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Representations of Real Reductive Lie Groups

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To Jonathan, for eating the first draft.

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Preface

Since this manuscript was completed in August, 1980, a great deal of progress has been made on some of the topics treated. Working independently, Brylinski-Kashiwara and Beilinson-Bernstein have established Conjecture 2.1.7 on the composition series of Verma modules. Lusztig and I have used the work of Beilinson and Bernstein to prove Conjectures 2.2.12 and 2.3.11 on Harish-Chandra modules, computing all composition series in the case of integral infinitesimal character. [This settles Problems 9 and 10 of Section 9.7.] The non-integral case remains open. The proofs rely on the Weil conjectures and the Deligne-Goresky-MacPherson intersection homology theory. It is not yet clear how these developments will affect the material presented in this book -- they do not provide any obvious substantial simplifications, although in some cases they suggest significant improvements in the formulation of the theory.

I would like to thank Janet Ellis for typing and proof-reading a long and messy manuscript with great care and skill in the midst of many other responsibilities. I have the dubious satisfaction of knowing that no errors remain which are not of my own devising.

Cambridge, Massachusetts

April, 1981

Introduction

This book is a survey of some recent work on the (non-unitary) infinite dimensional representations of a real reductive Lie group G. There are three major topics. The first is Langlands' theorem on the classification and realization of the irreducible representations of G (Theorem 6.5.12). They are described in terms of certain "standard representations" (Definition 6.5.2), which generalize the principal series and are sometimes reducible. The second topic is the reducibility of these standard representations. The main results (Theorem 8.6.6 and Proposition 8.7.6) are due to B. Speh and the author; they are not quite decisive. The third topic is a conjecture which describes explicitly the decomposition of standard representations into irreducible representations -- or, equivalently, the Harish-Chandra characters of the irreducible representations. This generalizes a conjecture of Kazhdan and Lusztig for Verma modules, and is described in Section 9.6.

Since the theory of non-unitary representations of G was created by Harish-Chandra essentially as a means to study unitary representations, some apology might seem to be required for a book in which unitary representations play almost no part. The first explanation for this omission is simply a lack of space. The first two topics at least are very important for recent work on unitary representations. For example, the study of unitary representations with non-zero continuous cohomology (see [3]) has been advanced by the algebraic study of certain representations which are still not known to be unitary. (They are included in the conjecturally unitary representations of (6.5.17) below.) The theory of "complementary series" of unitary representations depends on (among other things) an understanding of the reducibility of standard representations. Thus Theorem 8.6.6 provides a large number of unitary representations, and a proof of the conjecture of Section 9.6 would provide even more.

The real explanation, however, is that non-unitary representation theory is interesting enough not to require any such justification. Such a claim has to be supported in the text and not in the introduction; but Chapter 2 attempts to describe the nature of the main results without too much technical clutter.

In a little more detail, the book is organized as follows. The reader is assumed to be quite familiar with the structure and finite dimensional representation theory of complex reductive Lie algebras; this is really a prerequisite even for understanding the statements of most of the results. Logically, the book also depends on Harish-Chandra's basic theory relating group representations and Lie algebra representations ([12]) and on his subquotient theorem ([13]). These topics are treated in [49] or [50], and the results are summarized in Sections 0.3 and 4.1 of this book. Since the ideas needed to prove them will not be used here, the reader who is willing to take them on faith should have little difficulty. The first three sections of Chapter 1 summarize, with some proofs, the representation theory of $SL(2, \mathbb{R})$. This is intended to provide examples as guides to the rather abstract and technical treatment of the general case. Some results are proved in general by reduction to $SL(2, \mathbb{R})$, and the necessary special cases of these are discussed in the rest of Chapter 1.

Chapter 2 contains a detailed statement of the Langlands classification of irreducible representations of G in a special case; and a geometric formulation of the conjecture of Section 9.6 on composition series of standard representations. (The two formulations are not known to be equivalent.) The entire chapter is meant as an extended introduction to the rest of the book.

The main technical tool used here to study representations is Lie algebra cohomology (of the nil radical of a parabolic subalgebra, with coefficients in a representation). Chapter 3 contains two fundamental theorems in that subject: the Casselman-Osborne theorem relating cohomology and the center of the enveloping algebra, and Kostant's formulation of the Bott-Borel-Weil theorem.

Chapter 4 discusses that part of the classification of irreducibles which can be obtained from ordinary principal series representations. In addition to more standard intertwining operator techniques, it uses the Bernstein-Gelfand-Gelfand theory of fine representations (see [2]); a detailed account of this theory is given in Section 4.3.

Chapters 5 and 6 complete the classification of irreducible representations. The method is discussed in some detail in the introductions to those chapters. Essentially

it is a generalization of the highest weight theory of finite dimensional representations, with highest weight vectors replaced by a more general kind of cohomology classes for the representation. The main problem is to find a construction of representations, generalizing induction, which is nicely related to cohomology. This was done (for reductive groups) by G. Zuckerman (Definition 6.3.1). This book uses only one special case of this definition, in addition to ordinary induction. It seems likely that one can do much more with the idea.

Chapter 7 is devoted to the Jantzen-Zuckerman "translation principle" and related matters. This says that all irreducible representations come in nice families (like the principal series, or finite dimensional representations). Many results can therefore be proved by reducing to the case when the representation is in "general position" in some sense. This is helpful technically, but the translation principle plays an even more fundamental role. Roughly speaking, it provides a connection between the structure of irreducible representations, and the combinatorial structure of the Weyl group. This is discussed more carefully in Section 7.3; the key result is Theorem 7.3.16.

In Chapter 8, the basic theorem on reducibility of standard representations is proved. This is in principle a trivial consequence of the results of Chapter 7, but requires some messy calculations. (For example, we need the Hecht-Schmid character identities for discrete series from [15].) Exactly the same ideas, with the judicious addition of a technical conjecture (Conjecture 7.3.25), lead to the algorithm for computing composition series; this is the content of Chapter 9.

The book differs from the existing literature in several ways. Most importantly, the standard representations are not constructed by Langlands' method (that is, ordinary induction from discrete series). We use instead Zuckerman's "cohomological" induction from principal series. This gives isomorphic standard representations (Theorem 6.6.15), but that fact is not proved in the text (or used). As far as the classification itself goes, this choice is simply a matter of taste. However, the conjecture on composition series seems to be comprehensible only in the realization we have given. (There is an obvious way to try to use ordinary induction, but then any simple analogue of the critical Theorem 9.5.1 is false.) This is not to say that Zuckerman's realization is better; for analytic problems, it is Zuckerman's method which encounters obstacles.

There are many less substantive changes. What I had perceived as the main theorem of [42], which relates cohomology and $U(\mathfrak{g})^K$, does not appear here; the argument has been rearranged slightly to eliminate the need for it. The definition of lowest K-type in [42] (Definition 5.4.18 below) has been replaced by a much more technical one (Definition 5.4.1); they are equivalent for irreducibles but not for general $(\mathfrak{g},K)$ modules. The advantage of the new definition is that a number of subsequent technical arguments become much simpler when it is used.

The proof of the Knapp-Stein reducibility theorem (Corollary 4.4.11) is new, avoiding both the rather delicate analysis used in [26] and the long case-by-case computation in the unpublished second part of [42].

The main result about translation "across a wall" is Theorem 7.3.16; it is (or was) the only non-formal part of the proof of the theorem on reducibility of standard representations. About one-third of [39] is devoted to a proof of it, which might charitably be described as sketchy. A second proof was given in [45], which was fairly short, but used Duflo's main theorem on primitive ideals (see [8]). The proof given here is entirely trivial; but the result is labelled as a theorem in memory of [39].

Each chapter begins with an introduction; these provide a more detailed guide to the main results. Section 9.7 summarizes some open problems.

This book is based on lectures given at MIT during the 1979-80 academic year. I would like to thank those who attended for helpful comments and dogged perseverance. Much of Chapter 6 is unpublished work of G. Zuckerman. I thank him for explaining it to me, and allowing it to appear here. J. Vargas provided a list of errors in the first draft, which was very helpful. Many people have pointed out particular errors and shortcomings; I apologize for those which undoubtedly remain.

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Chapter 0. Preliminaries

General references for this chapter are [40] and [50].

§1. Assumptions on G

Notation 0.1.1 Suppose H is a real Lie group. Write

H_0 = identity component of H;

$\mathfrak{h}_0$ = Lie algebra of H;

Ad: $H \to \mathrm{End}(\mathfrak{h}_0)$, the adjoint representation of H;

$\mathfrak{h} = (\mathfrak{h}_0)_{\mathbb{C}}$, the complexification of $\mathfrak{h}$;

$U(\mathfrak{h})$ = universal enveloping algebra of $\mathfrak{h}$.

$\mathfrak{h}$ is given the complex structure bar, defined by

$$\overline{X + iY} = X - iY \qquad (X, Y \in \mathfrak{h}_0).$$

This notation will be applied to groups denoted by other Roman letters in the same way without comment.

By a real reductive linear group, we will mean a real Lie group G (not necessarily connected), a maximal compact subgroup K of G, and an involution θ of $\mathfrak{g}_0$, satisfying the following conditions.

(0.1.2)
a) $\mathfrak{g}_0$ is a real reductive Lie algebra;
b) If $g \in G$, the automorphism Ad(g) of $\mathfrak{g}$ is inner (for the corresponding complex connected group);
c) The fixed point set of θ is $\mathfrak{k}_0$;
d) Write $\mathfrak{p}_0$ for the -1 eigenspace of θ; then the map $K \times \mathfrak{p}_0 \to G$, $(k, X) \to k \cdot \exp(X)$ is a diffeomorphism;
e) G has a faithful finite dimensional representation;

f) Let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be a Cartan subalgebra, and let H be be the centralizer of $\mathfrak{h}_0$ in G. Then H is abelian.

(Notice that (d) forces G to have only finitely many components.)

Throughout this book, G will denote a real reductive linear group. We will from time to time assume that G satisfies additional conditions, but these at least must always be met. Conditions (a) - (d) define Harish-Chandra's category of real reductive groups (cf. [40], §5) and require no further justification here; we will use inductive arguments which lead from connected groups to disconnected ones. Assumption (e) is in some sense only a convenience -- most of the results we obtain can be gotten without it, although sometimes this requires more work and a less satisfactory formulation of the theorems. However, one of our main goals is the formulation of the Kazhdan-Lusztig conjectures discussed in the introduction; and this has not been carried out for non-linear groups. (The problems do not seem to be very deep, but they are quite messy.) Assumption (f) is included chiefly to make the Knapp-Stein "commutativity of intertwining operators" theorem (Corollary 6.5.14) hold. (The simplest case where it fails has $|G/G_0| = 4$, $G_0 = SL(2, \mathbb{R}) \times SL(2,\mathbb{R})$; (a) - (e) are satisfied, but (f) is not.)

Definition 0.1.3 Make θ an involution of G by setting

$$\theta(k \cdot \exp(X)) = k \cdot \exp(-X) \quad (X \in \mathfrak{p}_0,\ k \in K) .$$

We call θ the Cartan involution of G or $\mathfrak{g}_0$.

Example 0.1.4

a) G = connected real linear semisimple group;

b) $G = GL(n, \mathbb{R})$, $K = O(n)$, $\theta(g) = {}^t g^{-1}$;

c) G = real points of a reductive algebraic group defined over $\mathbb{R}$;

d) If $G_{\mathbb{C}}$ is a complex semisimple Lie algebra, and $\mathfrak{g}_0$ is a real form of its Lie algebra $\mathfrak{g}$, then

$$G = \text{normalizer in } G_{\mathbb{C}} \text{ of } \mathfrak{g}_0;$$

e) $G = SL(2, \mathbb{R}) \cup SL(2, \mathbb{R}) \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$;

f) Suppose G is a real reductive linear group, and H is a θ-stable abelian subgroup. Then the centralizer G^H of H in G is a real reductive linear group.

Of course, these examples overlap enormously. We will make constant use of example (f); its (easy) verification is left to the reader.

Example (b) illustrates an annoying technical problem: K need not be a real reductive group. The problem is that the orthogonal group O(n) does not satisfy (0.1.2)(b). Thus the Cartan-Weyl highest weight theory does not apply directly to K; we do not have a priori a perfect grasp on $\hat{K}$. This is circumvented by using the relation between K and G; a description of $\hat{K}$ is given in Section 5.1.

Definition 0.1.5 A Cartan subgroup of G is the centralizer in G of a Cartan subalgebra of $\mathfrak{g}_0$. A parabolic subgroup of G is the normalizer in G of a parabolic subalgebra of $\mathfrak{g}_0$.

Lemma 0.1.6

a) Suppose $P \subseteq G$ is a parabolic subgroup. Then $G = K \cdot P$.

b) Suppose $H \subseteq G$ is a Cartan subgroup. Then H is conjugate under G to a θ-stable Cartan subgroup H' (that is, $\theta H' = H'$).

c) Two θ-stable Cartan subgroups of G are conjugate under G if and only if they are conjugate under K.

These results are trivial consequences of the corresponding ones for connected groups (cf. [50], 1.2 and 1.3). They are included mostly as an illustration; we will use other structure theorems (Iwasawa decomposition, Langlands decomposition for parabolics, and so forth) as needed, although the references may treat only connected semisimple groups.

Finally, we fix once and for all a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, which we denote by $\langle\ ,\ \rangle$. We can and do choose this form so that

(0.1.7) the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is orthogonal for $\langle\ ,\ \rangle$; $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$.

This form will be restricted, complexified, and transferred to dual vector spaces without change of notation.

§2. Roots and $Z(\mathfrak{g})$.

Through Lemma 0.2.2, H will be a fixed θ-stable Cartan subgroup of G. We write V* for the dual of a vector space V.

