line $\mathbf{C}f_1$.

Here is an equivariant version of the Nullstellensatz; it is a trivial consequence of Theorem 1.9.

Theorem 1.21. Suppose V is a finite dimensional vector space, and G is a group of linear transformations of V. Then the map $I \rightarrow \underline{V}(I)$ of (1.8) defines a bijection from the set of G-invariant

prime ideals in S(V) onto the set of G-invariant irreducible algebraic varieties in V^*.

Taking into account this theorem, Lemma 1.19, and examples 1.16 and 1.20, it is now reasonable to formulate

Conjecture 1.22. Suppose $\underline{g}$ is a finite dimensional Lie algebra. With $Ad(\underline{g})$ as in Definition 1.18, there is a natural bijection from the set of completely prime ideals in $U(\underline{g})$ onto the set of $Ad(\underline{g})$-invariant irreducible algebraic varieties in $\underline{g}^*$.

The first thing to be said about this conjecture is that it is false, at least if "natural" is to be given any kind of reasonable meaning. In sections 4, 5, and 6, we will discuss the counterexamples assembled by Borho, Joseph, and others. The choice of "completely prime" over "prime" is dictated by the example of $\underline{sl}(2,\mathbb{C})$ (Example 1.27). For $\underline{g}$ solvable, Conjecture 1.22 is Dixmier's conjecture. It is true in that case; a proof and a discussion of the history may be found in [6]. For $\underline{g}$ equal to $\underline{sl}(n,\mathbb{C})$, no counterexamples are known; partial results have been given by Borho [4] and Borho-Jantzen [7]. For primitive ideals in $\underline{sl}(n,\mathbb{C})$ (cf. Conjecture 1.24(b) below), C. Moeglin has recently proved the conjecture completely ([18]).

Before embarking on a more specific discussion, it is helpful to narrow the problem somewhat. In the commutative case, a special role is played by the maximal ideals, which are always prime. The noncommutative generalization is subtle, but well known.

Definition 1.23. An ideal I in a ring R is called <u>primitive</u> if there is a simple R-module M such that

$$I = \{r \in R \mid rm = 0, \text{ all } m \in M\}.$$

We write

$$I = \text{Ann } M$$

By elementary arguments, maximal ideals are primitive, and primitive ideals are prime. Neither converse implication holds; the zero ideal in Example 1.16 is primitive (this requires a non-trivial argument) but not maximal. Primitive is unrelated to completely prime; the zero ideal in the ring of n by n matrices is maximal (hence primitive), but not completely prime unless $n = 1$.

Conjecture 1.24. In the setting of Conjecture 1.22, the bijection should have the following property: a completely prime ideal I is primitive (Definition 1.23) if and only if the corresponding variety is the (Zariski) closure of a single orbit of $Ad(\underline{g})$ on $\underline{g}^*$.

Roughly speaking, the known counterexamples to Conjectures 1.22 and 1.24 involve primitive ideals, and either orbit closures which are not normal, or orbits which are not simply connected. In geometric language, this suggests that one ought to allow some (possibly ramified) coverings of the varieties under consideration. Here is a version of the Nullstellensatz, altered to give such coverings equal status.

Theorem 1.25. Suppose V is a finite dimensional complex vector space. Then there is a natural bijection between the following two classes of objects:

a) commutative prime C algebras A, equipped with an algebra homomorphism

$$\Phi : S(V) \rightarrow A,$$

making A a finitely generated S(V)-module;

b) irreducible affine algebraic varieties X, equipped with a finite morphism

$$\pi : X \rightarrow V^*.$$

This bijection sends A to Spec A.

As stated, this is more nearly a definition than a theorem. It acquires content only through the addition of known geometric facts about finite morphisms - for example, that they have finite fibers. One of the objects in Theorem 1.25(b) is therefore a ramified finite covering of an irreducible subvariety of V^*. In light of the remarks preceding Theorem 1.25, this suggests the following revision of Conjecture 1.22.

Conjecture 1.26. Suppose $\underline{g}$ is a finite dimensional complex Lie algebra. Let G be a connected algebraic group equipped with a morphism

$$Ad:G \to Aut(\underline{g}),$$

the image of which contains $Ad(\underline{g})$. Then there is a natural bijection between the following two classes of objects:

a) completely prime **C** algebras A, equipped with

1) an algebra homomorphism

$$\Phi:U(\underline{g}) \to A,$$

making A a finitely generated $U(\underline{g})$-module; and

2) a (locally finite) algebraic action of G on A by automorphisms, extending the action Ad on the image of $U(\underline{g})$.

b) irreducible affine algebraic varieties X, equipped with

1) a finite morphism

$$\pi:X \to \underline{g}^*; \text{ and}$$

2) an algebraic action of G, compatible with the morphism π and the action of G on $\underline{g}^*$.

This bijection should have the following properties. Write A for an algebra as in (a), I for the kernel of the map Φ, X for the corresponding variety, and V for the image of π.

i) The following are equivalent: I is a primitive ideal; A is a primitive ring; V is the closure of a single $Ad(\underline{g})$ orbit. If G is a finite cover of $Ad(\underline{g})$, this is equivalent to requiring X to be the closure of a single G orbit.

ii) The Gelfand-Kirillov dimension (cf. section 2) of A (or of $U(\underline{g})/I$) is equal to the dimension of the variety X (or V).

iii) A is isomorphic as a module for G (but not as an algebra) to the ring of regular functions on X.

This is our proposed version of the Dixmier conjecture. It is **not** a refinement or extension of Dixmier's original suggestions for solvable groups, just as Theorem 1.25 is not an improvement on the Nullstellensatz. The difficulty is that, although the algebras allowed in A include the completely prime quotients of $U(\underline{g})$, we do not know how to identify the set of varieties in (b) to which they correspond. The natural guess would be that they are the subvarieties of $\underline{g}^*$; but this is ruled out by the examples of Borho and Joseph, as we shall see.

Example 1.27. Suppose $\underline{g}$ is $\underline{sl}(2,\mathbf{C})$, the Lie algebra of two by two matrices of trace zero. Write Ω for the Casimir element of $U(\underline{g})$. We normalize it so that its eigenvalue in the n dimensional representation of $\underline{g}$ is n^2-1. For $z\in\mathbf{C}$, write I_z for the ideal generated by $\Omega-(z^2-1)$. This ideal is always primitive and completely prime; it is maximal unless z is a non-zero integer. Let I^n denote the annihilator of the n dimensional irreducible representation. The quotient of $U(\underline{g})$ by this ideal is isomorphic to the ring of n by n matrices. Therefore I^n is always maximal, and it is completely prime only if $n=1$. This exhausts the primitive ideals. The only other prime ideal in $U(\underline{g})$ is the zero ideal; it is completely prime.

The space $\underline{g}^*$ may be identified with $\underline{g}$ using the trace form. The orbits of $Ad(\underline{g})$ on it are

$$\underline{O}^1 = \{0\}$$

$$\underline{O}_z = \text{orbit of} \begin{pmatrix} z/2 & 1 \\ 0 & -z/2 \end{pmatrix}.$$

All of these are closed except for $\underline{O}_\emptyset$, the closure of which is

$$V_\emptyset = \underline{O}_\emptyset \cup \underline{O}^1.$$

We can make an obvious correspondence between orbit closures and completely prime primitive ideals. By attaching all of $\underline{g}^*$ to the zero ideal, we see that Conjecture 1.22 holds in this case.

With respect to Conjecture 1.26, we concentrate on primitive algebras. If G is Ad($\underline{g}$), any X as in Conjecture 1.26(b) covering an orbit closure must be equal to the orbit closure. (The conjecture then predicts an analogous fact in (a); I do not know if it is true.) Let us take G to be SL(2,**C**), with the conjugation action on $\underline{g}$. Then we can take X to be $\mathbf{C}^2$ with the natural action, and π to be the map

$$\pi(z_1,z_2) = \begin{pmatrix} -z_1z_2 & (z_1)^2 \\ -(z_2)^2 & z_1z_2 \end{pmatrix}.$$

(This is the moment map for the standard symplectic structure on $\mathbf{C}^2$.) This is a two-fold cover of $V_\emptyset$, ramified at 0. The corresponding algebra is the Weyl algebra A_1, consisting of polynomial coefficient differential operators in one variable. We make G act on A_1 by identifying the two dimensional space of generators (spanned by x and d/dx) with $\mathbf{C}^2$; that G preserves the defining relation amounts to the fact that SL(2,**C**) preserves the symplectic form on $\mathbf{C}^2$. To define a map Φ of U($\underline{g}$) into A_1, we need to specify a Lie subalgebra of A_1 isomorphic to $\underline{g}$. Such a subalgebra is generated by x^2 and d^2/dx^2. The image of U($\underline{g}$) consists of the even operators; the kernel of Φ turns out to be $I_{1/2}$. (Notice that this is not the ideal associated to the image of π.)