Definition 0.2.1 The vector part of H is

$$A = H \cap \exp(\mathfrak{p}_0) = \exp(\mathfrak{h}_0 \cap \mathfrak{p}_0).$$

The compact part of H is

$$T = H \cap K.$$

The set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ is written $\Delta(\mathfrak{g},\mathfrak{h}) \subseteq \mathfrak{h}^*$. A root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is called real if $\alpha|_{\mathfrak{t}} = 0$; imaginary if $\alpha|_{\mathfrak{a}} = 0$; and complex if it is neither real nor imaginary. Suppose α is an imaginary root; let $X_\alpha \in \mathfrak{g}$ be a root vector for it. Then α is called compact imaginary if $X_\alpha \in \mathfrak{k}$, and noncompact imaginary otherwise. (Harish-Chandra calls these latter roots singular imaginary.)

Lemma 0.2.2 ([50], 1.3 and 1.4). In the notation just defined:

a) $H = T\,A$, a direct product of Lie groups. A is isomorphic to its Lie algebra via exp, and T is a compact abelian Lie group (possibly disconnected);

b) As linear functionals on $\mathfrak{h}_0$, the roots take real values on $\mathfrak{a}_0$ and purely imaginary values on $\mathfrak{t}_0$; so a root is real, imaginary, or complex as its values on $\mathfrak{h}_0$ are;

c) If we make the Cartan involution θ act on $\mathfrak{h}^*$ by duality, then $\Delta(\mathfrak{g},\mathfrak{h})$ is θ-stable. Furthermore, if α is a root, then

$$\alpha \text{ is real} \iff \theta\alpha = -\alpha$$

$$\alpha \text{ is imaginary} \iff \theta\alpha = \alpha$$

$$\alpha \text{ is complex} \iff \theta\alpha \neq \pm\alpha;$$

d) If we make the complex structure bar act on $\mathfrak{h}^*$ by duality (notation (0.1.1)), then

$$\bar{\alpha} = -\theta\alpha.$$

Sketch of Proof. Part (a) follows from hypothesis (0.1.2)(d) on G. One knows that $\mathfrak{a}_0$ has real eigenvalues on $\mathfrak{g}_0$; and a character of a compact group (such as a root regarded as a character of T) has a purely imaginary differential. This proves (b). Since θ is an automorphism of $\mathfrak{g}$, (c) is obvious. Part (d) follows from (b). Q.E.D.

Definition 0.2.3 Let H be an abelian Lie group, and V a finite dimensional semisimple representation of H. (This means that the action of H on V is diagonalizable.) If χ is a character of H (that is, $\chi \in \hat{H}$), define the χ weight space of V

$$V(\chi) = \{v \in V \mid h\cdot v = \chi(h)v, \text{ all } h \in H\},$$

and the set of weights of H in V

$$\Delta(V) = \Delta(V,H) = \{\chi \in \hat{H} \mid V(\chi) \neq 0\}.$$

We may often regard $\Delta(V)$ as a "multiset"; that is, we assume that χ occurs in $\Delta(V)$ with multiplicity $\dim V(\chi)$. Similarly, if V is just a representation of the Lie algebra $\mathfrak{h}$ of H, and $\lambda \in \mathfrak{h}^*$, define the λ-weight space and the set of weights of V by

$$V(\lambda) = \{v \in V \mid h\cdot v = \lambda(h)v, \text{ all } h \in \mathfrak{h}\}$$

$$\Delta(V) = \Delta(V,\mathfrak{h}) = \{\lambda \in \mathfrak{h}^* \mid V(\lambda) \neq 0\}.$$

Finally define,

$$\rho(\Delta(V)) = \rho(V) = \frac{1}{2}\sum_{\lambda\in\Delta(V)} \lambda \in \mathfrak{h}^*$$

(with multiplicities).

So if $h \in \mathfrak{h}$, $\rho(V)(h)$ is half the trace of h on V. We include multiplicities in $\Delta(V)$ so that formulas like

$$\det_V(h) = \prod_{\chi\in\Delta(V)} \chi(h) \qquad (h \in H)$$

hold. A very important special case is when $\mathfrak{h} \subseteq \mathfrak{g}$, and $\mathfrak{s} \subseteq \mathfrak{g}$ is stable under $\mathrm{ad}(\mathfrak{h})$; then expressions like $\rho(\mathfrak{s})$, half the sum of the weights of $\mathfrak{h}$ in $\mathfrak{s}$, will appear often. The most important special case is when $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$$

is a Borel subalgebra of $\mathfrak{g}$, with nil radical $\mathfrak{n}$. When the choice of such a $\mathfrak{b}$ is clear, we may write simply

$$(0.2.4) \qquad \rho = \rho(\mathfrak{n}) = \rho(\Delta^+(\mathfrak{g},\mathfrak{h})) = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha;$$

here $\Delta^+(\mathfrak{g},\mathfrak{h})$ is the positive root system corresponding to $\mathfrak{n}$.

Definition 0.2.5 Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra. If $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$, the simple reflection about α is the automorphism of $\mathfrak{h}^*$ given by

$$\lambda \mapsto s_\alpha(\lambda) = \lambda - (2\langle\alpha,\lambda\rangle/\langle\alpha,\alpha\rangle)\alpha.$$

The coroot corresponding to α is

$$\check{\alpha} = 2\alpha/\langle\alpha,\alpha\rangle.$$

(Thus we can write

$$s_\alpha(\lambda) = \lambda - \langle\lambda,\check{\alpha}\rangle\alpha.)$$

The <u>(complex) Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$</u>, $W = W(\mathfrak{g},\mathfrak{h})$ is the group generated by the various s_α; it is regarded as a group of automorphisms of $\mathfrak{h}^*$ or (by duality) of $\mathfrak{h}$.

<u>Definition 0.2.6</u> Suppose $H \subseteq G$ is a θ-stable Cartan subgroup. <u>The Weyl group of H in G,</u> $W(G,H)$ is by definition the quotient of the normalizer of H in G by H:

$$W(G,H) = N_G(H)/H$$
$$= N_K(H)/(H \cap K);$$

the second equality is [50], Proposition 1.4.2.1. We regard $W(G,H)$ as a subgroup of $W(\mathfrak{g},\mathfrak{h})$ when this is convenient.

<u>Definition 0.2.7</u> The center of the universal enveloping algebra of $\mathfrak{g}$ is written $Z(\mathfrak{g})$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ (with nil radical $\mathfrak{n}$). Let $\mathfrak{n}^-$ denote the opposite nil radical, so that

a) $$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}$$

[a <u>triangular decomposition</u> of G]. By Poincaré-Birkhoff-Witt, write

b) $$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$$

c) $$= U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}),$$

and let

$$\tilde{\xi} : Z(\mathfrak{g}) \to U(\mathfrak{h})$$

be the projection on the first factor in (c). Let

$$\rho = \rho(\mathfrak{n})$$

be as in (0.2.4). Regard $U(\mathfrak{h})$ as the algebra of polynomial functions on $\mathfrak{h}^*$, and define

$$(T_\rho f)(\lambda) = f(\lambda - \rho) \qquad (f \in U(\mathfrak{h}),\ \lambda \in \mathfrak{h}^*).$$

The <u>Harish-Chandra map</u> from $Z(\mathfrak{g})$ into $U(\mathfrak{h})$ is by definition

$$\xi = T_\rho \circ \tilde{\xi}.$$

<u>Theorem 0.2.8</u> (Harish-Chandra -- see [22], p. 130). <u>In the setting of Definition 0.2.7, the map ξ is an algebra isomorphism from $Z(\mathfrak{g})$ onto $U(\mathfrak{h})^W$, the algebra of Weyl group-invariant elements of $U(\mathfrak{h})$. It depends only on $\mathfrak{h}$ (and not on the choice of $\mathfrak{b}$).</u>

<u>Definition 0.2.9</u> Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra and

$$\xi : Z(\mathfrak{g}) \to U(\mathfrak{h})$$

is the Harish-Chandra map. If $\lambda \in \mathfrak{h}^*$, let

$$\xi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$$

be the composition of ξ with evaluation at λ; here we continue to regard elements of $U(\mathfrak{h})$ as functions on $\mathfrak{h}^*$.

Corollary 0.2.10 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Every homomorphism from $Z(\mathfrak{g})$ to $\mathbb{C}$ is of the form ξ_λ, for some $\lambda \in \mathfrak{h}^*$. Furthermore

$$\xi_\lambda = \xi_{\lambda'} \Leftrightarrow \text{there is } w \in W \text{ such that } \lambda' = w\lambda.$$

Definition 0.2.11 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. A weight $\lambda \in \mathfrak{h}^*$ is called nonsingular (or regular) if

$$\langle\alpha,\lambda\rangle \neq 0 \qquad (\text{all } \alpha \in \Delta(\mathfrak{g},\mathfrak{h})).$$

The infinitesimal character ξ_λ is called nonsingular if λ is; by Corollary 0.2.10, this concept depends only on ξ_λ, and not on the choice of $\mathfrak{h}$ and λ. If λ is nonsingular, we can associate to it a positive root system

a) $$\Delta^+_\lambda(\mathfrak{g},\mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \mid \mathrm{Re}\langle\alpha,\lambda\rangle > 0, \text{ or } \mathrm{Re}\langle\alpha,\lambda\rangle = 0, \text{ and } \mathrm{Im}\langle\alpha,\lambda\rangle > 0\};$$

we call this the positive system defined by λ. Conversely, if $\Delta^+ = \Delta^+(\mathfrak{g},\mathfrak{h})$ is a fixed positive root system, set

b) $$\mathcal{C}_{\Delta^+} = \{\lambda \in \mathfrak{h}^* \mid \lambda \text{ is nonsingular, and } \Delta^+ = \Delta^+_\lambda\} \text{ --}$$

this is the positive Weyl chamber defined by Δ^+. An element λ of the closure $\overline{\mathcal{C}}_{\Delta^+}$ of this positive Weyl chamber is called dominant for Δ^+; and Δ^+ is called positive for λ.

Lemma 0.2.12 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and $\Delta^+ \subseteq \Delta(\mathfrak{g},\mathfrak{h})$ a positive root system. Then the closed Weyl chamber $\overline{\mathcal{C}}_{\Delta^+}$ (Definition 0.2.11) is a fundamental domain for the action of W on $\mathfrak{h}^*$.

This result is most familiar for the real span of the roots in $\mathfrak{h}$; but the general case is easily reduced to that.

§3. Group representations and Harish-Chandra modules.

Definition 0.3.1 A (continuous) representation of a Lie group H is a pair $(\pi,\mathcal{H})$, with $\mathcal{H}$ a complex Hilbert space, and

$$\pi : H \to B(\mathcal{H})$$

a continuous homomorphism of G into the semigroup $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$ (endowed with the weak topology); we assume that $\pi(1)$ is the identity operator. An invariant subspace of $(\pi,\mathcal{H})$ is a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ which is left invariant by all the operators in $\pi(H)$. Such a subspace is called proper if it is not equal to $\{0\}$ or $\mathcal{H}$. The representation $(\pi,\mathcal{H})$ is called irreducible if $\mathcal{H} \neq 0$, and there are no proper closed invariant subspaces. It is called unitary if the operators in $\pi(H)$ are all unitary.

At some critical points in Chapter 4, we will need to deal with some representations on spaces of smooth functions, which are, of course, only Fréchet spaces. However, it seems reasonable to deal then with some minor technical problems, rather than outline the theory in that generality.

<u>Definition 0.3.2</u> Suppose $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are continuous representations of a Lie group H. Define

$$\begin{aligned}\mathrm{Hom}_H(\pi,\pi') &= \mathrm{Hom}_H(\mathcal{H},\mathcal{H}')\\ &= \{L : \mathcal{H} \to \mathcal{H}' \mid L \text{ is continuous, linear, and } \pi'(g)L = L\pi(g) \text{ for all } g \in H\},\end{aligned}$$

the space of <u>intertwining operators</u> between $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H})$. We say that the representations are <u>boundedly equivalent</u> if there is an invertible intertwining operator between them.

<u>Definition 0.3.3</u> Suppose H is a direct product of a compact group and an abelian group, so that every irreducible representation of H is finite dimensional. Define $\hat{H}$, the <u>dual object</u> of H, to be the set of bounded equivalence classes of irreducible representations of H. If H is abelian, so that every irreducible representation is one-dimensional, we make $\hat{H}$ a group as usual, and call it the <u>group of characters</u> of H.

For infinite dimensional (non-unitary) representations, bounded equivalence fails to identify some representations which morally ought to be equivalent. There is no completely satisfactory way around this in general; but for reductive groups, the right notion of equivalence is provided by the work of Harish-Chandra in [12]. Its formulation requires some preparation.

<u>Definition 0.3.4</u> Suppose $(\pi, \mathcal{H})$ is a continuous representation of G. A vector $v \in \mathcal{H}$ is called <u>K-finite</u> if

$$\dim\langle\pi(K)v\rangle < \infty;$$

here we write $\langle\pi(K)v\rangle$ for the linear span of all vectors of the form $\pi(k)v$, for $k \in K$. Put

$$\mathcal{H}_K = \{v \in \mathcal{H} \mid v \text{ is K-finite}\}.$$

Fix $(\delta, V_\delta) \in \hat{K}$, and define

$$\mathcal{H}_K(\delta) = \bigcup_{L \in \mathrm{Hom}_K(\delta,\pi)} L(V_\delta)$$

the <u>δ K-type</u> or <u>δ-primary subspace</u> of $\mathcal{H}$; it is a (possibly infinite) direct sum of copies of δ. We say that $(\pi, \mathcal{H})$ is <u>admissible</u> if

$$\dim \mathcal{H}_K(\delta) < \infty, \text{ all } \delta \in \hat{K}.$$

In that case, the <u>multiplicity of δ in π</u> is by definition $m(\delta,\pi) = \dim \mathrm{Hom}_K(V_\delta, \mathcal{H}_K)$.