We conclude this introduction with some remarks about infinite dimensional Lie algebras. The class of algebras for which an adjoint group is defined in Definition 1.18 includes the Kac-Moody algebras. Borho has informed me that these seem not to have an interesting primitive ideal theory. This is certainly consistent with the fact that the proof of the Weyl-Kac character formula uses a Casimir element which is not in the enveloping algebra, but only in some sort of completion. Still, the problems considered here do make sense whenever Definition 1.18 applies. In that direction, M. Artin has pointed out to me that Theorem 1.9 remains true if V is only assumed to be of countable dimension. (What matters is that the cardinality of the ground field should exceed the dimension of V.) Something like Conjecture 1.26 may therefore be true for Lie algebras generated by a finite (or even countable) set of ad-locally finite elements. The example of Kac-Moody algebras suggests that some ring other than U(g) may be more interesting.

Much of what is new in these notes comes from joint work in progress with Dan Barbasch; this applies particularly to the material on representation theory. It is a pleasure to thank him for countless productive discussions. Walter Borho has helped me to understand the subject of primitive ideals in general, and his own work in particular; the latter provides most of the evidence for Conjecture 1.26. Proposition 7.12 and the example in section 6 arose from suggestions of his. Michael Artin and Michel Duflo kindly provided helpful information in pleasant conversations, saving me from the daunting prospect of doing serious library research. Their insights into what ought to be true, what ought to be false, and what might be proveable, were also very useful.

2. Filtrations, Gelfand-Kirillov dimension, and differential operators

For the rest of these notes, g will denote a finite dimensional complex Lie algebra, and G a connected algebraic group equipped with a morphism

$$\mathrm{Ad}: G \to \mathrm{Aut}(\underline{g}), \tag{2.1}$$

the image of which contains $\mathrm{Ad}(\underline{g})$.

Definition 2.2. The <u>standard filtration</u> of $U(\underline{g})$ is the one defined by

$$U_n = \text{span of products of at most } n \text{ elements of } \underline{g}.$$

Recall that the Poincare-Birkhoff-Witt theorem says that the associated graded ring is naturally isomorphic to the symmetric algebra $S(\underline{g})$. Suppose M is a finitely generated $U(\underline{g})$-module. A <u>good filtration</u> of M is an increasing filtration by vector subspaces

$$M_0 \subseteq M_1 \subseteq \ldots$$

subject to the following conditions:

a) $U_n M_m \subseteq M_{n+m}$; and

b) the associated graded $S(\underline{g})$-module

$$\mathrm{gr}\, M = \sum_{n \geq 0} (M_n / M_{n-1})$$

is finitely generated.

If such a filtration exists, we define the <u>associated variety</u> Ass M of M to be the support of gr M (in $\underline{g}^*$). The <u>Gelfand-Kirillov dimension</u> Dim M is the Krull dimension of gr M. It is equal to the dimension of Ass M.

For more about this definition, see for example [8] or [19]. Good filtrations exist if and only if the module M is finitely generated. They are not unique, but the invariants Ass M and Dim M are independent of their choice.

Definition 2.2 immediately suggests a correspondence between ideals in $U(\underline{g})$ and varieties in $\underline{g}^*$, namely sending I to $\mathrm{Ass}(U(\underline{g})/I)$. This is not the one needed for Dixmier's conjecture, however. Even if

$\underline{g}$ is abelian, it produces something called the associated cone for $\underline{V}(I)$, and not $\underline{V}(I)$ itself. This does suggest a way to sharpen Conjecture 1.26, however.

Conjecture 2.3. In the setting of Conjecture 1.26, the correspondence should have the following additional properties:

iv) $\mathrm{Ass}(U(\underline{g})/I)$ is the associated cone for V in $\underline{g}^*$.

v) The algebras A and $R(X)$ (the regular functions on X) both have G-stable good filtrations such that

1) $\phi(U_n(\underline{g})) \subseteq A_n$;

2) $\pi^*(S^n(\underline{g})) \subseteq R(X)_n$;

3) $\mathrm{gr}\, A \cong \mathrm{gr}\, R(X)$,

the last as commutative algebras with G action.

Of course (v) implies (iv) and (ii), and even (iii) if G is reductive. One reason for keeping the various desiderata separate is that the evidence is too scanty for most of them; it may well be that many parts will have to be modified or discarded. In (1) and (2) of (v), it is important to note that we do not assume that, for example,

$$\phi(U) \cap A_n = \phi(U_n).$$

There are examples ([5], Example 3.10) which indicate that this would be an unreasonable requirement.

As an illustration, we will consider in section 3 the case of semisimple orbits in reductive groups. Some of the preliminary formalism is more general; to the extent that it is not standard, it is taken from [15]. We assume first of all that

$$\underline{g} = \mathrm{Lie}(G) \tag{2.4}$$

Fix an algebraic subgroup P of G, and a normal algebraic subgroup P_0 of P, so that the quotient

$$P/P_0 = T \tag{2.5}$$

is abelian. Write

$$
\begin{aligned}
Z &= G/P \\
\underline{n} &= T^*_e(Z) \\
Y &= G/P_0 \qquad (2.6)\\
\underline{u} &= T^*_e(Y) \\
\underline{t}^* &= \underline{u}/\underline{n}.
\end{aligned}
$$

We may identify $\underline{u}$, for example, with the annihilator of $\mathrm{Lie}(P_0)$ in $\underline{g}^*$. With such identifications, $\underline{n}$ becomes a subspace of it, so that the last formula in (2.6) makes sense. This shows that $\underline{t}^*$ may be identified with dual of Lie(T), as the notation suggests.

We want to define a pair of families of objects as in Conjecture 1.26, which ought to correspond under that conjecture. Both families will be indexed by points ξ of the vector space $\underline{t}^*$. We begin with the varieties. Set

$$
\begin{aligned}
\underline{n} + \xi &= \{y \in \underline{u} \mid y \text{ maps to } \xi \text{ in } \underline{t}^*\} \\
X_\xi &= G \times_P (\underline{n} + \xi) \qquad (2.7)\\
&= \{(g,y) \mid g \in G,\ y \in (\underline{n} + \xi)\}/\sim
\end{aligned}
$$

Here $\sim$ denotes the equivalence relation

$$(g,y) \sim (gp^{-1}, p\cdot y), \text{ all } p \in P.$$

This makes X_ξ a homogeneous affine bundle over G/P. If ξ is zero, it is the cotangent bundle. There is a natural invariant symplectic structure on X_ξ; the moment map for this structure is

$$
\begin{aligned}
\pi_\xi &: X_\xi \to \underline{g}^* \qquad (2.8)\\
\pi_\xi(g,y) &= g\cdot y
\end{aligned}
$$

Here we use the identification of $\underline{u}$ as a subspace of $\underline{g}^*$ given after (2.6).

These varieties appear at first glance to be of the type required by Conjecture 1.26(b). Two things are lacking. First, the varieties are only quasi-projective, and need not be affine. (In our main example, G/P will be proper, so its cotangent bundle X_0 cannot be affine. This is probably more a reflection on the defects in Conjecture 1.26 than on the example, but we can in any case easily adjust the example: simply replace X_ξ by its affinization (the spectrum of the ring of global regular functions). More seriously, the mapping π_ξ need not be finite. (To guarantee this, it is enough to have π_ξ proper before affinization.) This problem can only be cured by imposing additional assumptions on G, P, P_0, and ξ. It is certainly sufficient (but not necessary - see Example 2.13) to require that

(2.9) G/P is projective.

Next, we describe the family of algebras. Roughly speaking, they will be algebras of differential operators on section of line bundles on G/P, attached to characters of P/P_0. Since not all elements of $\underline{t}^*$ actually exponentiate to characters, we need to be a little more roundabout (following [2]). In addition, it is helpful for technical reasons to modify the parameter ξ slightly. Write $-2\rho_1$ for the differential of the character of P on the top exterior power of $\underline{n}$. We will assume that

(2.10)(a) ρ_1 is trivial on $\mathrm{Lie}(P_0)$,

so that we can write

(2.10)(b) $\rho_1 \in \underline{t}^*$.

Translation by ρ_1 is the modification of ξ to be introduced later.

On G/P_0, the group T acts by right translation; this commutes with the G action on the left. Consider the algebra

$$\mathrm{Diff}(G/P_0)$$

of algebraic differential operators on G/P_0. We have an algebra homomorphism

$$U(\underline{g}) \otimes U(\underline{t}) \dashrightarrow \mathrm{Diff}(G/P_0),$$

given by these two actions. Set

(2.11) A = centralizer of T in $\mathrm{Diff}(G/P_0)$

Since the G and T actions commute, $U(\mathfrak{g})$ maps into A. Define I_ξ to be the ideal in A generated by elements of the form

$$h + (\xi - \rho_1)(h) \quad (h \in \mathrm{Lie}(T))$$

Notice that these elements are central in A, since the differential of the T action on differential operators is commutation with Lie(T). Finally, define

$$A_\xi = A/I_\xi \tag{2.12}$$
$$= \mathrm{Diff}_\xi(G/P),$$

an algebra of <u>twisted</u> <u>differential</u> <u>operators</u> on G/P. The map Φ_ξ required by Conjecture 1.26 exists by the remark after (2.11).

We consider now the requirements of Conjecture 2.3 in connection with this construction. First, it is standard that the algebra A_ξ has a filtration (by degree of the operators), and a symbol calculus. Together these define an embedding

$$\mathrm{gr}\, A_\xi \hookrightarrow R(T^*(G/P)). \tag{2.13}$$

(Recall that $T^*(G/P)$ is isomorphic to the variety X_0.) On the other hand, there is a natural filtration of the regular functions on X_ξ as well: we say that f belongs to the nth level of the filtration if the restriction of f to each fiber of the map

$$X_\xi \to G/P$$

has degree at most n. Since the fibers are affine spaces (that is, principal homogeneous spaces for vector spaces), this makes sense. It is easy to see that the associated graded ring for the filtered ring of functions on an affine space is the ring of polynomial functions on the attached vector space. We therefore get an embedding

$$\mathrm{gr}\, R(X_\xi) \hookrightarrow R(X_0) \tag{2.14}$$

The inclusions (2.13) and (2.14) come from isomorphisms on the level of coherent sheaves on G/P. We can therefore give an easy sufficient condition for equality.

Lemma 2.15. With notation as above, make the following assumption: the sheaf cohomology of all the symmetric powers of the cotangent bundle of G/P vanishes in positive degree. Then the inclusions (2.13) and (2.14) are isomorphisms. In particular,

$$\mathrm{gr}\, A_\xi = \mathrm{gr}\, R(X_\xi).$$

We will look at two examples of this construction.

Example 2.16. Let G be the group of three by three upper triangular matrices with ones on the diagonal. Its Lie algebra $\underline{g}$ consists of strictly upper triangular matrices. In terms of the usual basis of matrices, $\underline{g}$ has basis

$$x = e_{12},\ y = e_{23},\ z = e_{13}.$$

We take P to be the connected subgroup with Lie algebra spanned by x and z, and P_0 to be the one parameter subgroup generated by x. We may identify T, the quotient of P by P_0, with the one parameter subgroup generated by z; it is the center of $\underline{g}$. This identifies

$$\underline{n} + \xi = \{\lambda \in \underline{g}^* \mid \lambda(x) = 0,\ \lambda(z) = \xi\}$$

Write S for the one parameter subgroup generated by y. Multiplication defines an algebraic isomorphism

$$P \times S \to G.$$

Consequently, the bundle product of (2.7) simplifies to

$$X_\xi \cong S \times (\underline{n} + \xi)$$

If we identify S and $\underline{n}+\xi$ with $\mathbf{C}$, and $\underline{g}^*$ with $\mathbf{C}^3$ (all in more or less obvious ways), then we compute

$$\mathrm{Ad}(s)(x) = x - sz \quad (s \in S)$$

$$\pi_\xi(s,t) = (s\xi, t, \xi).$$

Evidently π_ξ is a finite map if and only if ξ is not zero. In that case, it is an isomorphism onto a single orbit.

Similarly, the algebra $\mathrm{Diff}(G/P_0)$ may be identified with the Weyl algebra of polynomial coefficient differential operators in two variables. The action of T is by translation in the second variable; so the ring A consists of operators in two variables, with coefficients depending only on the first. The map of (2.11) is

$$x \dashrightarrow -sD_t$$

$$y \dashrightarrow -D_s$$

$$z \dashrightarrow -D_t$$

$$t \dashrightarrow D_t.$$

The remarks above now allow us to compute that A_ξ may be identified with a Weyl algebra in one variable (which should be thought of as Diff(S). Specifically,

$$\Phi_\xi(x) = s\xi$$

$$\Phi_\xi(y) = -D_s$$

$$\Phi_\xi(z) = \xi.$$

Thus A_ξ is a finitely generated $U(\underline{g})$-module if and only if ξ is not zero; and in that case it is actually a homomorphic image of $U(\underline{g})$.

This example shows that the finiteness properties of π_ξ and Φ_ξ are closely related, as they should be if the differential operator construction is to be well related to Conjecture 1.26. It also shows that G/P need not be projective for π_ξ to be finite.

Example 2.17. Let $\underline{g}$ be as in Examples 1.16 and 1.20, and $G=Ad(\underline{g})$. We use the notation from those examples. Up to G conjugacy, there are only two connected proper subgroups of G: the strictly upper triangular matrices N, and the diagonal matrices H. We identify these with **C** and $\mathbf{C}^\times$, respectively. Take P and T to be equal to N, and P_0 to be trivial. Arguing as in the previous example, we can identify X_ξ with H×**C**, and A_ξ with Diff(H). The maps π_ξ and Φ_ξ in coordinates are

$$\pi_\xi(s,t) = te_1 + s^{-1}\xi e_2 \qquad (s\in \mathbb{C}^\times,\ t\in \mathbb{C})$$

$$\phi_\xi(e_1) = -sD_s$$

$$\phi_\xi(e_2) = s^{-1}\xi$$

The first map is not finite - the image is the complement of a line (if ξ is not zero), and so is not closed. It is also easy to see that A_ξ is not finitely generated as a $U(\underline{g})$ module. We therefore get neither kind of object needed for Conjecture 1.26. It would of course be interesting to find a general modification of the differential operator construction which **does** work here.

In the language of geometric quantization, Example 2.17 corresponds to a polarization satisfying the Pukanszky condition, at least if ξ is not zero. This corresponds to the fact that the image of π_ξ is a single orbit. One can get even worse behavior by taking P to be A. In this case the polarization does not satisfy the Pukanszky condition. The map π_ξ then hits one point on the line of G-fixed points in $\underline{g}^*$, in addition to the open orbit.

3. Differential operators on flag varieties

For the remainder of these notes, we will assume that

(3.1) G is a connected reductive algebraic group.

For the rest of this section, we fix a parabolic subgroup P of G, and let P_0 be the commutator subgroup of P. Choose a Levi decomposition

(3.2)(a) $$P = MN$$

of P. If we write M_0 for the commutator subgroup of M, then

(3.2)(b) $$P_0 = M_0N$$

Choose a Cartan subgroup H of M, and write $\underline{h}$ for its Lie algebra. Similarly, write Z(M) for the center of M, and $\underline{z}(\underline{m})$ for its Lie algebra. Then

(3.2)(c) $$\underline{h} = (\underline{h} \cap \underline{m}_0) + \underline{z}(\underline{m}).$$

On the other hand, the group T of (2.5) may be identified with the quotient of $Z(M)$ by its (finite) intersection with M_0. Combining this with (3.2)(c) gives identifications

(3.2)(d) $$\underline{t}^* = \underline{z}(\underline{m})^*$$

(3.2)(e) $$\underline{h}^* = \underline{t}^* + (\underline{h} \cap \underline{m}_0)^*$$

Write

(3.3)(a) $$\Delta(\underline{g},\underline{h})$$

for the set of roots of $\underline{h}$ in $\underline{g}$, regarded as a subset of $\underline{h}^*$. Fix a set $\Delta^+(\underline{m},\underline{h})$ of positive roots of $\underline{h}$ in $\underline{m}$. Define

(3.3)(b) $$\Delta^+(\underline{g},\underline{h}) = \Delta^+(\underline{m},\underline{h}) \cup (-\{\text{roots of } \underline{h} \text{ in } \underline{n}\})$$

(3.3)(c) $$\rho_0 = \tfrac{1}{2} \sum_{\alpha \in \Delta^+(\underline{m},\underline{h})} \alpha$$

(3.3)(d) $$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

It follows that

(3.3)(e) $$\rho_0 \in (\underline{h} \cap \underline{m}_0)^*,$$

and

(3.3)(f) $$\rho = \rho_0 + \rho_1$$

(cf. (3.2)(e) and the definition of ρ_1 before (2.10)).

We begin with some standard facts about the maps π_ξ (cf. (2.8)) in this situation. We will often make use of a nondegenerate symmetric invariant bilinear form $\langle , \rangle$ on $\underline{g}$ and related spaces; it is often helpful to assume (as we may) that the roots have positive length in this form.