<u>Theorem 0.3.5</u> (Harish-Chandra [12]). <u>Let $(\pi, \mathcal{H})$ be an admissible representation of G. If $v \in \mathcal{H}_K$ (Definition 0.3.4) and $x \in \mathfrak{g}_0$, the limit</u>

$$\lim_{t\to 0} \frac{1}{t}\left(\pi(\exp(tx))v - v\right)$$

<u>exists; call it $\pi(x)v$. Then</u>

$$\pi(x)v \in \mathcal{H}_K,$$

<u>and this defines a representation of the Lie algebra $\mathfrak{g}_0$ on $\mathcal{H}_K$. There is a lattice isomorphism between</u>

the closed invariant subspaces of $\mathcal{H}$, and (arbitrary) K-invariant, $\mathfrak{g}_0$-invariant subspaces of $\mathcal{H}_K$; if $\mathcal{S} \subseteq \mathcal{H}$ is closed and invariant, then

$$\mathcal{S} \leftrightarrow \mathcal{S}_K = \mathcal{S} \cap \mathcal{H}_K; \quad \mathcal{S} = \overline{\mathcal{S}_K}.$$

A nice exposition of this may be found in [49]. One does not know whether an arbitrary irreducible representation is admissible; but there are various technical results which show that admissibility is a very weak condition on an irreducible representation (cf. [12]). As an illustration, we mention

Theorem 0.3.6 (Harish-Chandra [12]). Any irreducible unitary representation of G is admissible.

Theorem 0.3.5 gives to $\mathcal{H}_K$ the structure of a representation of $\mathfrak{g}_0$; and since $\mathcal{H}$ is complex, this structure can be complexified to a (complex linear) representation of $\mathfrak{g}$. At the same time $\mathcal{H}_K$ is a representation of the group K. These two representations satisfy the following (obvious) conditions:

(0.3.7)

a) Every vector $v \in \mathcal{H}_K$ is K-finite; that is,

$$\dim \langle \pi(K)v \rangle < \infty .$$

b) The differential of the representation of K is the restriction of the $\mathfrak{g}$ representation to $\mathfrak{k}_0$; that is, if $v \in \mathcal{H}_K$ and $x \in \mathfrak{k}_0$,

$$\pi(x)v = \lim_{t\to 0} \frac{1}{t} (\pi(\exp(tx))v - v).$$

c) If $x \in \mathfrak{g}$ and $k \in K$, then

$$\pi(\mathrm{Ad}(k)x) = \pi(k)\pi(x)\pi(k)^{-1}.$$

Definition 0.3.8 (Lepowsky) A $(\mathfrak{g},K)$ module is a pair (π,X) with X a complex vector space and π a map

$$\pi: \mathfrak{g} \cup K \to \mathrm{End}\,(X),$$

satisfying

a) $\pi|\mathfrak{g}$ is a complex linear Lie algebra representation, and $\pi|_K$ is a group representation; and

b) the compatibility conditions 0.3.7 (a) - (c) hold, with X replacing $\mathcal{H}_K$.

The K-types of X and their multiplicities, and the notion of admissible, are defined exactly as for group representations (Definition 0.3.4). We write $\mathcal{M}(\mathfrak{g},K)$ for the category of $(\mathfrak{g},K)$ modules, and $\mathcal{A}(\mathfrak{g},K)$ for the subcategory of admissible modules. The $(\mathfrak{g},K)$ module $(\pi,\mathcal{H}_K)$ attached to an admissible group representation $(\pi,\mathcal{H})$ by Theorem 0.3.5 is called the Harish-Chandra module of π.

We will often use module notation for $(\mathfrak{g},K)$ modules, writing $x \cdot v$ instead of $\pi(x)v$. The category structure referred to above is the obvious one. Part of it is given by

Definition 0.3.9 Suppose (π,X) and (π',X') are two $(\mathfrak{g},K)$ modules. Define

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi,\pi') = \mathrm{Hom}_{\mathfrak{g},K}(X,X') = \{L : X \to X' \mid L \text{ is complex linear, and } \pi' L = L\pi'\},$$

the $(\mathfrak{g},K)$-module maps (or intertwining operators) from π to π'. We call X and X' equivalent if there is an invertible $(\mathfrak{g},K)$-module map between them. We write $\hat{G}$ for the set of equivalence classes of irreducible $(\mathfrak{g},K)$ modules. Two representations of G are called infinitesimally equivalent if their Harish-Chandra modules are equivalent.

The study of $(\mathfrak{g},K)$ modules instead of group representations is justified by Theorems 0.3.5 and 0.3.6, and the next result.

Theorem 0.3.10 (Harish-Chandra, Lepowsky, Rader -- see [49]). Every irreducible $(\mathfrak{g},K)$ module is the Harish-Chandra module of an irreducible admissible representation of G. Two irreducible unitary representations of G are boundedly equivalent if and only if they are infinitesimally equivalent.

In particular, an irreducible $(\mathfrak{g},K)$ module is automatically admissible.

When G is connected, the group K plays a less serious role in the structure of a $(\mathfrak{g},K)$ module: for example, condition (0.3.7)(c) is automatic, and $(\mathfrak{g},K)$-module maps are just $\mathfrak{g}$-module maps. Best of all is

Theorem 0.3.11 (Harish-Chandra [12]). Suppose G is connected. Let (π,X) and (π',X') be irreducible $(\mathfrak{g},K)$ modules. Fix $(\delta,V_\delta) \in \hat{K}$, and suppose

$$X(\delta) \neq 0.$$

Let $U(\mathfrak{g})^{\mathfrak{k}}$ denote the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$. Then X and X' are equivalent if and only if $X(\delta)$ and $X'(\delta)$ are isomorphic $U(\mathfrak{g})^{\mathfrak{k}}$ modules. Furthermore, the $U(\mathfrak{g})^{\mathfrak{k}}$ module

$$\mathrm{Hom}_K(V_\delta,X)$$

(where $U(\mathfrak{g})^{\mathfrak{k}}$ acts on the range only) is irreducible (and finite dimensional).

Probably one can find a good substitute for this theorem in the case of disconnected groups, but I have not been able to do so. To some extent this may be an advantage, because it prevents us from spending a lot of effort trying to understand the (hideously complicated) non-commutative ring $U(\mathfrak{g})^{\mathfrak{k}}$. For the reader whose tastes run in that direction, however, we point out that [42] contains a proof of the classification of $(\mathfrak{g},K)$ modules (for connected G), in which $U(\mathfrak{g})^{\mathfrak{k}}$ plays a central role.

We need some generalities about composition series. Although the admissible representations we need all have finite composition series, this will not be proved for some time, and we need to work with them to prove it; so our definitions allow for infinite composition series.

<u>Definition 0.3.12</u> Suppose (π,X) is a $(\mathfrak{g},K)$ module, and (π^0,X^0) is an irreducible $(\mathfrak{g},K)$ module. We say that π^0 is a <u>subquotient</u> or <u>composition factor</u> of π, or that π^0 <u>occurs in π</u> if X has two submodules M_1 and M_2 such that

$$M_1 \supseteq M_2,\ M_1/M_2 \cong X^0 \quad (\text{as } (\mathfrak{g},K) \text{ modules}.$$

Suppose (π,X) is admissible. The <u>multiplicity of π^0 in π</u> is the largest integer n such that there is a chain of submodules

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N \subseteq X,$$

with

$$M_i/M_{i-1} \cong X^0$$

for n distinct values of i. We write

$$n = m(\pi^0,\pi) = m(X^0,X),$$

and say that <u>π^0 occurs n times in π</u>. If (π',X') is another admissible $(\mathfrak{g},K)$ module, we say that π and π' have <u>equivalent composition series</u> if for every irreducible $(\mathfrak{g},K)$ module (π^0,X^0),

$$m(\pi^0,\pi) = m(\pi^0,\pi').$$

To see that this all makes sense, one looks at one K-type at a time. For example, to prove that the largest integer n exists, pick $\delta \in K$ so that $X(\delta) \neq 0$; then clearly

$$n \leq \text{multiplicity of } \delta \text{ in } \pi.$$

<u>Definition 0.3.13</u> The $(\mathfrak{g},K)$ module (π,X) is said to have <u>finite length</u> if it admits a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N = X,$$

such that M_i/M_{i-1} is irreducible for $i = 1,\ldots,N$. We write $\mathcal{F}(\mathfrak{g},K)$ for the category of $(\mathfrak{g},K)$ modules of finite length.

By the remark after Theorem 0.3.10, any $(\mathfrak{g},K)$ module of finite length is admissible; so

(0.3.14) $$\mathcal{F}(\mathfrak{g},K) \subseteq \mathcal{A}(\mathfrak{g},K) \subseteq \mathcal{M}(\mathfrak{g},K).$$

<u>Lemma 0.3.15</u> <u>Suppose (π^0,X^0) is a fixed irreducible $(\mathfrak{g},K)$ module. Then</u>

$$X \to m(X^0,X) \quad (X \in \mathcal{A}(\mathfrak{g},K))$$

<u>is exact in X; that is, if</u>

$$0 \to X^1 \to X^2 \to X^3 \to 0$$

<u>is an exact sequence in $\mathcal{A}(\mathfrak{g},K)$</u>,

$$m(X^0,X^2) = m(X^0,X^1) + m(X^0,X^3).$$

This is trivial.

<u>Definition 0.3.16</u> The <u>Grothendieck group</u> of $\mathcal{F}(\mathfrak{g},K)$ is the abelian group generated by $(\mathfrak{g},K)$ modules of finite length, modulo the equivalence relations

$$X \sim Y + Z$$

whenever there is a short exact sequence

$$0 \to Y \to X \to Z \to 0 \quad .$$

We write it as $\mathcal{V}(\mathfrak{g},K)$, and call it the group of *virtual $(\mathfrak{g},K)$ modules* or *virtual characters of G*. Although we will not, strictly speaking, need to do so, it may be helpful to think of virtual characters in more concrete terms. The first level of concreteness is

Lemma 0.3.17 Suppose X is a $(\mathfrak{g},K)$ module of finite length. Then, in the Grothendieck group $\mathcal{V}(\mathfrak{g},K)$, we have the relation

$$X = \sum_{Y \in \hat{G}} m(Y,X)Y \ .$$

This gives rise to an isomorphism of $\mathcal{V}(\mathfrak{g},K)$ with the free abelian group having $\hat{G}$ as its set of generators; so for any $X \in \mathcal{V}(\mathfrak{g},K)$ and $Y \in \hat{G}$, $m(Y,X)$ is a well-defined integer.

Again, we can leave the easy proof to the reader. We can be still more concrete.

Definition 0.3.18 A $\mathfrak{g}$ module X is called *quasisimple* if $Z(\mathfrak{g})$ acts by scalars on X (Definition 0.2.7). The corresponding homomorphism

$$\chi: \ Z(\mathfrak{g}) \to \mathbb{C} \ ,$$

defined by

$$z \cdot x = \chi(z)x \qquad (x \in X, \ z \in Z(\mathfrak{g}))$$

is called the *infinitesimal character* of X. More

generally, X is called *$Z(\mathfrak{g})$-finite* if there is an ideal $I \subseteq Z(\mathfrak{g})$ of finite codimension which annihilates X. If I can be chosen to be primary (that is, $I \subseteq (\ker \chi)^n$ for some positive integer n and homomorphism

$$\chi : \ Z(\mathfrak{g}) \to \mathbb{C}) \ ,$$

then we say that X has *generalized infinitesimal character* χ. This means that

$$(z - \chi(z))^n \, x = 0, \quad \text{all } x \in X, \ z \in Z(\mathfrak{g}).$$

If X is a $Z(\mathfrak{g})$-finite $\mathfrak{g}$ module, and χ is a character of $Z(\mathfrak{g})$ as above, we define

$$P_\chi(X) = \{x \in X \mid \text{for some } n > 0, \text{ and all } z \in Z(\mathfrak{g}), \ (z - \chi(z))^n \, x = 0\},$$

the *χ-primary component of X*. If χ corresponds to $\lambda \in \mathfrak{h}^*$ (cf. 0.2.9), we may write $P_\lambda(X)$ instead of $P_\chi(X)$.

Proposition 0.3.19

a) *Any irreducible $(\mathfrak{g},K)$ module is quasisimple.*

b) *Any $(\mathfrak{g},K)$ module of finite length is $Z(\mathfrak{g})$-finite.*

c) *If X is a $Z(\mathfrak{g})$-finite $\mathfrak{g}$ module, then*

$$X = \bigoplus_{\chi \in \operatorname{Spec} Z(\mathfrak{g})} P_\chi(X).$$

d) *If X is a $Z(\mathfrak{g})$-finite $(\mathfrak{g},K)$ module, then each $P_\chi(X)$ is a $(\mathfrak{g},K)$-submodule of X.*

Proposition 0.3.23 The map

$$V \to \Theta(V)$$

from $\mathcal{V}(\mathfrak{g},K)$ to distributions on G is an isomorphism onto the free abelian group with generators

$$\{\Theta(\pi) \mid \pi \in \hat{G}\}.$$

Proof. We only need to check that $\Theta(V)$ is non-zero when V is; but this follows from the linear independence of the irreducible characters (Theorem 0.3.21). Q.E.D.

We may therefore refer to $\mathcal{V}(\mathfrak{g},K)$ or the corresponding distributions as the lattice of virtual characters of G.

Proposition 0.3.24 Suppose $(\pi,\mathcal{H})$ and $(\pi',\mathcal{H}')$ are admissible representations of G, satisfying the conditions of Theorem 0.3.21. Then $\mathcal{H}_K$ and $\mathcal{H}'_K$ have equivalent composition series (Definition 0.3.12) if and only if $\Theta(\pi) = \Theta(\pi')$.

The "only if" part follows from the formula for $\Theta(\pi)$ in Theorem 0.3.21. The "if" part is a slight strengthening of the linear independence of irreducible characters, and can be proved by exactly the same method (cf. [14]). We leave this to the reader.

We conclude with a simple technique for constructing representations of disconnected groups; it is used in Definition 1.4.7 to construct the discrete series for $SL^{\pm}(2,\mathbb{R})$ from those for $SL(2,\mathbb{R})$, and for analogous purposes later in the book.

Definition 0.3.25 Suppose G_1 is a subgroup of G of finite index (which therefore contains the identity component G_0). Let $K_1 = K \cap G_1$. Suppose (π_1,X_1) is a $(\mathfrak{g},K_1)$ module. Define a $(\mathfrak{g},K)$ module

$$(\pi,X) = \mathrm{Ind}_{\mathfrak{g},K_1}^{\mathfrak{g},K}(\pi_1,X_1)$$

as follows.

$$X = \mathrm{Ind}_{K_1}^{K}(X_1)$$
$$= \{f : K \to X \mid f(kk_1) = \pi_1(k_1^{-1})f(k) \text{ for } k \in K, k_1 \in K_1\}.$$

If f is in X, k and k_0 are in K, and x is in $\mathfrak{g}$,

$$[\pi(k)f](k_0) = f(k^{-1}k_0)$$

$$[\pi(x)f](k_0) = \pi_1(\mathrm{Ad}(k_0^{-1})x) \cdot (f(k_0)).$$

It is easy to check that (π,X) is really a $(\mathfrak{g},K)$ module. Its basic properties are given by

Lemma 0.3.26 (Frobenius reciprocity). Fix notation as in Definition 0.3.25.

a) Suppose $\delta \in \hat{K}$. In the notation (0.3.4),

$$m(\delta,\mathrm{Ind}_{\mathfrak{g},K_1}^{\mathfrak{g},K}(\pi_1)) = \sum_{\delta_1 \in \hat{K}_1} m(\delta_1,\delta)m(\delta_1,\pi_1).$$

More precisely, there is a natural isomorphism

$$\mathrm{Hom}_K(V_\delta,\mathrm{Ind}_{\mathfrak{g},K_1}^{\mathfrak{g},K}(X_1)) \cong \mathrm{Hom}_{K_1}(V_\delta|_{K_1},X_1).$$

b) Let (ρ, Y) be any $(\mathfrak{g}, K)$ module. Then there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K}(Y, \mathrm{Ind}_{\mathfrak{g},K_1}^{\mathfrak{g},K}(X_1)) \cong \mathrm{Hom}_{\mathfrak{g},K_1}(Y|_{(\mathfrak{g},K_1)}, X_1).$$

The proof, which is exactly like that of Frobenius reciprocity for compact groups, is left to the reader.

§4. Finite dimensional representations.

Because G is not necessarily connected, the extension of the Cartan-Weyl theory of finite dimensional representations to G requires a few comments.

Definition 0.4.1 A θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is called maximally split if

$$\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$$

is a maximal abelian subalgebra of $\mathfrak{p}_0$. A Cartan subgroup is called maximally split if its Lie algebra is. We often write H^s for such a subgroup.

Lemma 0.4.2 Any two θ-stable maximally split Cartan subgroups of G are conjugate under K_0. Fix such a subgroup H^s, and write

$$T^s = H^s \cap K, \qquad A^s = \exp(\mathfrak{h}_0^s \cap \mathfrak{p}_0).$$

Then

a) T^s meets every component of G.

b) $W(G,H) = W(G_0, H^s \cap G_0)$

$$= \{w \in W(\mathfrak{g}, \mathfrak{h}^s) \mid \theta w = w\theta\}$$

$$= \{w \in W(\mathfrak{g}, \mathfrak{h}^s) \mid w(\mathfrak{a}_0^s) = \mathfrak{a}_0^s\}.$$

Proof. The conjugacy statement is in [50], Corollary 1.3.1.5. The last two equalities in (b) are [50], Proposition 1.1.3.3. For (a), fix a component C of G, and pick an element $k \in K \cap C$. (This is possible by (0.1.2)(d).) Then $\mathrm{Ad}(k)(H^s)$ is another maximally split Cartan subgroup of G, so we can find $k_0 \in K_0$ so that

$$\mathrm{Ad}(k_0)[\mathrm{Ad}(k)H^s] = H^s .$$

So $k_0 k$ normalizes H^s. But $k_0 k$ also lies in C; so we may as well replace k by $k_0 k$, and assume that k normalizes H^s. Thus k defines an element

$$w = kH^s \in W(G,H^s) .$$

Since $\theta k = k$, w commutes with the action of θ on $\mathfrak{h}^s$. By the part of (b) we already know, it follows that

$$w \in W(G_0, H^s \cap G_0).$$

(This step in the argument completes the proof of (b)). So we can find $k_0 \in K_0$ normalizing $H^s \cap G_0$, so that $k_0 k$ actually centralizes $\mathfrak{h}_0$ (that is, defines the trivial element of $W(G,H^s)$). Thus $k_0 k$ is an element of T^s. Since $k_0 k$ also lies in C, this proves (a). Q.E.D.

Next, we recall the theory for Lie algebras.

Proposition 0.4.3 (Cartan-Weyl) Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra, and F an irreducible finite dimensional representation of $\mathfrak{g}$. Write $W = W(\mathfrak{g}, \mathfrak{h})$, and let $\Delta(F)$ denote the set of weights of $\mathfrak{h}$ in F (Definition 0.2.3).

a) $\Delta(F)$ is stable under W. If $\mu \in \Delta(F)$, and α is a root,

$$\langle \check{\alpha}, \mu \rangle \in \mathbb{Z} \qquad \text{Definition (0.2.5).}$$

b) There is a unique W orbit

$$W \cdot \lambda \subseteq \Delta(F)$$

characterized by

$$\langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle \geq 0, \quad \text{all } \mu \in \Delta(F);$$

equality holds if and only if $\mu \in W \cdot \lambda$. The weights in $W \cdot \lambda$ are called extremal weights of F. Every weight lies in the convex hull of the extremal weights.

c) Every extremal weight has multiplicity one.

d) If F' is any other irreducible finite dimensional representation of $\mathfrak{g}$, then F' is isomorphic to F if and only if F' and F have the same extremal weights.

e) Fix a positive root system $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$. Then there is a unique extremal weight λ which is dominant for Δ^+:

$$\langle \lambda, \alpha \rangle \geq 0, \quad \text{all } \alpha \in \Delta^+.$$

We call λ the highest weight of F.

f) The highest weight space of F is characterized by

$$F(\lambda) = \{v \in F \mid X_\alpha v = 0, \text{ all } \alpha \in \Delta^+\} ;$$

here X_α denotes a root vector for the root α.

g) If μ is any weight of F, then

$$\mu = \lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha \quad (n_\alpha \text{ a non-negative integer}).$$

There is also an existence result.

Proposition 0.4.4 (Cartan) Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra, and $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ a positive root system. Fix a weight $\lambda \in \mathfrak{h}^*$, and suppose that

$$\langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}, \qquad \text{all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}).$$

Then there is a unique irreducible finite dimensional representation F of extremal weight λ. If λ is dominant, i.e.,

$$\langle \alpha, \lambda \rangle \geq 0, \qquad \text{all } \alpha \in \Delta^+,$$

then λ is the highest weight of F.

Lemma 0.4.5 Let $H \subseteq G$ be a Cartan subgroup, and let $\Lambda^r \subseteq \mathfrak{h}^*$ be the lattice generated by the roots. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$; let X_α be a root vector. Define $\tilde{\alpha} \in \hat{H}$ to be the character by which H acts on $\mathbb{C}X_\alpha$. Suppose

$$\sum_{\alpha \in \Delta} n_\alpha \alpha = \sum_{\alpha \in \Delta} m_\alpha \alpha$$

for some integers n_α, m_α. Then (writing $\hat{H}$ additively)

$$\sum n_\alpha \tilde{\alpha} = \sum m_\alpha \tilde{\alpha}.$$

Proof. Because the simple roots form a $\mathbb{Z}$ basis of Λ^r, and the root vectors for simple roots and their negatives generate $\mathfrak{g}$, the relations of the form in the lemma are all integer combinations of relations

$$\alpha + \beta = \gamma \qquad (\alpha,\beta,\gamma \in \Delta(\mathfrak{g},\mathfrak{h}))$$
$$\alpha + (-\alpha) = 0 \qquad (\alpha \in \Delta(\mathfrak{g},\mathfrak{h}))$$

So it is enough to prove the corresponding relations for the $\tilde{\alpha}$. In the first case,

$$[X_\alpha, X_\beta] = cX_\gamma \qquad (c \neq 0);$$

so if $h \in H$,

$$\begin{aligned} c\tilde{\gamma}(h)X_\gamma &= c\,\mathrm{Ad}(h)X_\gamma \\ &= \mathrm{Ad}(h)[X_\alpha, X_\beta] \\ &= [\mathrm{Ad}(h)X_\alpha, \mathrm{Ad}(h)X_\beta] \\ &= [\tilde{\alpha}(h)X_\alpha, \tilde{\beta}(h)X_\beta] \\ &= \tilde{\alpha}(h)\tilde{\beta}(h)[X_\alpha, X_\beta] \\ &= (\tilde{\alpha} + \tilde{\beta})(h)(cX_\gamma). \end{aligned}$$

It follows that $\tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}$. In the second case, $[X_\alpha, X_{-\alpha}]$ is a non-zero element of $\mathfrak{h}$, and the proof is similar. Q.E.D.

Definition 0.4.6 Let $H \subseteq G$ be a Cartan subgroup. The root lattice in $\mathfrak{h}^*$ is the lattice generated by the roots; and the root lattice in $\hat{H}$ is the subgroup of $\hat{H}$ generated by the elements $\tilde{\alpha}$ of Lemma 0.4.5. We identify these lattices by

$$\Sigma\, n_\alpha\, \alpha \leftrightarrow \Sigma\, n_\alpha\, \tilde{\alpha}$$

(which makes sense by the lemma), and may therefore write

$$\Sigma\, n_\alpha\, \alpha \in \hat{H}\ .$$

Proposition 0.4.7 Let F be a finite dimensional irreducible representation of G. Then the restriction of F to G_0, or equivalently, the differentiated representation of $\mathfrak{g}$, is irreducible.

Proof. Choose a maximally split Cartan subgroup H^s of G, and a positive root system Δ^+ for $\mathfrak{h}^s$ in $\mathfrak{g}$. Fix an irreducible $\mathfrak{g}$ subrepresentation F_0 of F, and let $\gamma \in (\mathfrak{h}^s)^*$ be the highest weight of F_0. By Lemma 0.4.5 (f),

$$F^\gamma = \{v \in F(\gamma) \mid X_\alpha v = 0 \text{ for all } \alpha \in \Delta^+\}$$

is non-zero. Clearly F^γ is stable under H^s, so we can choose a weight vector $v \in F^\gamma$ for H^s. Let

$$F_1 = U(\mathfrak{g}) \cdot v,$$

the smallest $\mathfrak{g}$-invariant subspace containing v. Because it is generated by a highest weight vector, F_1 is an irreducible representation of $\mathfrak{g}$. We will show that $F_1 = F$. If $h \in H^s$, and $u \in U(\mathfrak{g})$,

$$\begin{aligned} h \cdot (uv) &= (\mathrm{Ad}(h)u)(hv) \\ &= c(\mathrm{Ad}(h)u)v \in F_1 \end{aligned}$$

since v is a weight vector for H^s. So F_1 is H^s-invariant By exponentiation, F_1 is G_0 invariant; so by Lemma 0.4.2, F_1 is invariant under $H^s \cdot G_0 = G$. Since F is irreducible, $F = F_1$. Q.E.D.

Definition 0.4.8 Suppose F is an irreducible representation of G, $H \subseteq G$ is a Cartan subgroup, and $\Delta^+ \subseteq \Delta(\mathfrak{g},\mathfrak{h})$ is a positive root system. The extremal weights of H

in F are the weights of H on the extremal $\mathfrak{h}$-weight spaces (Proposition 0.4.3(b)). The highest weight of H in F is the weight of H on the highest $\mathfrak{h}$-weight space (Proposition 0.4.3(e)).

These definitions make sense because of Proposition 0.4.7.

Proposition 0.4.9 Suppose $H \subseteq G$ is a Cartan subgroup, and $\Delta^+ \subseteq \Delta(\mathfrak{g},\mathfrak{h})$ is a positive system. Let F be an irreducible finite dimensional representation of G, of highest weight $\lambda \in \hat{H}$.

a) Any two weights of H in F are congruent modulo the root lattice in $\hat{H}$.

b) Every weight of H in F is of the form

$$\lambda - \sum_{\alpha\in\Delta^+} n_\alpha \alpha.$$

c) H acts by scalars on each $\mathfrak{h}$-weight space of F.

d) Suppose H^s is maximally split, and F' is another irreducible finite dimensional representation of G. Then F and F' are isomorphic if and only if they have the same highest H^s-weight.

Proof. For (a), let $\Lambda^r \subseteq \hat{H}$ denote the root lattice. Define

$$F_0 = \bigoplus_{\gamma\in(\lambda+\Lambda^r)} F(\gamma),$$

the span of the H-weights of F congruent to λ mod Λ^r. We want to show $F = F_0$; by Proposition 0.4.7, it is enough to show that F_0 is $\mathfrak{g}$-invariant. Clearly, it is $\mathfrak{h}$-invariant; and if X_α is a root vector for the root α, $\gamma \in (\lambda+\Lambda^r)$, and

$v \in F(\gamma)$, then clearly

$$X_\alpha \cdot v \in F(\gamma+\alpha).$$

Since $\alpha \in \Lambda^r$, $(\gamma+\alpha) \in (\lambda+\Lambda^r)$; so

$$X_\alpha \cdot v \in F_0 .$$

Since $\mathfrak{g}$ is spanned by $\mathfrak{h}$ and the root vectors, F_0 is $\mathfrak{g}$ invariant. Part (b) follows from (a) and Proposition 0.4.3(g). For (c), suppose $\mu_0 \in \mathfrak{h}^*$ is a weight of F. Let λ_0 be the differential of λ. Then we can write

$$\mu_0 = \lambda_0 - \Sigma\, n_\alpha \alpha .$$

By (a) and Lemma 0.4.5, H acts on $F(\mu_0)$ by the weight

$$\mu = \lambda + \Sigma\, n_\alpha\, \alpha \in \hat{H}.$$

For (d), the "only if" is clear; so suppose F' also has highest weight λ. Let

$$I : F \to F'$$

be an isomorphism of $\mathfrak{g}$ modules. Then I preserves the $\mathfrak{h}^s$ weight spaces. By the proof of (c), H^s acts by the same scalars in a given $\mathfrak{h}^s$ weight space for F and F'; so I intertwines the action of H^s. By exponentiation, I interwines the action of G_0; so it intertwines $H^sG_0 = G$. Q.E.D.

The existence result corresponding to Proposition 0.4.4 is a little tricky. The following result about Lie algebra representations will be needed.