Lemma 3.4. With the notation (3.1)-(3.3), the constructions of (2.5)-(2.12) have the following properties.

a) The homogeneous space G/P is a simply connected projective variety, so all the maps π_ξ are proper.

b) The image of π_ξ is the closure of a single orbit of G on $\underline{g}^*$, which we call $\underline{O}_\xi$. Over $\underline{O}_\xi$, the map π_ξ is a finite covering.

c) The orbit $\underline{O}_\xi$ is semisimple (or, equivalently, closed) if and only if $\langle\alpha,\xi\rangle$ is not zero for any root α of $\underline{h}$ in $\underline{n}$. In that case, π_ξ is an isomorphism onto its image.

d) $\underline{O}_\xi$ is nilpotent if and only if ξ is zero.

The analogous facts about the differential operator rings A_ξ are slightly more technical, though not really difficult. The experts may find them a little more comprehensible after a short technical digression. Write

(3.5) $$Z(\underline{g}) = \text{center of } U(\underline{g})$$

(3.6) $$W(\underline{g},\underline{h}) = \text{Weyl group of } \underline{h} \text{ in } \underline{g} = W.$$

The Harish-Chandra isomorphism identifies maximal ideals in $Z(\underline{g})$ (<u>infinitesimal characters</u>) with W orbits on $\underline{h}^*$.

Lemma 3.7. **With notation as above and in (2.12), the $U(\underline{g})$-module A_ξ has infinitesimal character $\xi+\rho_\theta$. That is, the kernel of Φ_ξ meets $Z(\underline{g})$ in the maximal ideal corresponding to the weight $\xi+\rho_\theta$.**

Notice that the shift here is not the same as the one introduced before (2.12). It is exactly the infinitesimal character of the augmentation ideal in $U(\underline{m}_\theta)$.

Definition 3.8. Fix a weight λ in $\underline{h}^*$. We say that λ is <u>integral on the root α</u> if

$$2\langle\alpha,\lambda\rangle/\langle\alpha,\alpha\rangle \in Z.$$

We say it is <u>non-negative on α</u> if the same number is not a negative integer. We say λ is <u>dominant</u> if it is non-negative on every positive root, and <u>integral</u> if every root is integral. Finally, we say that λ is <u>regular</u> if it is not orthogonal to any root.

Theorem 3.9 ([2], [5]). **With notation as above, the differential operator ring**

$$A_\xi = \mathrm{Diff}_\xi(G/P)$$

has the following properties.

a) It is finitely generated as a $U(\underline{g})$-module.

b) It is isomorphic as a module for G to the ring of regular functions on the variety X_ξ (cf. (2.7)). More precisely,

$$\mathrm{gr}\, A_\xi \cong \mathrm{gr}\, R(X_\xi).$$

c) Assume that $\xi+\rho_0$ is dominant. Then Φ_ξ is a surjection from $U(\underline{g})$ onto A_ξ.

d) Assume that for every positive root α which is integral for $\xi+\rho_0$, we have

$$\langle\alpha,\xi\rangle \geq 0.$$

Then A_ξ is a primitive algebra.

Part (a) is rather easy. Part(b) relies on the fact, proved by Elkik, that the cotangent bundle of G/P satisfies the condition in Lemma 2.15. Part (c) is fairly subtle, and is one of the keys to the Beilinson-Bernstein localization theory. Part (d) is a consequence of Proposition 8.5 of [23]. (Bernstein has pointed out to me that the same proposition allows one to recover half of the Beilinson-Bernstein theory: for example, (under the hypothesis of (d)), the space of global sections of an irreducible sheaf of $\underline{\mathrm{Diff}}_\xi(G/P)$-modules is an irreducible A_ξ-module (or zero). A special case of this is part of Theorem 8.6 below.) It seems very likely that A_ξ is always primitive, but I do not know how to prove this.

We can now make precise the contribution which these differential operator rings should make to Conjecture 1.26 (always in the setting of (3.1)-(3.3)). One should restrict attention to those ξ satisfying the condition of Theorem 3.9(d). If we take the algebra of (1.26)(a) to be A_ξ (cf. (2.12)), then the variety of (1.26)(b) should be the affinization of X_ξ (cf. (2.7)). This pair satisfies (i)-(iii) of Conjecture 1.26, and (iv)-(v) of Conjecture 2.3. (It should be emphasized again that the proofs of these assertions are due to Borho and Brylinski and others; in any case they are not original.)

There is one case in which this construction should be the whole story.

Theorem 3.10 ([16]). Suppose G is $GL(n)$ or $PGL(n)$. Let (X,π) be a pair satisfying the hypotheses of Conjecture 1.26(b); and assume that $\pi(X)$ is the closure of a single orbit of G. Then π is an isomorphism onto that orbit closure. Furthermore, there is a parabolic subgroup P of G, and a weight ξ in $\underline{t}^*$ (cf. (3.2)(d)) satisfying the hypothesis of Theorem 3.9(d), such that (X,π) is isomorphic in a G-equivariant way to (X_ξ,π_ξ) (cf. (2.7) and (2.8)).

In conjunction with the results of [5], this theorem implies that the algebras A_ξ are always primitive quotients of $U(\underline{g})$ (still for $GL(n)$). Moeglin's result [18] says that they exhaust the completely prime primitive quotients of $U(\underline{g})$. That the correspondence is one to one is verified in [7]. For $GL(n)$ and primitive algebras, Conjecture 1.26 is therefore reduced to the conjecture that any primitive algebra as in Conjecture 1.26(a) must be a quotient of $U(g)$. As example 1.27 shows, the situation is somewhat more complicated for $SL(n)$.

4. Borho's counterexamples in B_2

In this section, we will make explicit some aspects of the general constructions in Sections 2 and 3 in the case of the simple Lie algebra of type B_2. Then, following Borho [3], we will see how this leads to problems in Conjecture 1.22 (but not in Conjecture 1.26). It is convenient to take G to be $Sp(4)$, the group of linear transformations preserving the standard symplectic form on $\mathbb{C}^4$. Fix a Borel subgroup B of G, and a Cartan subgroup H contained in B. We use notation as in (3.3); in particular, Δ^+ refers to the roots of $\underline{h}$ in

$\underline{g}/\underline{b}$. There are exactly two proper parabolic subgroups strictly containing B. Write P^1 for the one with a long root in its Levi factor, and P^2 for the other. We will use all the notation of Section 3, with superscripts to distinguish between the two parabolics as necessary. We may identify $\underline{h}^*$ with $\mathbf{C}^2$ in such a way that the positive roots are

$$(2,0),\ (0,2),\ (1,1),\ (1,-1). \tag{4.1}$$

The Weyl group acts by permuting and changing the signs of the two coordinates. One computes

$$\begin{aligned} \rho &= (2,1) \\ (\rho_0)^1 &= (0,1) \\ (\rho_0)^2 &= (½,-½). \end{aligned} \tag{4.2}$$

We have

$$\begin{aligned} (\underline{t}^1)^* &= \{(\xi,0)\,|\,\xi\in\mathbf{C}\} \\ (\underline{t}^2)^* &= \{(\xi/2,\xi/2)\,|\,\xi\in\mathbf{C}\} \end{aligned} \tag{4.3}$$

We will use these identifications of the two spaces with $\mathbf{C}$. Lemma 3.7 shows that

(4.4) $(A_\xi)^1$ has infinitesimal character $(\xi,1)$

$(A_\xi)^2$ has infinitesimal character $((\xi+1)/2,(\xi-1)/2)$.

Define

$$(I_\xi)^n = \operatorname{Ker}\,(\Phi_\xi)^n \quad (n=1,2), \tag{4.5}$$

a completely prime primitive ideal in $U(\underline{g})$. Recall the orbits $\underline{O}_\xi{}^n$ defined in Lemma 3.4. In Conjecture 1.22, the ideal $(I_\xi)^n$ should correspond to this orbit (or rather to its closure). The difficulty is that this leads to a correspondence which is not well-defined. Everything in sight is unchanged if ξ is replaced by $-\xi$; and there are also the following coincidences:

$$\begin{aligned} (\underline{O}_0)^1 &= (\underline{O}_0)^2 \\ (I_0)^1 &= (I_{½})^2. \end{aligned} \tag{4.6}$$

These are the only coincidences. In the first case, we have therefore associated two different ideals to a single orbit; and in the second, two different orbits correspond to the same ideal. These problems can be cured by fiddling with the ρ shift before (2.12), but not in any reasonable way.