Lemma 0.4.10 *Suppose $\mathfrak{g}$ is a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, and F is a finite dimensional representation of $\mathfrak{g}$. Fix a weight $\lambda \in \Delta(F, \mathfrak{h})$. Then there is an integer n so that the finite dimensional irreducible representation of extremal weight λ occurs in*

$$F \otimes T^n(\mathfrak{g}) .$$

Here $T^n(\mathfrak{g})$ is the n^{th} tensor power of $\mathfrak{g}$, and $\mathfrak{g}$ acts on it by the adjoint representation.

The reader can supply a proof without serious difficulty.

Definition 0.4.11 Suppose $H \subseteq G$ is a Cartan subgroup. The *weight lattice in $\hat{H}$* is the subgroup Λ of $\hat{H}$ consisting of weights of finite dimensional representations of G. The *weight lattice in $\mathfrak{h}^*$* is the additive subgroup of $\mathfrak{h}^*$ consisting of differentials of elements of the weight lattice in $\hat{H}$. If $\lambda \in \Lambda$, write $\overline{\lambda} \in \mathfrak{h}^*$ for the differential of λ.

If G has compact center, the weight lattice in $\mathfrak{h}^*$ really is a lattice; and if in addition G is connected, the weight lattice in $\hat{H}$ is really a lattice.

Proposition 0.4.12 *Suppose $H \subseteq G$ is a Cartan subgroup, and $\lambda \in \hat{H}$ lies in the weight lattice. Then there is an irreducible finite dimensional representation F of G of extremal weight λ. If H is maximally split, F is unique.*

Proof. Fix a finite dimensional representation F_1 of G containing the weight λ; we may as well assume F_1 is irreducible. By Proposition 0.4.9 (a), all weights of F_1 are congruent to λ modulo the root lattice. Choose n as in Lemma 0.4.10, and regard $T^n(\mathfrak{g})$ as a representation of G by the adjoint action. Then every weight of $T^n(\mathfrak{g})$ lies in the root lattice for H; so all weights of

$$F_2 = F_1 \otimes T^n(\mathfrak{g})$$

are congruent to λ modulo the root lattice. By Lemma 0.4.10, F_2 contains an irreducible G module F with an extremal weight μ having the same differential as λ. By Lemma 0.4.5, $\lambda = \mu$; so F has the desired property. The uniqueness statement is Proposition 0.4.9(d). Q.E.D.

In general, it can be shown that the number of distinct irreducible representations of extremal weight λ is

$$|W(G,H)/W(G_0, H \cap G_0)| .$$

For example, $SL^{\pm}(2, \mathbb{R})$ (see Section 1.4) has two representations of each positive dimension, and these have the same weights under the compact Cartan subgroup SO(2). We will not use this, so the proof is left to the reader.

Chapter 1. SL(2, $\mathbb{R}$).

One of the most pleasant techniques in the theory of reductive groups is reduction to three dimensional simple subgroups. We will need a very detailed understanding of the structure and representation theory of the group SL(2, $\mathbb{R}$), and various closely related groups. This material will be quite familiar to most readers, except perhaps in the disconnected case; and they can refer to this chapter as needed or not at all. For other readers, the first three sections of this chapter can be taken as an introduction; some material from Chapter 0 has been repeated here for clarity.

§1. Group representations and Lie algebra representations.

Throughout the first three sections, G will denote the group

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

Definition 1.1.1 A (continuous) representation of the group G on a Hilbert space is a pair $(\pi, \mathcal{H})$, with $\mathcal{H}$ a complex Hilbert space and

$$\pi : G \to B(\mathcal{H})$$

a homomorphism from G to the semigroup of bounded operators on $\mathcal{H}$. We assume that π is continuous with respect to the weak topology on $B(\mathcal{H})$; that is, we assume that the map

$$G \times \mathcal{H} \to \mathcal{H}, \quad (g,v) \mapsto \pi(g)v$$

is continuous. The representation is called unitary if, for each $g \in G$, $\pi(g)$ is a unitary operator on $\mathcal{H}$; it is called (topologically) irreducible if there is no proper closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ which is invariant under all of the operators $\pi(g)$.

We may often refer to π or $\mathcal{H}$ alone as a representation of G, if the other is clear from the context.

Example 1.1.2 Let

$$\mathcal{H} = L^2\left(\mathbb{R}, \frac{dx}{x^2+1}\right)$$

be the Hilbert space of measurable functions f on the real line such that

$$||f||^2 = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{x^2+1}\, dx < \infty .$$

If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad f \in \mathcal{H}$$

define

$$[\pi(g)f](x) = f\left(\frac{dx - b}{-cx + a}\right) .$$

Then $(\pi, \mathcal{H})$ is a continuous representation of G, as can be checked by a messy calculus computation (the only messy part is continuity). It is not irreducible: the constant functions clearly form a one-dimensional invariant subspace $\mathcal{H}_0$. It is less obvious that the natural quotient representation $(\bar{\pi}, \mathcal{H}/\mathcal{H}_0)$ (whose definition we leave to the reader) is also reducible: it is a direct sum of two infinite dimensional irreducible representations. We will prove this in a moment.

For the rest of this section, K will denote the circle group

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\} \subseteq SL(2, \mathbb{R}).$$

Lemma 1.1.3 Suppose $\mathcal{H}$ is a Hilbert space, and

$$\rho: \quad K \to U(\mathcal{H})$$

is a (continuous) unitary representation of K (that is, a weakly continuous homomorphism into the group of unitary operators on $\mathcal{H}$). Then there is a unique collection of closed, mutually orthogonal subspaces

$$\{\mathcal{H}_n \subseteq \mathcal{H} \mid n \in \mathbb{Z}\}$$

such that

a) $\mathcal{H} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$ (Hilbert space direct sum);

b) Each $\mathcal{H}_n$ is an invariant subspace for $\rho(K)$;

c) If $v \in \mathcal{H}_n$, and

$$k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K,$$

then

$$\rho(k)v = e^{in\theta} v.$$

This can be regarded as a very special case of the spectral theorem. The subspace $\mathcal{H}_n$ is the image of the projection operator P_n defined as follows. Set

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then

$$P_n(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \rho(k(\theta))v \, d\theta.$$

We leave the details to the reader.

Definition 1.1.4 In the setting of Lemma 1.1.3, define the space of K-finite vectors in $\mathcal{H}$ by

$$\mathcal{H}_K = \{v \in \mathcal{H} \mid \text{the span of } \rho(K) \cdot v \text{ is finite dimensional}\}$$
$$= \text{algebraic direct sum of the } \mathcal{H}_n,$$

a dense subspace of $\mathcal{H}$.

Definition 1.1.5 Let $(\pi, \mathcal{H})$ be a representation of G, and suppose that the restriction of π to K is unitary. Define the subspaces $\mathcal{H}_n$ as in Lemma 1.1.3. Then π is called admissible if

$$\dim \mathcal{H}_n < \infty, \quad \text{all } n \in \mathbb{Z}.$$

Write $\mathfrak{g}_0$ for the Lie algebra of G.

Proposition 1.1.6 (Harish-Chandra [12]). Suppose $(\pi, \mathcal{H})$ is an admissible representation of G. Fix $x \in \mathfrak{g}_0$ and $v \in \mathcal{H}_K$ (Definition 1.1.4). Then

$$\lim_{t \to 0} \frac{1}{t}(\pi(\exp tx)v - v) = \pi(x)\, v$$

exists, and defines a Lie algebra representation π of $\mathfrak{g}_0$ on $\mathcal{H}_K$. There is a bijection between the set of closed, G-invariant subspaces of $\mathcal{H}$, and arbitrary $\mathfrak{g}_0$-invariant subspaces of $\mathcal{H}_K$; a closed subspace $\mathcal{S}$ corresponds to

$$\mathcal{S}_K = \mathcal{S} \cap \mathcal{H}_K,$$

and

$$\mathcal{S} = \overline{\mathcal{S}_K}.$$

<u>In particular, $\mathcal{H}$ is topologically irreducible if and only if $\mathcal{H}_K$ is algebraically irreducible.</u>

<u>Example 1.1.2 continued.</u> In the representation defined earlier,

$$[k(\theta) \cdot f](0) = f(-\tan\theta) .$$

So the condition $f \in \mathcal{H}_n$ means at least that

$$f(-\tan\theta) = [k(\theta) \cdot f](0) = e^{in\theta} \cdot f(0)$$

Taking $\theta = \pi$, we find

$$f(0) = (-1)^n f(0) ,$$

which is possible only if $n = 2m$ is even. In that case, f must be $f(0)$ times the function f_{2m} defined by

$$f_{2m}(x) = e^{-2im(\tan^{-1} x)} .$$

An elementary calculation shows that

$$\left\{ \frac{1}{\sqrt{\pi}} f_{2m} \;\middle|\; m \in \mathbb{Z} \right\}$$

is an orthonormal basis of $\mathcal{H}$. We can now compute the action of $\mathfrak{g}_0$ in this basis. Consider first the element

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}_0 ; \quad \exp(ta) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} .$$

We have

$$\begin{aligned} [\pi(a)f](x) &= \lim_{t\to 0} \frac{1}{t} [\pi(\exp ta)f(x) - f(x)] \\ &= \lim_{t\to 0} \frac{1}{t} \left\{ f(e^{-2t}x) - f(x) \right\} \\ &= -2x \frac{df}{dx} \end{aligned}$$

by the chain rule. So

$$[\pi(a)f_{2m}](x) = -\frac{4imx}{x^2+1} e^{2im(\tan^{-1} x)} .$$

Write $\theta = \tan^{-1} x$; then

$$\begin{aligned} \frac{x}{x^2+1} &= (\tan\theta)/(\tan^2\theta + 1) = \sin\theta\cos\theta , \\ &= \frac{1}{4i} \left[e^{2i\theta} - e^{-2i\theta} \right] . \end{aligned}$$

This leads to the formula

$$\pi(a)\, f_{2m} = -m\, f_{2m+2} + m\, f_{2m-2} .$$

Similarly one calculates

$$\left[\pi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f \right](x) = -\frac{df}{dx}$$

$$\pi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f_{2m} = \frac{im}{2} f_{2m+2} + im\, f_{2m} + \frac{im}{2} f_{2m-2}$$

$$\left[\pi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} f \right](x) = x^2 \frac{df}{dx}$$

$$\pi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} f_{2m} = +\frac{im}{2} f_{2m+2} - im\, f_{2m} + \frac{im}{2} f_{2m-2} .$$

From these formulas, one checks easily that $\mathcal{H}_K$ has exactly three proper invariant subspaces: $\mathcal{H}_0$ (as defined earlier, the span of f_0), and

$$\mathcal{H}_K^+ = \text{span } \{f_{2m} \mid m \geq 0\}$$

$$\mathcal{H}_K^- = \text{span } \{f_{2m} \mid m \leq 0\} .$$

Write $\mathcal{H}^\pm$ for the closure of $\mathcal{H}_K^\pm$ in $\mathcal{H}$. Then Proposition 1.1.6 implies that the representations of G on $\mathcal{H}^\pm/\mathcal{H}_0$ are both irreducible, as claimed earlier.

Proposition 1.1.6 provides an algebraic way to study representations of G. To complete this method, a converse of some sort is required. This can be formulated as follows.

Definition 1.1.7 A $(\mathfrak{g}_0,K)$ module (π,V) is a Lie algebra representation of $\mathfrak{g}_0$, and a representation of K, both denoted by π and both on the same complex vector space V. We require also that there be subspaces

$$\{V_n \subseteq V \mid n \in \mathbb{Z}\}$$

such that

a) $V = \bigoplus_{n\in\mathbb{Z}} V_n$ (algebraic direct sum);

b) each V_n is invariant under $\pi(K)$;

c) if $v \in V_n$ and $\theta \in \mathbb{R}$, then

$$\pi(k(\theta))v = e^{in\theta}\, v;$$

d) the action of the Lie algebra $\mathfrak{k}_0$ of K in the Lie algebra representation is the differential of the action of K. More explicitly, if $v \in V_n$, and

$$h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}_0,$$

$$\pi(h)v = in \cdot v.$$

We say that (π,V) is admissible if

$$\dim V_n < \infty \qquad (n \in \mathbb{Z}).$$

If $(\pi,\mathcal{H})$ is an admissible representation of G, the $(\mathfrak{g}_0,K)$ module $(\pi,\mathcal{H}_K)$ defined by Proposition 1.1.6 is called the Harish-Chandra module of π. If $(\pi_1,\mathcal{H}_1)$ and $(\pi_2,\mathcal{H}_2)$ are two admissible representations of G, we say that they are infinitesimally equivalent if their Harish-Chandra modules $(\pi_i,(\mathcal{H}_i)_K)$ are isomorphic.

The converse we want for Proposition 1.1.6 is

Proposition 1.1.8 Every irreducible admissible $(\mathfrak{g}_0,K)$ module arises as the Harish-Chandra module of an irreducible representation of G.

The representations of G in question will be constructed in Section 1.3.

There is an obvious notion of equivalence of representations of G (the existence of a bounded invertible intertwining operator). This implies infinitesimal equivalence. The converse is not obvious, and may in fact be false; but it is true in an important special case.

Proposition 1.1.9 (Harish-Chandra [12]). Any unitary irreducible representation of G is admissible. Two such representations are infinitesimally equivalent if and only if they are unitarily equivalent.

The importance of this proposition is that it allows us to study unitary representations by algebraic methods: the definitions of "admissible" and "infinitesimal equivalence", which were arranged for algebraic convenience, neither omit any unitary representations, nor overlook some subtlety of their structure. Since it is unitary representations which arise most often in applications, this shows that we have not defined all of the interesting problems out of existence.

§2. Structure of Harish-Chandra modules.

Because of Proposition 1.1.8, we want to describe all the irreducible admissible $(\mathfrak{g}_0, K)$ modules. (Actually, an irreducible $(\mathfrak{g}_0, K)$ module is automatically admissible, but we will not need this fact.) I have made no attempt to attribute the ideas of this section; most go back to [1]. Let $G = SL(2,\mathbb{R})$, $K = SO(2)$ as before, and put

$$\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}} ,$$

the complexified Lie algebra of G. Representations of a real Lie algebra on a complex vector space can be complexified; so we may as well consider $(\mathfrak{g}, K)$ modules (defined in precisely the same way, with $\mathfrak{g}$ replacing $\mathfrak{g}_0$ in Definition 1.1.7). Our first step is to choose a convenient basis of $\mathfrak{g}$. We take

$$H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad [H,X] = 2X$$

$$X = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \qquad [H,Y] = -2Y \tag{1.2.1}$$

$$Y = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \qquad [X,Y] = H,$$

which are easily seen to be a basis satisfying the indicated commutation relations.

Lemma 1.2.2 (cf. [22], Lemma 7.2) *Let (π, V) be a $(\mathfrak{g}, K)$ module (Definition 1.1.7), with*

$$V = \bigoplus_{n\in\mathbb{Z}} V_n .$$

Then if $v \in V_n$,

$$\pi(H)v = nv$$

$$\pi(X)v \in V_{n+2}$$

$$\pi(Y)v \in V_{n-2} .$$

Proof. The first formula is (d) of Definition 1.1.7. For the second,

$$\begin{aligned}\pi(H)\pi(X)v &= \pi(X)\pi(H)v + 2\pi(X)v \qquad \text{(by 1.2.1)}\\ &= n\pi(X)v + 2\pi(X)v \\ &= (n+2)\pi(X)v .\end{aligned}$$

So, again, by 1.1.7(d),

$$\pi(X)v \in V_{n+2}.$$

The last formula is proved in the same way. Q.E.D.

Example 1.2.3 Let (π, V) be the Harish-Chandra module of the representation of Example 1.1.2. If we combine the calculations before (1.1.7) with the definitions in (1.2.1), we find the formulas

$$\pi(H)f_{2m} = 2m \cdot f_{2m}$$

$$\pi(X)f_{2m} = -m \cdot f_{2m+2}$$

$$\pi(Y)f_{2m} = m\, f_{2m-2} .$$

Lemma 1.2.4 The element

$$\begin{aligned}\Omega &= H^2 - 2H+1 + 4XY \\ &= H^2 + 1 + 2XY + 2YX \\ &= H^2 + 2H+1 + 4YX \\ &= (H-1)^2 + 4XY \\ &= (H+1)^2 + 4YX\end{aligned}$$

is in the center of the enveloping algebra $U(\mathfrak{g})$. It acts by a scalar on any irreducible admissible $(\mathfrak{g},K)$ module (π,V). Fix a square root λ of this scalar. Then if $v \in V_n$, we have

$$\pi(XY)v = \frac{1}{4}(\lambda^2 - (n-1)^2)v$$

$$\pi(YX)v = \frac{1}{4}(\lambda^2 - (n+1)^2)v .$$

Proof. The fact that the various formulas for Ω are equivalent follows from the identity in $U(\mathfrak{g})$

$$XY = YX + H,$$

which is part of (1.2.1). To see that Ω is in the center of $U(\mathfrak{g})$, we must show that it commutes with X,Y, and H. Using the commutation relations in (1.2.1), we find

$$\begin{aligned}[X,\Omega] &= [X,H^2] - 2[X,H] + 4[X,XY] \\ &= [X,H]H + H[X,H] - 2[X,H] + 4X[X,Y] \\ &= -2XH - 2HX + 4X + 4XH \\ &= -2XH - (2XH + 4X) + 4X + 4XH \\ &= 0 .\end{aligned}$$

The calculations for Y and H are similar. (The element Ω is the Casimir element plus one -- cf. [22], 22.1.)

Since Ω is central in $U(\mathfrak{g})$, any eigenspace of $\pi(\Omega)$ is a $\mathfrak{g}$-invariant subspace of V, and so (by irreducibility) is 0 or V. So to show that $\pi(\Omega)$ is a scalar, we only have to show that it has an eigenvector. Fix n so that V_n is non-zero. Since Ω commutes with H, it preserves the finite dimensional space V_n; so it must have an eigenvector.

The last two formulas are simply the last two formulas for Ω, together with the information

$$\pi(\Omega)v = \lambda^2 v, \quad \pi(H)v = n\ v.$$

Q.E.D.

Before treating arbitrary $(\mathfrak{g},K)$ modules, we need to take care of a small technical problem.

Lemma 1.2.5 Suppose (π,V) is an irreducible admissible $(\mathfrak{g},K)$ module, and $v \in V_n$ is non-zero.

a) If $\pi(X)v = 0$, then the elements

$$w_{n-2m} = \pi(Y)^m v$$

span V; and in particular, V_{n+2} is zero.

b) If $\pi(Y)v = 0$, then the elements

$$w_{n+2m} = \pi(X)^m v$$

span V; and in particular, V_{n-2} is zero.

c) If $\pi(YX)v = 0$, then $\pi(X)v = 0$.

d) If $\pi(XY)v = 0$, then $\pi(Y)v = 0$.

Proof. Consider (a). Let W denote the span of $\{w_{n-2m}\}$. Since v is in W, W is non-zero. We claim that W is invariant under $\mathfrak{g}$. Since V is irreducible, it will follow that

$W = V$. By Lemma 1.2.2,

$$w_{n-2m} \in V_{n-2m} ;$$

so

$$\pi(H) w_{n-2m} = (n - 2m) w_{n-2m} .$$

So $\pi(H)$ leaves W invariant; and (trivially) $\pi(Y)$ does also. By hypothesis

$$\pi(X) w_n = \pi(X) v = 0 \in W;$$

and if $m > 0$,

$$\pi(X) w_{n-2m} = \pi(XY) w_{n-2(m-1)} = c \cdot w_{n-2(m-1)},$$

where c is a scalar (computed in Lemma 1.2.4). So $\pi(X)$ also leaves W invariant, proving that $W = V$. Since w_{n-2m} lies in V_{n-2m}, V_{n+2} is zero. Part (b) is proved in exactly the same way. For (c), suppose (to get a contradiction) that

$$\pi(X) v = v_1 \neq 0.$$

By Lemma 1.2.2, v_1 lies in V_{n+2}; and by hypothesis

$$\pi(Y) v_1 = \pi(YX) v = 0.$$

If we apply (b) with v replaced by v_1, we conclude that

$$V_{(n+2)-2} = V_n = 0 ,$$

contradicting $v \neq 0$. This proves (c), and (d) is identical.

Q.E.D.

Obviously, parts (a) and (b) of this lemma depend on the irreducibility of V. What is not so obvious is that (c) and (d) do as well. This can be seen (for (c), say) by taking (π, V) as in Example 1.2.3, with $v = f_{-2}$.

Lemma 1.2.6 (cf. [22], Lemma 7.2). _Suppose (π, V) is an irreducible admissible $(\mathfrak{g}, K)$ module, λ is as in Lemma 1.2.4, and $v \in V_n$ is non-zero. Define (inductively, for $m \geq 0$)_

$$w_n = v$$

$$w_{n+2(m+1)} = \begin{cases} \left(\frac{2}{\lambda + (n+2m+1)}\right) \pi(X) w_{n+2m} & \text{if } \lambda \neq -n-2m-1 \\ 0 & \text{if } \lambda = -n-2m-1 . \end{cases}$$

$$w_{n-2(m+1)} = \begin{cases} \left(\frac{2}{\lambda - (n-2m-1)}\right) \pi(Y) w_{n-2m} & \text{if } \lambda \neq n-2m-1. \\ 0 & \text{if } \lambda = n-2m-1. \end{cases}$$

Then, whenever $w_j \neq 0$,

a) $$\pi(H) w_j = j \cdot w_j$$

b) $$\pi(X) w_j = \frac{1}{2}(\lambda + (j+1)) \, w_{j+2}$$

c) $$\pi(Y) w_j = \frac{1}{2}(\lambda - (j-1)) \, w_{j-2} .$$

In particular, the non-zero w_j form a basis of V.

Proof. Since w_j is a multiple of either

$$\pi(X)^{\frac{1}{2}(j-n)} v$$

or of

$$\pi(Y)^{\frac{1}{2}(n-j)} v ,$$

formula (a) follows from Lemma 1.2.2. Consider next (b). There are several possibilities. Suppose first that $j \geq n$; write $j = n + 2m$. If $-\lambda \neq n+2m+1$, then (b) is just the definition of w_{n+2m+2}. If $-\lambda = n+2m+1$, then by Lemma 1.2.4,

$$\pi(YX)w_{n+2m} = \frac{1}{4}(\lambda^2 - (n+2m+1)^2)w_{n+2m}$$
$$= 0 .$$

By Lemma 1.2.5(c),

$$\pi(X)w_{n+2m} = 0 .$$

Since w_{n+2m+2} is zero in this case, (b) holds. Next, suppose j is negative; write

$$j = n-2m-2 ,$$

with $m \geq 0$. If $\lambda \neq n-2m-1$, then by definition of w_{n-2m-2},

$$\pi(X)w_{n-2m-2} = \pi(X)\pi(Y)\left(\frac{2}{\lambda-(n-2m-1)}\right) w_{n-2m} .$$

By Lemma 1.2.4, the right side is

$$\frac{1}{4}(\lambda^2 - (n-2m-1)^2) \frac{2}{\lambda-(n-2m-1)} w_{n-2m}$$
$$= \frac{1}{2}(\lambda + (n-2m-1)) w_{n-2m}$$
$$= \frac{1}{2}(\lambda + (j+1)) w_{j+2} ,$$

as claimed. So suppose finally that $\lambda = n-2m-1$. In this case $w_j = w_{n-2m-2}$ is zero by definition, so there is nothing to prove. So formula (b) is completely established. The proof of (c) is very similar. These formulas show that the span of $\{w_j\}$ is invariant under $\mathfrak{g}$. Since it contains the non-zero element $w_n = v$, it must be all of V. Q.E.D.

<u>Lemma 1.2.7</u> <u>In the setting of Lemma 1.2.6, suppose $m \geq 0$. Then</u>

a) <u>$w_{n+2(m+1)} = 0$ if and only if $w_{n+2m} = 0$, or $\lambda = \pm(n+2m+1)$.</u>

b) <u>$w_{n-2(m+1)} = 0$ if and only if $w_{n-2m} = 0$, or $\lambda = \pm(n-2m-1)$.</u>

<u>Proof.</u> Consider (a). We may as well suppose $w_{n+2m} \neq 0$ (for, if not, w_{n+2m+2} is also zero, and (a) holds). If $\lambda = -(n+2m+1)$, then by definition $w_{n+2(m+1)} = 0$, and (a) holds; so we may suppose that $\lambda \neq -(n+2m+1)$. In this case, $w_{n+2(m+1)}$ is a non-zero multiple of $\pi(X)w_{n+2m}$; so we want to show

$$\pi(X)w_{n+2m} = 0 \iff \lambda = n+2m+1.$$

By Lemma 1.2.5(c), and the last formula in Lemma 1.2.4,

$$\pi(X)w_{n+2m} = 0 \iff \pi(YX)\, w_{n+2m} = 0$$
$$\iff (\lambda^2 - (n+2m+1)^2)\, w_{n+2m} = 0$$
$$\iff \lambda = \pm(n+2m+1), \text{ or } w_{n+2m} = 0.$$

The proof of (b) is similar. Q.E.D.

Thus the entire structure of V can be deduced from two things: the scalar $\pi(\Omega)$, and a single integer n such that $V_n \neq 0$. We state that as

<u>Corollary 1.2.8</u> <u>Suppose (π,V) and (π',V') are irreducible admissible $(\mathfrak{g},K)$ modules. Then they are isomorphic if and only if</u>

a) $\pi(\Omega) = \pi'(\Omega)$ <u>(cf. Lemma 1.2.4);</u>

b) there is an $n \in \mathbb{Z}$ such that V_n and V'_n are both non-zero.

Lemma 1.2.9 For each complex number c, and each integer n, there is an irreducible admissible $(\mathfrak{g},K)$ module (π,V) such that

a) $$\pi(\Omega) = c,$$

b) $$V_n \neq 0.$$

Proof. Lemmas 1.2.6 and 1.2.7 describe the structure of V very explicitly. One only has to check that the formulas there really define a $(\mathfrak{g},K)$ module. This amounts to checking that 1.2.6(a) - (c) define a Lie algebra representation. The only relation which offers any difficulty is

$$(\pi(X)\pi(Y) - \pi(Y)\pi(X))w_j = \pi(H)w_j .$$

Here one must consider several cases, depending on which of $w_{j\pm 2}$ are zero. In all cases the calculation is easy. Q.E.D.

These results completely describe all the irreducible admissible $(\mathfrak{g},K)$ modules. It is convenient, however, to organize the results a little differently.

Definition 1.2.10 Let (π,V) be a $(\mathfrak{g},K)$ module. The set of K-types of (π,V) is

$$\{n \in \mathbb{Z} \mid V_n \neq 0\}.$$

An integer $n \in \mathbb{Z}$ is called a lowest K-type of (π,V) if it is a K-type of (π,V) and $|n|$ is minimal with respect to that property.

Lemma 1.2.11 Suppose (π,V) is an irreducible admissible $(\mathfrak{g},K)$ module of lowest K-type n, and $|n| > 1$. Then

$$\pi(\Omega) = (|n| - 1)^2,$$

so that, in particular, (π,V) is uniquely determined by n.

Proof. Suppose for definiteness that $n > 0$; the other case is similar. Fix $v \in V_n$ not equal to zero. Since $|n| > 1$,

$$|n - 2| < |n|;$$

so by Definition 1.2.10, V_{n-2} is zero. By Lemma 1.2.2,

$$\pi(Y)v = 0.$$

By the next to last formula for Ω in Lemma 1.2.4,

$$\begin{aligned}\pi(\Omega)v &= \pi((H-1)^2)v + 4\pi(XY)v \\ &= \pi((H-1)^2)v \\ &= (n-1)^2 v;\end{aligned}$$

so

$$\pi(\Omega) = (n-1)^2,$$

as claimed. The uniqueness statement is Corollary 1.2.8.

Q.E.D.