Conjecture 1.26 disposes of these difficulties. There are no coincidences among the algebras $(A_{\xi})^n$ or the corresponding varieties. The first formula of (4.6) is replaced by the existence (after affinization) of a two-fold ramified covering

(4.7)(a) $$(X_0)^1 \to (X_0)^2.$$

The second corresponds to a proper inclusion

(4.7)(b) $$(A_0)^2 \subseteq (A_{\chi})^1.$$

These examples show that $(A_{\chi})^1$ cannot be equal to the image of $U(\mathfrak{g})$ inside it, and that $(X_0)^1$ cannot map isomorphically onto an orbit closure. They are the only examples of these two phenomena (for twisted differential operator algebras) in B_2.

We should mention that this example also shows that the differential operator construction of section 2 is not adequate to give all completely prime primitive ideals. Of course one must also use the parabolic subgroups B and G; but even after these are included, one finds that one orbit of G on $\mathfrak{g}^*$ (the minimal nilpotent one, which is four dimensional), and one completely prime primitive ideal (the Joseph ideal) are missing. Of course they should correspond in the Dixmier conjecture; but the theory lacks a general method like that of section 2 to implement this correspondence.

5. Joseph's counterexamples in G_2.

In this section, we take G to be the complex group of type G_2, and recall Joseph's counterexample to Conjecture 1.22 from [14]. The point of it is that, unlike Borho's counterexamples described in

Section 4, it does not require any precise guess about what the correspondence between orbits and ideals ought to be. (Recall that Borho's counterexamples could be repaired by an (apparently unnatural) change in the ρ shift in the definition of the correspondence.)

Here is an outline of the argument. It is known from the work of Dynkin [10] that there is exactly one orbit of G on $\underline{g}^*$ of dimension 8. Joseph shows that there are exactly two completely prime primitive ideals of that Gelfand-Kirillov dimension. This means that there cannot be any dimension preserving bijection between completely prime primitive ideals and orbits on $\underline{g}^*$ in this case. The calculation begins with the following result, which is well known.

Lemma 5.1. Suppose G is a connected reductive algebraic group with Lie algebra $\underline{g}$. Use notation as in (3.3). Fix a regular weight λ in $\underline{h}^*$. Set

$R(\lambda)$ = set of integral roots for λ (cf. Definition 3.8)

$W(\lambda)$ = Weyl group of $R(\lambda)$

Let I be the unique maximal ideal in $U(\underline{g})$ of infinitesimal character λ (cf. Lemma 3.7).

a) The Gelfand-Kirillov dimension of $U(\underline{g})/I$ is equal to

$$|\Delta(\underline{g},\underline{h})| - |R(\lambda)|;$$

here we use vertical bars to denote the cardinality of a set.

b) Suppose F is an irreducible finite dimensional representation of G. Write $m(\xi,F)$ for the multiplicity of a weight ξ in F. Then the multiplicity of F in $U(\underline{g})/I$ is equal to

$$\sum_{w\in W(\lambda)} \det(w)m(w\lambda-\lambda,F).$$

For G of type G_2, we can identify $\underline{h}^*$ with

$$\{(x,y,z)\in\mathbb{C}^3 \mid x+y+z = 0\}$$

The Weyl group is generated by permutations of the coordinates and (scalar multiplication by) -1; it has order 12. The roots are

(5.2) $\alpha = (2,-1,-1)$, $\beta = (0,1,-1)$

and their various conjugates under the Weyl group; thus there are 12 roots in all. A weight is integral exactly when all its coordinates are integers.

Define

(5.3)(a) $\lambda_1 = (1,1/2,-3/2)$, $\lambda_2 = (2,-1/2,-3/2)$.

Because these differ by an integral weight, they have the same system R of integral roots; and in fact one calculates easily that

(5.3)(b) $R = \{\pm\alpha, \pm\beta\}$,

a system of type $A_1 \times A_1$. Define

(5.3)(c) I_j = maximal ideal in $U(\underline{g})$ of infinitesimal character λ_j.

Since λ_j is regular, Lemma 5.1(a) and (5.3)(b) show that I_j has Gelfand-Kirillov dimension 8. One of the main results of [14] is

Proposition 5.4. The primitive ideals I_j (j=1,2) defined above are prime. They are the only completely prime primitive ideals in the enveloping algebra of G_2 having Gelfand Kirillov dimension 8.

It is a straightforward consequence of Joseph's work on Goldie ranks that either this proposition holds, or there are <u>no</u> completely prime primitive ideals of Gelfand-Kirillov dimension 8. The point of [14] is therefore to construct such an ideal, more or less by hand.

Write $\underline{O}$ for the eight dimensional G orbit in $\underline{g}^*$, and V for its closure. Joseph goes on to claim in [14] that

(5.5) $\mathrm{gr}\, U(\underline{g})/I_j = R(V)$,

the ring of regular functions on V. This is the same as the claim that $\mathrm{gr}\, I_j$ is a prime ideal in $S(\underline{g})$. He offers no proof of this claim; and it contradicts Conjecture 1.26. Fortunately, (5.5) is false. To see this, we make explicit the formula in Lemma 5.1(b) for these cases. Fix an irreducible finite dimensional representation F of G. Then its multiplicity in $U(\underline{g})/I_1$ is

$$m(0,F) - m(\alpha,F) - m(2\beta,F) + m(\alpha+2\beta,F);$$

and in $U(\underline{g})/I_2$,

$$m(0,F) - m(2\alpha,F) - m(\beta,F) + m(2\alpha+\beta,F).$$

The only weights of the seven dimensional representation F_7 of G_2 are 0 and the W conjugates of β, all occurring once. It follows that F_7 occurs once in $U(\underline{g})/I_1$, but not at all in $U(\underline{g})/I_2$. This contradicts (5.5).

On the other hand, Joseph's analysis does show that the two graded algebras gr $U(\underline{g})/I_j$ have the same multiplicity (leading term of the Hilbert polynomial). Conjecture 1.26 therefore forces

Conjecture 5.6. The ideal gr I_2 in $S(\underline{g})$ is completely prime; it is the ideal of the eight-dimensional variety V defined above. The ring of functions on the orbit $\underline{O}$ itself - equivalently, the normalization of $R(V)$ - is isomorphic as a G-module to gr $U(\underline{g})/I_1$.

Because of the multiplicity formulas given above, these are purely algebro-geometric questions. In particular, the conjecture asserts that V is not normal.

There is another interesting phenomenon which first appears in G_2: in the setting of (say) Theorem 3.9(d), it can happen that the geometric map Φ_ξ is an isomorphism onto a semisimple orbit while π_ξ is not surjective. This is implicit in [1], and represents joint work with Dan Barbasch. Consider

(5.7)(a) $\quad I$ = maximal ideal in $U(\underline{g})$ of infinitesimal character β

(5.7)(b) $\quad A = U(\underline{g})/I$.

The results of Lemma 5.1 do not apply here, since β is not regular. However, appropriate generalizations may be found in [1]. In particular, one can compute that the multiplicity of a typical irreducible finite dimensional representation F of G in A_0 is

$$m(0,F) - m(\beta,F). \qquad (5.8)$$

We now place ourselves in the setting of section 3, making P a parabolic subgroup with the roots

$$\pm(-1,2,-1)$$

in $\Delta(\underline{m},\underline{h})$. We take

$$\xi = (1/2,0,-1/2);$$

there is now a unique choice of P satisfying the condition in Theorem 3.9(d). By Lemma 3.7, the differential operator ring A_ξ has infinitesimal character

$$\xi + \rho_0 = (1/2,0,-1/2) + (-1/2,1,-1/2)$$
$$= (0,1,-1) = \beta.$$

Because A_ξ and A both have Gelfand-Kirillov dimension 10, it follows that I is the kernel of Φ_ξ. That is, there is an inclusion

(5.9) $$A \subseteq A_\xi.$$

This is the map Φ_ξ. Theorem 3.9(b) calculates the multiplicity of any irreducible F in A_ξ. It is

(5.10) $$m(0,F) - m(\alpha,F).$$

This differs from (5.8) (for example on the seven dimensional representation), so the inclusion in (5.9) is strict. On the other hand, Lemma 3.4 shows that π_ξ is an isomorphism onto a single semisimple orbit.

This same example (still joint work with Barbasch) contradicts an aspect of the philosophy of coadjoint orbits (cf. section 7). There is a one dimensional unitary character χ of P, such that the unitarily induced representation

$$\pi = \text{Ind}\ \chi$$

of G has infinitesimal character (β,β). (A unitary representation of a complex Lie group comes equipped with two infinitesimal characters.) The method of coadjoint orbits would associate this representation to the orbit $\underline{O}_\xi$, and predict that it should be irreducible. In fact π

splits into two irreducible pieces. However, π itself (or at least its Harish-Chandra module) actually admits a natural structure of irreducible module for the algebra $A_\xi \otimes A_\xi$. In this picture, the enveloping algebra acts as the subalgebra $A \otimes A$. We will pursue the ideas suggested by this example in section 8.