Lemma 1.2.12 Suppose $n \in \mathbb{Z}$, and $|n| \geq 1$. Let (π,V) be the irreducible admissible $(\mathfrak{g},K)$ module such that $V_n \neq 0$, and

$$\pi(\Omega) = (|n| - 1)^2.$$

Then the set of K-types of (π,V) is

$$\{n + 2(\operatorname{sgn} n)m \mid m = 0,1,2,\ldots\}.$$

In particular, (π,V) has lowest K-type n.

Proof. Again we suppose for definiteness that n is positive. Using (say) Lemma 1.2.6, we see that it is enough to show that

$$w_{n-2} = 0, \quad w_{n+2m} \neq 0 \qquad (m = 1,2,3,\ldots).$$

But this is immediate from Lemma 1.2.7. Q.E.D.

Definition 1.2.13 Suppose $n \neq 0$. Define

$$(\pi, X_d(n)),$$

the discrete series representation with parameter n, to be the unique irreducible admissible $(\mathfrak{g},K)$ module with lowest K-type $n + \operatorname{sgn} n$. (This exists by the two preceding lemmas.) If $\mu = 0,1$, or -1, and λ is a complex number, define the continuous series representation

$$(\pi, \overline{X}_c(\lambda)(\mu))$$

to be the unique irreducible admissible $(\mathfrak{g},K)$ module containing the K-type μ, such that $\pi(\Omega) = \lambda^2$. (This exists by Lemma 1.2.9).

Proposition 1.2.14 Let (π,V) be an irreducible admissible $(\mathfrak{g},K)$ module, and let $\mu \in \mathbb{Z}$ be a lowest K-type of it.

a) If $|\mu| > 1$, then

$$V \cong X_d(\mu - \operatorname{sgn} \mu)$$

In this case, μ is the unique lowest K-type of V.

b) Suppose $|\mu| \leq 1$; let λ be a square root of $\pi(\Omega)$ (cf. Lemma 1.2.4). Then

$$V \cong \overline{X}_c(\lambda)(\mu).$$

Furthermore,

$$\overline{X}_c(\lambda)(\mu) \cong \overline{X}_c(\lambda')(\mu) \iff \lambda = \pm\lambda'.$$

If $\mu = 0$, then μ is the unique lowest K-type of V. If $\mu = \pm 1$, and $\lambda \neq 0$, then the set of lowest K-types of V is

$$\{\mu, -\mu\}.$$

In this case

$$\overline{X}_c(\lambda)(\mu) \cong \overline{X}_c(\lambda)(-\mu) \qquad (\lambda \neq 0).$$

If $\mu = \pm 1$ and $\lambda = 0$, then μ is the unique lowest K-type of V.

This proposition gives a precise classification of the irreducible admissible $(\mathfrak{g},K)$ modules, although the more complete structural information of Lemmas 1.2.6 and 1.2.7 has been lost. A result very similar to this proposition will be proved for general reductive groups, however (Theorem 6.5.12); and there seems to be little hope at present of finding reasonable generalizations of Lemmas 1.2.6 and 1.2.7.

Proof. Part (a) is Lemma 1.2.11. For (b), the fact that V is isomorphic to the indicated standard representation is

Corollary 1.2.8; and since

$$\pi(\Omega) = \lambda^2,$$

the second assertion is also included in that corollary. If $\mu = 0$, the remaining assertion is trivial. Suppose then for definiteness that $\mu = 1$. Obviously, the only other candidate for a lowest K-type of V is -1. By Lemma 1.2.7(b) (with $n = 1$ and $m = 0$), this K-type occurs if and only if

$$\lambda \neq \pm(n+2m-1) = 0 .$$

It remains only to show that if $\lambda \neq 0$,

$$\overline{X}_c(\lambda)(\mu) \cong \overline{X}_c(\lambda)(-\mu).$$

By what we have just shown, these representations have the K-types $\pm\mu$ in common; and Ω acts by λ^2 in both. By Corollary 1.2.8, they are isomorphic. Q.E.D.

Lemma 1.2.12 gives a satisfactory description of the discrete series of $(\mathfrak{g},K)$ modules; we want to record the corresponding result for the continuous series.

Lemma 1.2.15 Fix $\mu = 0, 1$, or -1, and $\lambda \in \mathbb{C}$.

a) If λ is not an integer of the same parity as $\mu+1$, then the K-types of $\overline{X}_c(\lambda)(\mu)$ are

$$\{\mu + 2m \mid m \in \mathbb{Z}\}.$$

b) If λ is a non-zero integer of the same parity as $\mu+1$, then $\overline{X}_c(\lambda)(\mu)$ is the finite dimensional representation of highest weight $|\lambda| - 1$ (cf. [22], Theorem 7.2). Its K-types are

$$\{|\lambda| - 1, |\lambda| - 3, \ldots -|\lambda| + 1\}.$$

c) If $\lambda = 0$ and $\mu = \pm 1$, then the K-types of $\overline{X}_c(\lambda)(\mu)$ are

$$\{\mu + 2(\mathrm{sgn}\ \mu)m \mid m = 0,1,2,\ldots\}.$$

Proof. Take $n = \mu$ in Lemma 1.2.6, and construct a basis of $\overline{X}_c(\lambda)(\mu)$. By Lemma 1.2.7, all the $w_{\mu+2m}$ are non-zero unless

$$\lambda = \pm\mu \pm 2m \pm 1$$

for some m. This proves (a). In case (b), suppose for definiteness that $\mu = 0$ and λ is positive; the other cases are similar. By hypothesis, λ is odd; so

$$\lambda = 2r + 1,$$

with r a non-negative integer. By Lemma 1.2.7,

$$w_{\mu+2m} = w_{2m} \neq 0 \iff m = 0, \pm 1, \ldots, \pm r .$$

Thus the K-types of $\overline{X}_c(\lambda)(\mu)$ are

$$\{2r, 2r-2, \ldots , -2r+2, -2r\},$$

which agrees with the statement in the lemma. Finally, (c) is contained in Lemma 1.2.12. Q.E.D.

§3. The principal series.

The formulas of Lemma 1.2.6 are so nice that the somewhat complicated condition for w_j to be non-zero seems unnatural. In fact, one can make a perfectly good representation without this condition. We retain the notation of Sec.2.

Definition 1.3.1 Fix $\lambda \in \mathbb{C}$ and a "parity" $\varepsilon = \pm 1$. The principal series representation (or $(\mathfrak{g}, K)$ module) with parameters ε and λ, $X_c(\varepsilon \otimes \lambda)$, is defined as follows

a) $X_c(\varepsilon \otimes \lambda)$ has a basis

$$\{w_n \mid n \equiv \varepsilon \bmod 2\mathbb{Z}\}.$$

b) If $k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then

$$\pi(k(\theta)) \cdot w_n = e^{in\theta} w_n.$$

c) The basis (1.2.1) of $\mathfrak{g}$ acts by the formulas of Lemma 1.2.6; that is,

$$\pi(H)w_n = n\, w_n$$
$$\pi(X)w_n = \tfrac{1}{2}(\lambda + (n+1))\, w_{n+2}$$
$$\pi(Y)w_n = \tfrac{1}{2}(\lambda - (n-1))\, w_{n-2}\ .$$

To see that this definition makes sense, we need only check that the formulas of (c) define a Lie algebra representation. This is very easy to do, but the following argument also works. The relations to be proved are obviously algebraic in λ. If λ is not an integer, then Lemma 1.2.15(a) shows that the module we want is a certain $\overline{X}_c(\lambda)(\mu)$ (whose existence was proved by the calculation we are now avoiding). So the relations in question hold when λ is not an integer; so they always hold.

Example 1.3.2 Take $\lambda = -1$, $\varepsilon = +1$. Then the defining relations for $X_c(1 \otimes (-1)) = V$ are

$$\pi(H)w_{2m} = 2m \cdot w_{2m}$$
$$\pi(X)w_{2m} = m \cdot w_{2m+2}$$
$$\pi(Y)w_{2m} = -m \cdot w_{2m-2}\ .$$

Comparing this with the calculation in Example 1.2.3, we see that (π, V) is isomorphic to the Harish-Chandra module of the representation defined in Example 1.1.2; the isomorphism takes w_{2m} to $(-1)^m f_{2m}$.

All of the principal series have a similar analytic realization, as we will see in a moment. First we will describe the subspace structure of the principal series representations.

Proposition 1.3.3 Fix $\lambda \in \mathbb{C}$ and a parity $\varepsilon = \pm 1$; and let (π, V) denote the principal series representation $X_c(\varepsilon \otimes \lambda)$ (Definition 1.3.1). Let μ be 0 if $\varepsilon = 1$, and either ± 1 if $\varepsilon = -1$.

a) $\pi(\Omega) = \lambda^2$ (cf. Lemma 1.2.4).

b) If λ is not an integer of parity $-\varepsilon$, then V is irreducible, and isomorphic to $\overline{X}_c(\lambda)(\mu)$ (cf. Definition 1.2.13).

c) If λ is a positive integer of parity $-\varepsilon$, then V contains as a submodule $X_d(\lambda) \oplus X_d(-\lambda)$ (cf. Definition 1.2.13). This submodule, and its two summands, are the only proper invariant subspaces of V. The quotient is

$$V/(X_d(\lambda) \oplus X_d(-\lambda)) \cong \overline{X}_c(\lambda)(\mu),$$

the finite dimensional module of highest weight $\lambda - 1$.

d) *If λ is a negative integer of parity $-\varepsilon$, then V contains as its unique irreducible submodule the finite dimensional representation $\overline{X}_c(\lambda)(\mu)$ of highest weight $-\lambda - 1$. The quotient is*

$$V/\overline{X}_c(\lambda)(\mu) \cong X_d(\lambda) \oplus X_d(-\lambda).$$

e) *If $\lambda = 0$ and $\varepsilon = -1$, then*

$$V \cong \overline{X}_c(0)(\mu) \oplus \overline{X}_c(0)(-\mu).$$

Proof. For (a), we use the last formula for Ω in Lemma 1.2.[illegible] and Definition 1.3.1. This gives

$$\begin{aligned}\pi(\Omega)w_n &= \pi((H+1)^2)w_n + 4\pi(YX)w_n \\ &= (n+1)^2 w_n + 2\pi(Y)(\lambda+(n+1))w_{n+2} \\ &= [(n+1)^2 + (\lambda-n-1)(\lambda+n+1)]w_n \\ &= [(n+1)^2 + \lambda^2 - (n+1)^2]w_n \\ &= \lambda^2 w_n ,\end{aligned}$$

as claimed. For the rest of the proposition, it is an elementary exercise to read off the invariant subspace structure of $X_c(\varepsilon \otimes \lambda)$ from the structure constants given in Definition 1.3.1. To identify the various irreducible composition factors, we only need to know their K-types and $\pi(\Omega)$ (by Corollary 1.2.8). We have that information, so the results can easily be checked. We leave the details to the reader. Q.E.D.

Corollary 1.3.4 *Fix $\lambda \in \mathbb{C}$, and $\varepsilon = \pm 1$; and suppose $\mathrm{Re}\,\lambda > 0$. Let $\mu = 0$ if $\varepsilon = +1$, and $\mu = \pm 1$ if $\varepsilon = -1$. Then there is a unique (up to a scalar) non-zero homomorphism of $(\mathfrak{g},K)$ modules*

$$A : X_c(\varepsilon \otimes \lambda) \to X_c(\varepsilon \otimes -\lambda).$$

The image of A is isomorphic to $\overline{X}_c(\lambda)(\mu)$; in particular, A is non-zero on the K-type μ.

Proof. Suppose first that λ is not an integer of parity $-\varepsilon$. By Proposition 1.3.3(b) and Proposition 1.2.14(b),

$$X_c(\varepsilon \otimes \lambda) \cong \overline{X}_c(\lambda)(\mu) \cong \overline{X}_c(-\lambda)(\mu) \cong X_c(\varepsilon \otimes -\lambda).$$

So the result is true in this case; A must even be an isomorphism. Next suppose λ is an integer of parity $-\varepsilon$; by hypothesis it is positive. By Proposition 1.3.3(c) and (d), $\overline{X}_c(\lambda)(\mu)$ is a quotient of $X_c(\varepsilon \otimes \lambda)$ and a submodule of $X_c(\varepsilon \otimes -\lambda)$; so there is a map A with the desired properties. Now let B be any other homomorphism between these principal series. By Proposition 1.3.3(c), $X_c(\varepsilon \otimes \lambda)$ contains $X_d(\pm\lambda)$ as a submodule; so $X_c(\varepsilon \otimes -\lambda)$ contains

$$B(X_d(\pm\lambda)) = Y_\pm$$

as a submodule. But the only irreducible submodule of $X_c(\varepsilon \otimes -\lambda)$ is $\overline{X}_c(-\lambda)(\mu)$ (by Proposition 1.3.3(d)); so $Y_\pm = 0$. So B factors as

$$\begin{array}{ccc} X_c(\varepsilon \otimes \lambda) & \xrightarrow{B} & X_c(\varepsilon \otimes -\lambda) \\ & \searrow \quad \overline{X}_c(\lambda)(\mu) \quad \nearrow_{\overline{B}} & \end{array}$$

Since $\overline{X}_c(\lambda)(\mu)$ is irreducible and occurs just once as a submodule of $X_c(\varepsilon \otimes -\lambda)$, the map

$$\overline{X}_c(\lambda)(\mu) \to X_c(\varepsilon \otimes -\lambda)$$

is unique up to a scalar. So $\bar{B}$ must be a multiple of $\bar{A}$, so B is a multiple of A. Q.E.D.

We turn now to the analytic description of the principal series representations.

Definition 1.3.5 Define the following subgroups of $G = SL(2, \mathbb{R})$:

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \; a \in \mathbb{R},\ a > 0 \right\}$$

$$M = \text{centralizer of } A \text{ in } K = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}$$

$$P = MAN = \left\{ \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} \middle| \; 0 \neq a \in \mathbb{R},\ t \in \mathbb{R} \right\}.$$

If $\varepsilon = \pm 1$, and $\lambda \in \mathbb{C}$, define a character

$$\varepsilon \otimes \lambda : P \to \mathbb{C}$$

by

$$(\varepsilon \otimes \lambda)\begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} = (\varepsilon)^{(\operatorname{sgn} a)} |a|^{\lambda}.$$

Let $\mathcal{H}_{\varepsilon\otimes\lambda}$ denote the Hilbert space of complex valued functions f on G, satisfying

a) if $g \in G$ and $p \in P$,

$$f(gp) = [\varepsilon \otimes (\lambda+1)](p^{-1})\, f(g);$$

b) the restriction of f to K lies in $L^2(K)$.