6. Sp(8) and the role of non-normality

Proposition 5.4 and Conjecture 5.6 suggest that the non-normality of orbit closures can influence primitive ideal theory, as Conjecture 1.26 implies it must. Since I have not checked whether the orbit in G_2 is actually non-normal, however, this example is hardly conclusive. In this section we will investigate an orbit whose closure **is** known to be non-normal, and look for effects on primitive ideal theory. We take G to be Sp(8), the group of linear transformations preserving the standard symplectic form on $\mathbf{C}^8$; it is simple of type C_4. Fix a Borel subgroup B of G, and a Cartan subgroup H contained in B. We use notation as in (3.3). We may identify $\underline{h}^*$ with $\mathbf{C}^4$ in such a way that the positive roots are

(6.1) $$e_i \pm e_j,\ 2e_i \qquad (i<j).$$

The Weyl group acts by permuting and changing the signs of the coordinates. Define P to be the parabolic subgroup containing B, containing in its Levi factor the simple roots

(6.2) $$e_1 - e_2,\ e_3 - e_4,\ 2e_4.$$

Adopt the notation of section 3; then we compute

(6.3) $$\begin{aligned} \rho &= (4,3,2,1) \\ \rho_0 &= (\tfrac{1}{2},-\tfrac{1}{2},2,1) \\ \underline{t}^* &= \{(\xi/2,\xi/2,0,0) \mid \xi\in \mathbf{C}\} \end{aligned}$$

Proposition 6.4. The map

$$\Phi_\xi : U(\underline{g}) \to A_\xi$$

is surjective except when $\xi=\pm 1$. For those values, the image is a proper subalgebra of full multiplicity in A_ξ; that is, the $U(\underline{g})$ module

$$A_\xi/\mathrm{im}\ \Phi_\xi$$

has strictly smaller Gelfand-Kirillov dimension than A_ξ.

We omit the proof, but hints will be given when we discuss the representation-theoretic aspects of this example in section 7.

One interesting consequence of Proposition 6.4 is that A_ξ and im Φ_ξ have the same faithful simple modules. This follows from

Proposition 6.5. **Suppose R_1 and R_2 are primitive rings. Assume that the left and right annihilators of R_2/R_1 in R_1 are not zero. Then any faithful simple R_2 module is also simple as an R_1 module; and conversely, any faithful simple R_1 module has a unique R_2 module structure.**

Proof. Suppose M is a faithful simple R_1 module. Write J for the right annihilator of R_2/R_1 in R_1. Then JM is equal to M. It follows easily that $(R_2/R_1)\ \ M$ is zero. (All tensor products will be over R_1.) By left exactness, we therefore have a surjection

$$R_1\ \ M \twoheadrightarrow R_2\ \ M.$$

The first term here is M itself. Write N for the second; it is an R_2 module. Since M is simple, N is either isomorphic to M or zero. Now the Kernel K of the map from M to N is a quotient of $\mathrm{Tor}^1(R_2/R_1,M)$. As an R_1 module, K is annihilated by the left annihilator of R_2/R_1, which is not zero. Since M is faithful, K is not equal to M. This proves the second part of the proposition. The first, which is easier, is left to the reader. Q.E.D.

Recall now from (2.7) and (2.8) the geometric objects corresponding to the algebras A_ξ.

Proposition 6.6 ([17]). In the setting of (6.1) to (6.3), the map π_ξ is an isomorphism onto its image except when $\xi=0$. In that case, its image V_0 is a non-normal variety (the closure of the nilpotent orbit $\underline{O}_0$); and the affinization of π_0 is the normalization map for V_0.

Proof. Since the variety X_ξ is smooth, it and its affinization are normal. It is shown in [17] that V_0 is not normal. The proposition will follow from this fact and Lemma 3.4, once we know that π_0 is birational. This follows from the fact that $\underline{O}_0$ is simply connected; and this in turn is a consequence of the connectedness of the stabilizer of an element of it, which can be computed directly. Q.E.D.

Because of Propositions 6.4 and 6.6, it is clear how to extend the suggestions before Theorem 3.10 towards Conjecture 1.26 in this case. The primitive quotient im Φ_1 of $U(\underline{g})$ should correspond to the subvariety V_0 of $\underline{g}^*$. It is not a trivial matter to compute either $R(V_0)$ or im Φ_1 as a G module, so property (iii) of the conjecture is already hard to verify.

7. Representation theory: complex groups

In this section and the next, we consider the relationship between ideal theory and representation theory.

Definition 7.1. Suppose H is a topological group. A unitary representation of H is a pair (π, V), with V a complex Hilbert space and π a homomorphism of H into the group of invertible linear transformations of V, subject to the following conditions:

a) The map

$$H \times V \to V, \quad (h,v) \to \pi(h)v$$

is continuous.

b) For every h in H, the operator $\pi(h)$ is unitary. That is,

$$\langle \pi(h)v, \pi(h)w \rangle = \langle v,w \rangle$$

for all v and w in V.

We say that the representation is irreducible if V is not zero, and no proper closed subspace of V is invariant under all the operators $\pi(h)$.

The following result is included to indicate the kind of connection one would like between group representations and Lie algebra representations.

Proposition 7.2. Suppose H is a connected, simply connected real Lie group. Then there is a bijection (defined by differentiation) between finite dimensional unitary representations of H, and finite dimensional skew-adjoint representations of Lie(H). (By the latter, we understand Lie algebra homomorphisms π of Lie(H) into the operators on a finite dimensional Hilbert space V, satisfying

$$\langle \pi(X)v,w\rangle = -\langle v,\pi(X)w\rangle$$

for all X in Lie(H) and v,w in V.) This bijection preserves irreducibility.

Corollary 7.3. In the setting of Proposition 7.2, write $\underline{h}$ for the complexified Lie algebra of H. Then the annihilator of an irreducible unitary representation of H is a well-defined primitive ideal in $U(\underline{h})$.

There are several obstacles to extending these results to infinite dimensional representations. The differentiation referred to in the first is

$$\pi(X)v = \lim_{t\to 0} (\pi(\exp(tX))v-v)/t. \tag{7.4}$$

This limit exists only as for v in a dense subspace of V, which leads to serious technical problems when one tries to recover a group representation from a Lie algebra representation. Secondly, the natural common domain V^{∞} for all the Lie algebra operators $\pi(X)$ is not an irreducible Lie algebra representation in the algebraic sense. Its annihilator in $U(\underline{h})$ is therefore not obviously a primitive ideal.

Dixmier has dealt with the second problem in general (cf. [9]), but for our purposes the following result of Harish-Chandra will suffice.

Theorem 7.5 ([12]). Suppose G_1 is a real connected reductive Lie group. Write $\underline{g}_1$ for Lie(G) and $\underline{g}$ for its complexification. Let (π,V) be an irreducible unitary representation of G_1. Then there is a dense subspace V_0 of V (contained in V^∞), with the following properties:

a) For v in V_0 and X in $\underline{g}_1$, the limit (7.4) exists and belongs to V_0.

b) The representation (π,V_0) of $\underline{g}_1$ is algebraically irreducible.

c) For X in $\underline{g}_1$, the operator $\pi(X)$ is skew-adjoint with respect to the pre-Hilbert space structure on V_0.

Harish-Chandra describes precisely how to find V_0; it is unique up to the choice of a maximal compact subgroup of G. He goes on to characterize precisely which Lie algebra representations can arise in this way, giving a dictionary exactly analogous to the one in Proposition 7.2 between group representations and Lie algebra representations. This will not concern us directly, however.

Definition 7.6. Suppose (π,V) is an irreducible unitary representation of the connected reductive real Lie group G_1. The <u>annihilator of π</u>, $\mathrm{Ann}(\pi)$, is the annihilator in $U(\underline{g})$ of the irreducible Lie algebra representation (π,V_0) given by Theorem 7.5.

It is an easy consequence of Harish-Chandra's results that $\mathrm{Ann}(\pi)$ is well-defined, even though V_0 is not unique.

Definition 7.7. Suppose $\underline{h}_1$ is a real Lie algebra, and $\underline{h}$ is its complexification. The <u>star anti-automorphism</u> of $U(\underline{h})$ is the anti-linear antiautomorphism characterized by

$$(X + iY)^* = -X + iY \quad (X, Y \in \underline{h}_1).$$

The point of this definition is the formula

(7.8) $$\langle \pi(u)v, w\rangle = \langle v, \pi(u^*)w\rangle,$$

valid for example for v and w in V_0 and u in $U(\underline{g})$, in the context of Theorem 7.5. An immediate consequence is

Corollary 7.9. Suppose π is an irreducible unitary representation of the connected reductive Lie group G_1. Then the primitive ideal $\mathrm{Ann}(\pi)$ is invariant under the star anti-automorphism of $U(\underline{g})$ (Definition 7.7).

Lemma 7.10. Suppose H is a complex Lie group. Write $\underline{h}$ for the Lie algebra of H (a complex Lie algebra), and $\underline{h}_1$ for the underlying real Lie algebra (obtained by forgetting the scalar multiplication by i). Write $\underline{h}_C$ for the complexification of $\underline{h}_1$. Then there is a natural isomorphism

$$\underline{h}_C = \underline{h}_L + \underline{h}_R,$$

a direct sum of Lie algebras. The first summand is isomorphic to $\underline{h}$, and the second to the complex conjugate algebra. The star anti-automorphism (Definition 7.7) interchanges these two summands.

Proof. Write j for the operation of multiplication by i on $\underline{h}$; this is a real linear transformation of $\underline{h}_1$, and so defines a complex linear transformation (still called j) of $\underline{h}_C$. The two summands are the plus and minus i eigenspaces of j:

$$\underline{h}_L = \{\tfrac{1}{2}(X - ijX) \mid X \in \underline{h}\}$$

$$\underline{h}_R = \{\tfrac{1}{2}(X + ijX) \mid X \in \underline{h}\}$$

These formulas are written to exhibit the required isomorphisms of the summands with $\underline{h}$ and its complex conjugate. The assertions of the lemma are all now easy to verify. Q.E.D.

Proposition 7.11. Suppose G is a complex reductive algebraic group. With notation as in Lemma 7.10, any primitive ideal I in $U(\mathfrak{g}_C)$ is of the form

$$I = I_L \otimes U(\mathfrak{g}_R) + U(\mathfrak{g}_L) \otimes I_R,$$

with I_L and I_R primitive ideals in $U(\mathfrak{g}_L)$ and $U(\mathfrak{g}_R)$ respectively. Consequently, the primitive quotient is of the form

$$U(\mathfrak{g}_C)/I = (U(\mathfrak{g}_L)/I_L) \otimes (U(\mathfrak{g}_R)/I_R).$$

Proof. By Duflo's theorem, I is the annihilator of an irreducible highest weight module V for $\mathfrak{g}_C$. It is an elementary exercise to show that such a module must be of the form $V_L \otimes V_R$, with the factors irreducible highest weight modules for $\mathfrak{g}_L$ and $\mathfrak{g}_R$, respectively. The proposition follows. Q.E.D.

The result is presumably true for any complex Lie algebra; one should use the results of Moeglin and Rentschler on primitive ideals in general Lie algebras to replace the application of Duflo's theorem.

Proposition 7.12. Suppose G is a complex reductive algebraic group, and π is an irreducible unitary representation of G. Write $I = \mathrm{Ann}(\pi)$ (Definition 7.6), and use the notation of Proposition 7.11. Then I_L and I_R are completely prime primitive ideals; and

$$I_R = (I_L)^*$$

(Definition 7.7).

Proof. By Proposition 7.11, the ideals in question are primitive, and the last assertion follows from (7.8). So we only have to show that there are no zero divisors in $U(\mathfrak{g}_L)/I_L$. This is simplified by the following observation, for which I am grateful to I. Kaplansky.

Lemma 7.13. Let R be a prime ring without non-zero nilpotents. Then R has no zero divisors; that is, R is completely prime.

We postpone the proof for a moment, and complete the argument for Proposition 7.12. Let u be an element of $U(\mathfrak{g}_L)$ not belonging to I_L; we need only show that u^2 does not belong to I_L. Let V_0 be as in Theorem

7.5. By assumption, u does not annihilate V_0; so there is an element v of V_0 such that $u.v$ is not zero. (We have dropped the π used earlier, returning to module notation.) Since V_0 is a pre-Hilbert space, it follows that

$$\langle u.v, u.v\rangle > 0.$$

By (7.8), this amounts to

$$\langle (u^*u).v, v\rangle > 0.$$

It follows that $(u^*u).v$ is not zero. Repeating the argument, we find

$$(u^*u)(u^*u).v = 0.$$

By the last assertion of Lemma 7.10, u^* belongs to $U(\underline{g}_R)$. Consequently it commutes with u, and the formula above may be rewritten as

$$(u^*)^2(u)^2.v = 0.$$

In particular, u^2 does not annihilate V_0; so it does not belong to I_L, as we wished to show. Q.E.D.

Proof of Lemma 7.13. Suppose a and b are non-zero elements of R. We want to show that ab is not zero. Because R is prime, we can find an element x of R such that bxa is not zero (cf. Definition 1.14(a).) Because R is assumed to contain no nilpotents, it follows that

$$(bxa)^2 = (bx)(ab)(xa)$$

is non-zero; so ab is non-zero. Q.E.D.

Proposition 7.12 provides a very simple direct correspondence from irreducible unitary representations to completely prime primitive ideals, in the case of complex reductive algebraic groups. (For general complex groups, only the generalization of Lemma 7.11 is lacking, and this should not be serious.) The correspondence is not surjective (that is, certain completely prime primitive ideals do not arise in this way), but only for rather dull reasons: in the setting of section 3, one gets ideals parametrized roughly by all elements of $\underline{t}^*$, but unitary representations parametrized only by those whose imaginary parts are differentials of unitary characters. This is the difference between $\mathbf{C}^n$ and $\mathbf{R}^n \times \mathbf{Z}^n$.

There is another way to view these ideas. A very crude statement of the Kirillov-Kostant "philosophy of coadjoint orbits" is

Conjecture 7.14. Suppose G is a complex reductive algebraic group. Then there is a natural one-to-finite correspondence from the set of integral orbits of G on $\underline{g}^*$, to the set of finite length unitary representations of G.

(An element λ of $\underline{g}^*$ is said to be <u>integral</u> if $i(\mathrm{Im}\ \lambda)$ is the differential of a unitary character of the identity component of the stabilizer $G(\lambda)$.)

The Kirillov-Kostant philosophy is actually much more general and (in favorable circumstances) more specific than Conjecture 7.14 would indicate. Our purpose here, however, is not to rehearse the great successes of this philosophy, but to indicate how Conjecture 1.26 may suggest ways to sharpen it. A reasonable treatment of that question is beyond the scope of these notes; but here is a first hint. If A is an algebra over $\mathbf{C}$, we write A^* for the algebra with the same underlying structure of algebra over $\mathbf{R}$, but with the operation of scalar multiplication by i replaced by its negative.

Conjecture 7.15. Suppose G is a complex reductive algebraic group. Let $\underline{O}$ be an integral orbit of G on $\underline{g}^*$, and let X be one of the ramified covers of the closure of $\underline{O}$ considered in Conjecture 1.26(2). Let A be the corresponding algebra (Conjecture 1.26(1)). To X and some additional data there corresponds a unitary representation (π, V) of G, of finite length. The Harish-Chandra module V_0 for $U(\underline{g}) \otimes U(\underline{g})^*$ (Theorem 7.5 and Lemma 7.10) is endowed also with the structure of a faithful $A \otimes A^*$ module, compatible with the $\underline{g}_C$ action and the algebra homomorphism Φ of Conjecture

1.26(1). As an $A \otimes A^{*}$ module, V_{0} is irreducible.

There are a number of things to notice here. First, it should be emphasized that one cannot expect to get all irreducible unitary representations of G from Conjecture 7.15: this fails even for $GL(2,\mathbb{C})$ (where the complementary series representations are not accounted for). Still, one should get enough unitary representations to solve most interesting harmonic analysis problems. (Of course "interesting" here has no precise meaning, but it does not seem inconceivable that such a meaning could be found.) In any case, this is a familiar problem for the orbit method, and needs no further comment here.

Second, the orbit method as it is usually formulated asks for an **irreducible** unitary representation attached to $\underline{O}$. As the example at the end of section 5 shows, this is not always an easily attainable goal. Conjecture 7.15 tries to indicate why this is the case: the representation is naturally irreducible only for some larger ring than $U(\underline{g}_{\mathbb{C}})$. A good way to study irreducibility questions is to prove irreducibility under the bigger algebra, then study the relationship between the bigger algebra and the image of $U(\underline{g}_{\mathbb{C}})$ inside it. An example is given in section 8.

Finally, it may be that Conjecture 7.15 will contribute more to primitive ideal theory than to unitary representation theory, for the following reason. Suppose that, starting from $\underline{O}$ and the other data, one manages to construct a unitary representation. Then it is reasonable to hope that the algebra $A \otimes A^{*}$ should be more or less visible in the construction. One could then prove A to be completely prime by the argument of Propositititon 7.12. This would be a start in the direction of Conjecture 1.26. In fact the partial results of section 3 arose in more or less this way: the study of the differential operator rings on G/P has close ties -- at least historical ones --

with the theory of unitary representations induced from P to G (cf. section 8). From this perspective, the paper [21] (which discusses attaching a unitary representation to the minimal coadjoint orbit) could be regarded as an alternate approach to the Joseph ideal ([13]).

8. Representation theory: real groups

The philosophy of orbits for real groups is much more subtle, and I do not know how to formulate even a conjecture as precise as Conjecture 7.15. A few things are clear, however. (See for example the chapter on geometric quantization in [11] for more details on the general philosophy.) Let G_1 be a connected real reductive Lie group, and $\underline{g}_1$ its Lie algebra. Fix an orbit $\underline{O}_1$ of G_1 on $(\underline{g}_1)^*$, and a point λ of $\underline{O}_1$. Write $G(\lambda)_1$ for the identity component of the stabilizer of λ. This group preserves a symplectic form ω on

$$E = \underline{g}_1/\underline{g}(\lambda)_1,$$

giving rise to a homomorphism of $G(\lambda)_0$ into the symplectic group $Sp(E,\omega)$. The metaplectic double cover of $Sp(E)$ now induces a central extension

$$1 \to \{1,\epsilon\} \to G^{\sim}(\lambda)_1 \to G(\lambda)_1 \to 1.$$

We say that λ (or $\underline{O}_1$) is <u>admissible</u> if $i\lambda$ is the differential of a character Λ of $G^{\sim}(\lambda)_1$, such that

$$\Lambda(\epsilon) = -1.$$

For complex groups, the metaplectic double cover splits, so admissible is the same as integral (defined after Conjecture 7.14).

Here is the analogue of Conjecture 7.14.

Conjecture 8.1 (Kirillov-Kostant). Suppose G_1 is a real reductive Lie group. Then there is a natural one-to-finite correspondence from the set of admissible orbits of G_1 on $(\underline{g}_1)^*$, to the set of finite length unitary representations of G_1.

The main weakness of this formulation is that one should probably start with some sort of union of several orbits, all with the same complexification, having a connected closure. In $SL(2,\mathbb{R})$, for example, the union of the two nilpotent half cones should correspond to the spherical principal series representation with parameter zero (which is irreducible). I do not know a precise formulation which is reasonable. This interferes with generalizing Conjecture 7.15; but one can say something. First, it is helpful to define the analogue of the star anti-automorphism of $U(\underline{g})$ (Definition 7.7) on the symmetric algebra. For p in $S(\underline{g})$ and f in $\underline{g}^*$, we can define

$$p^*(f) = \sigma(p(-\sigma f)).$$

Here we write σ for complex conjugation. (Alternatively, the definition given for $U(\underline{g})$ works equally well here.) Notice that star is anti-linear; but it is actually an automorphism (and not just an anti-automorphism) since $S(\underline{g})$ is commutative. Finally, notice that if Z is any subset of $(\underline{g}_1)^*$, then the ideal defined by iZ is stable under star.

Conjecture 8.2. Suppose G_1 is a real reductive Lie group, $\underline{g}_1$ its Lie algebra, and $\underline{g}$ the complexification of $\underline{g}_1$. Let G be a complex reductive algebraic group with Lie algebra $\underline{g}$, such that the inclusion of Lie algebras exponentiates to a group homomorphism of G_1 into G. Fix an admissible coadjoint orbit $\underline{O}_1$ of G_1, and write $\underline{O}$ for the complexification of $i\underline{O}_1$. Let X be one of the ramified covers of the closure of $\underline{O}$ considered in Conjecture 1.26(2). Assume that $R(X)$ is endowed with an anti-linear (anti-)automorphism star, which is compatible (via the map from X to $\underline{g}^*$) with star on $S(\underline{g})$. Under these assumptions, the corresponding algebra A is (that is, is conjectured to be) endowed with an anti-linear anti-automorphism "star", consistent with the one on $U(\underline{g})$ defined in Definition 7.7.

To all this and some additional data, there corresponds a finite length unitary representation (π, V) of G_1. The Harish-Chandra module V_0 of V (Theorem 7.5) is endowed with the structure of an A module, extending the $U(\underline{g})$ action and satisfying (7.8) for all u in A. Under some mild geometric hypotheses on the other data, V_0 should be irreducible or zero as an A module.

All of the comments made after Conjecture 7.15 apply here as well. Rather than expand on them, we turn to a setting where the conjecture is as nice as possible.

We continue to assume that G_1 is a connected real reductive Lie group, and that G is as in Conjecture 8.2. Let $\underline{p}$ be a parabolic subalgebra of $\underline{g}$. Assume that

$$\text{the intersection } \underline{m} \text{ of } \underline{p} \text{ with its complex conjugate } \underline{p}^- \text{ is a Levi factor of } \underline{p}. \tag{8.3}$$

Now use the notation of section 3. The Lie algebras $\underline{m}$ and $\underline{t}$ are defined over R; we use a subscript 1 to indicate their intersections with $\underline{g}_1$.

Lemma 8.4. With notation as above, the normalizer of $\underline{p}$ in G_1 is a connected subgroup M_1, with Lie algebra $\underline{m}_1$. Fix a weight ξ in $\underline{t}^*$. Extend ξ to a linear functional on all of $\underline{g}$ by making it zero on the commutator subalgebra $\underline{m}_0$ of $\underline{m}$ (cf. (3.2)), and on the nil radicals $\underline{n}$ and $\underline{n}^-$ of $\underline{p}$ and $\underline{p}^-$. Write λ for the restriction to $\underline{g}_1$ of $-i\xi$.

a) The stabilizer of λ in G_1 contains M_1; they are equal if and only if $\langle\alpha,\xi\rangle$ is not zero for any root α of $\underline{h}$ in $\underline{g}$ (cf. Lemma 3.4).

b) The linear functional λ is real-valued if and only if ξ takes purely imaginary values on $\underline{t}_1$.

c) Assume that the conditions in (a) and (b) are fulfilled.

Then λ is admissible (defined before Conjecture 8.1) if and only if $i\lambda-\rho_1$ (cf.3.3) is the differential of a unitary character $\Lambda^{\sim}$ of M_1.

This is straightforward. Part(c) is (at least) a folk theorem.

Definition 8.5 In the setting (8.3), the weight ξ in $\underline{t}^*$ is called <u>admissible</u> if $\xi-\rho_1$ is the (restriction to $\underline{t}$ of the) differential of a unitary character $\Lambda^{\sim}(\xi)$ of M_1.

Lemma 8.4(c) shows that this notion of admissibility is closely related to the one defined at the beginning of this section.

Theorem 8.6. **Suppose we are in the setting (8.3), and ξ is an admissible weight in $\underline{t}^*$ (Definition 8.5). Assume that ξ satisfies the positivity condition**

$$\mathrm{Re}\ \langle\alpha,\xi\rangle \geq 0$$

for every root α of $\underline{t}$ in $\underline{p}^-$. Then there is a natural unitary representation $V(\xi)$ attached to $\underline{p}$ and ξ. Its Harish-Chandra module carries a natural action of the differential operator algebra A_ξ (cf. Theorem 3.9), satisfying the requirements of Conjecture 8.2.

Sketch of proof. The unitary representation is the one whose Harish-Chandra module was denoted $A_{\underline{p}^-}(\Lambda^{\sim})$ in [24]; it is obtained by applying a "cohomological parabolic induction functor" (cf. [22]) to $\Lambda^{\sim}$. Its unitarity is established in [23]. That same paper essentially proves that $V(\xi)$ is obtained by a translation functor from a corresponding representation $V(\xi')$, with ξ' very regular. It is known that $V(\xi')$ is irreducible under $A_{\xi'}$ (which is just the image of $U(\underline{g})$ by Theorem 3.9(c)). Finally, it is easy to show that the differential operator algebras behave well under the translation principle. Putting all these things together, we find by a formal argument that A_ξ acts irreducibly on the Harish-Chandra module of $V(\xi)$. Q.E.D.

Although he might wish to disown this incarnation of it, I am grateful to Joseph Bernstein for showing me the preceding argument.

We can now explain part of the proof of Proposition 6.4. The most difficult part is to show that (say) Φ_1 is not surjective. Of course it suffices to exhibit a simple module for A_1 which is not simple as a $U(\underline{g})$ module. We use the preceding theorem, with G_1 the split real form of G (the symplectic group in eight real dimensions), and a well-chosen $\underline{p}$. The fact that V(1) is reducible is in principle computable from the Kazhdan-Lusztig conjectures; in fact the much simpler tricks from [20] suffice. What is interesting is that this module has smaller Gelfand-Kirillov dimension than one might first guess -- that is, than half the dimension of the orbit. Because of Proposition 6.5, this is a necessary feature of an example: any faithful simple A_1 module is also simple as a $U(\underline{g})$ module.

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