(We make $\mathcal{H}_{\varepsilon\otimes\lambda}$ a Hilbert space using the restriction mapping into the Hilbert space $L^2(K)$.) The principal series (group) representation with parameters ε and λ, $(\pi, \mathcal{H}_{\varepsilon\otimes\lambda})$, is defined by

$$[\pi(g)f](g_0) = f(g^{-1}g_0) \qquad (g, g_0 \in G,\ f \in \mathcal{H}_{\varepsilon\otimes\lambda}).$$

If λ is purely imaginary, so that $\varepsilon \otimes \lambda$ is a unitary character of P, then

$$(\pi, \mathcal{H}_{\varepsilon\otimes\lambda}) = \operatorname{Ind}_P^G(\varepsilon \otimes \lambda),$$

the usual induced representation in Mackey's sense [50]. If λ is not purely imaginary, $\varepsilon \otimes \lambda$ is not unitary; and there is no good general definition of non-unitary induced representations. In the present setting, however, Definition 1.3.5 is a reasonable one, and we may speak of $\mathcal{H}_{\varepsilon\otimes\lambda}$ as the induced representation.

There are still several things to check: that the operators $\pi(g)$ are bounded, that the representation is continuous, and that its Harish-Chandra module is $X_c(\varepsilon\otimes\lambda)$. All of this is based on

Lemma 1.3.6 (Iwasawa decomposition). Any element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2, \mathbb{R})$ can be written uniquely as a product

$$g = k(\theta)\begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} \qquad (\theta \in \mathbb{R},\ y > 0,\ x \in \mathbb{R}).$$

Here

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$y = \sqrt{a^2 + c^2},\quad \cos\theta = \frac{a}{\sqrt{a^2 + c^2}},\quad \sin\theta = -\frac{c}{\sqrt{a^2 + c^2}}$$

Proof. By matrix multiplication,

$$k(\theta)\begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix} = \begin{pmatrix} y\cos\theta & x\cos\theta + y^{-1}\sin\theta \\ -y\sin\theta & -x\sin\theta + y^{-1}\cos\theta\end{pmatrix}.$$

If this is equal to g, then clearly

$$a^2 + c^2 = y^2\cos^2\theta + y^2\sin^2\theta = y^2;$$

so since y is assumed to be positive,

$$y = \sqrt{a^2 + c^2}.$$

(Since g is invertible, a and c are not both zero.) This, in turn, forces

$$\cos\theta = \frac{a}{y}, \quad \sin\theta = -\frac{c}{y}.$$

So $k(\theta)$ and y are uniquely determined; and x is fixed by

$$(k(\theta))^{-1}g = \begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix}.$$

So the decomposition is unique. To prove that it exists, define $k(\theta)$ by the formulas above. Then

$$(k(\theta))^{-1}g = \begin{pmatrix} ay^{-1} & cy^{-1} \\ -cy^{-1} & ay^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

$$= \begin{pmatrix} (a^2+c^2)y^{-1} & x \\ 0 & (ad-bc)y^{-1}\end{pmatrix}$$

$$= \begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix}$$

since $a^2 + c^2 = y^2$, and $ad - bc = 1$; here x is some real number. Multiplying by $k(\theta)$ proves the lemma. Q.E.D.

Corollary 1.3.7 *If $\varepsilon = \pm 1$, let $\mathcal{H}_\varepsilon$ denote the Hilbert space of functions in $L^2(K)$ of parity ε, i.e.,*

$$\mathcal{H}_\varepsilon = \{f \in L^2(K) \mid f(-x) = \varepsilon f(x)\}.$$

Then restriction to K defines an isomorphism

$$\mathcal{H}_{\varepsilon\otimes\lambda} \cong \mathcal{H}_\varepsilon .$$

Proof. The element $\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$ of G lies in $P \cap K$, and acts by ε in the character $\varepsilon \otimes \lambda$ of P. So, by definition 1.3.5, if $f \in \mathcal{H}_{\varepsilon\otimes\lambda}$ and $x \in K$,

$$f(-x) = f\left(x\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\right) = \varepsilon f(x).$$

Conversely, suppose $\bar{f}$ lies in $\mathcal{H}_\varepsilon$. Extend $\bar{f}$ to a function f on G by

$$f(g) = \bar{f}(k)y^{-\lambda-1} \qquad \left(g = k \cdot \begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix}, \quad k \in K\right).$$

Then f restricted to K is $\bar{f}$, so f satisfies condition (b) of Definition 1.3.5; and f was constructed to satisfy (a). Q.E.D.

Proposition 1.3.8 *The principal series representation $(\pi, \mathcal{H}_{\varepsilon\otimes\lambda})$ of Definition 1.3.5 is an admissible group representation (Definition 1.1.5). Its Harish-Chandra module (Definition 1.1.7) is isomorphic to $X_C(\varepsilon \otimes \lambda)$ of Definition 1.3.1.*

Proof. Lemma 1.3.6 shows that the homogeneous space G/AN (with A and N as in Definition 1.3.5) is naturally isomorphic to K, by the inclusion

$$K = K/(K \cap AN) \to G/AN .$$

Therefore, there is an action of G on K, by

$$(1.3.9) \qquad g \cdot k = k' \qquad \left(gk = k'\begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix},\ y > 0\right)$$

Define a function

$$a:\ G \to \{y \in \mathbb{R} \mid y > 0\}$$

by

$$(1.3.10)\ a(g) = y \text{ if } g = k \cdot \begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix} \quad (k \in K,\ y > 0)\ .$$

Lemma 1.3.6 gives a formula for this function:

$$a\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \sqrt{a^2 + c^2}\ .$$

It also gives a formula for the action of G on K, which we will not write down; but we conclude by inspection of these formulas that the map

$$G \times K \to (K \times \mathbb{R})$$
$$(g,k) \to (g^{-1} \cdot k,\ a(g^{-1}k))$$

is continuous. Suppose now that f lies in $\mathcal{H}_\varepsilon \subseteq L^2(K)$. By Definition 1.3.5 and (1.3.9) and (1.3.10),

$$[\pi(g)f](k) = f(g^{-1} \cdot k)a(g^{-1}k)^{-\lambda-1}.$$

It is now clear that the operator $\pi(g)$ is bounded (by the maximum of the Jacobian of the diffeomorphism of K given by the action of g, times the maximum on K of the function $a(g^{-1}k)^{-\lambda-1}$); one simply performs a change of variable in the integration over K which defines $||\pi(g)f||$. For similar reasons π is continuous. Recall that $\mathcal{H}_\varepsilon$ is the space of functions on K (the circle group) of parity ε. By Fourier series, $\mathcal{H}_\varepsilon$ has an orthonormal basis given by functions

$$\{f_n \mid n \in \mathbb{Z} \text{ has parity } \varepsilon, \text{ and } f_n(k(\theta)) = e^{in\theta}\}.$$

Defining $\mathcal{H}_n$ as in Lemma 1.1.3 (for $\mathcal{H}_\varepsilon$), we find

$$\mathcal{H}_n = \begin{cases} 0 \text{ if } n \text{ has parity } -\varepsilon \\ \mathbb{C} \cdot f_n \text{ if } n \text{ has parity } \varepsilon. \end{cases}$$

In particular, $(\pi, \mathcal{H}_{\varepsilon\otimes\lambda})$ is admissible, and its Harish-Chandra module has basis $\{f_n\}$. By Lemma 1.2.2

$$(1.3.11) \qquad \begin{aligned} \pi(H)f_n &= nf_n \\ \pi(X)f_n &= a_n f_{n+2} \\ \pi(Y)f_n &= b_n f_{n-2}\ ; \end{aligned}$$

so we want to compute a_n and b_n. Since all the functions f_m are one at the identity element,

$$(1.3.12) \qquad a_n = [\pi(X)f_n](1)\ .$$

To compute this, we regard f_n as extended to a function on all of G, in accordance with condition (a) of Definition 1.3.5. Then

$$f_n\begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix} = y^{-\lambda-1} \qquad (y > 0).$$

Therefore

$$\begin{aligned}\left(\pi\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}f_n\right)(1) &= \lim_{t\to 0} \frac{1}{t}\left[f_n\left(\exp\begin{pmatrix} 0 & -t \\ 0 & 0\end{pmatrix}\right) - f_n(1)\right] \\ &= \lim_{t\to 0} \frac{1}{t}\left[f_n\begin{pmatrix} 1 & -t \\ 0 & 1\end{pmatrix} - f_n(1)\right] \\ &= \lim \frac{1}{t}[1 - 1] \\ &= 0;\end{aligned}$$

and similarly,

$$\left(\pi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} f_n\right)(1) = \lim_{t\to 0}\frac{1}{t}\left(f_n\begin{pmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{pmatrix} - f_n(1)\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\left(e^{t(\lambda+1)} - 1\right)$$

$$= \lambda + 1.$$

Finally, the first formula of 1.3.11 gives

$$\left(\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} f_n\right)(1) = i\ n.$$

By (1.2.1),

$$X = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{i}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + i\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} ;$$

so the computations above and (1.3.12) show that

$$a_n = \frac{1}{2}(\lambda + 1) + \frac{n}{2} = \frac{1}{2}(\lambda + (n+1)).$$

Similarly, we find

$$b_n = \frac{1}{2}(\lambda - (n-1)).$$

These are exactly the structure constants for $X_c(\varepsilon \otimes \lambda)$ (Definition 1.3.1), completing the proof of the proposition.

Q.E.D.

§4. $SL^{\pm}(2, \mathbb{R})$

For the remainder of this chapter, we return to our previous hypotheses on the level of experience of the reader; and in particular we provide no more detailed proofs. In

this section, G will denote the group

$$SL^{\pm}(2, \mathbb{R}) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \middle| a,b,c,d \in \mathbb{R},\ ad - bc = \pm 1\right\}.$$

Obviously G has two connected components, and the identity component G_0 is $SL(2, \mathbb{R})$. Define

$$(1.4.1)\quad \begin{cases} K = O(2) = \left\{\begin{pmatrix}\cos\theta & \sin\theta\\ \mp\sin\theta & \pm\cos\theta\end{pmatrix}\right\} \\ T = SO(2) = K_0 \\ A = \left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} \middle| a > 0\right\} \\ N = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \middle| x \in \mathbb{R}\right\} \\ M = \left\{\begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix}\right\} = \text{centralizer of } A \text{ in } K. \\ P = MAN = \left\{\begin{pmatrix}a & x\\ 0 & \pm a^{-1}\end{pmatrix} \middle| 0 \neq a \in \mathbb{R},\ x \in \mathbb{R}\right\}. \end{cases}$$

Lemma 1.4.2 The set of irreducible unitary representations of K, $\hat{K}$, may be parametrized as follows: if $n \in \mathbb{Z}$, let

$$\chi_n : K_0 \to \mathbb{C},\ \chi_n(k(\theta)) = e^{in\theta}$$

be the indicated character; here

$$k(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

If n is a positive integer, define

$$\mu_n = \mathrm{Ind}_{K_0}^{K}(\chi_n) \cong \mathrm{Ind}_{K_0}^{K}(\chi_{-n});$$

then μ_n is a two-dimensional irreducible unitary representation, and

$$\mu_n\Big|_{K_0} = \chi_n \oplus \chi_{-n}.$$

Let μ_0^+ denote the trivial representation, and μ_0^- the other one dimensional representation of K (which is trivial on K_0); then

$$\mu_0^{\pm}\Big|_{K_0} = \chi_0, \quad \mathrm{Ind}_{K_0}^{K}(\chi_0) = \mu_0^+ \oplus \mu_0^- .$$

Then

$$\hat{K} = \{\mu_0^+, \mu_0^-, \mu_1, \mu_2, \mu_3, \ldots\}.$$

The proof is left to the reader.

Definition 1.4.3 Identify the set $\hat{M}$ of the characters of M with pairs

$$\delta = (\delta_1, \delta_2) \qquad (\delta_i = 0 \text{ or } 1)$$

by

$$\delta\begin{pmatrix}\varepsilon_1 & 0 \\ 0 & \varepsilon_2\end{pmatrix} = \Pi \varepsilon_i^{\delta_i}.$$

(For example, if $\delta = (1,0)$, then

$$\delta\begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix} = -1, \quad \delta\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} = 1. \;)$$

If $\delta \in \hat{M}$ and $\nu \in \mathbb{C}$, define the principal series representation with parameters δ and ν by

$$(\pi, \mathcal{H}_{\delta\otimes\nu}) = \mathrm{Ind}_P^G(\delta \otimes \nu).$$

Here we regard ν as a character of A as in Definition 1.3.5, and we make $\delta \otimes \nu$ a character of P = MAN by making it trivial on N. More specifically,

$$\mathcal{H}_{\delta\otimes\nu} = \{f\colon G \to \mathbb{C} \mid f(gp) = f(g)[(\delta \otimes (\nu+1))(p^{-1})] \text{ for all } g \in G \text{ and } p \in P; \text{ and } f|_K \text{ lies in } L^2(K)\}$$

$$[\pi(g)f](g_0) = f(g^{-1}g_0) \qquad (g, g_0 \in G,\ f \in \mathcal{H}_{\delta\otimes\nu}).$$

Lemma 1.4.4 Fix $\delta \in \hat{M}$ and $\nu \in \mathbb{C}$.

a)
$$\mathcal{H}_{\delta\otimes\nu}\Big|_K = \begin{cases} \mu_0^+ \oplus \mu_2 \oplus \mu_4 \oplus \cdots & (\delta = (0,0)) \\ \mu_1 \oplus \mu_3 \oplus \mu_5 \oplus \cdots & (\delta = (1,0) \text{ or } (0,1)) \\ \mu_0^- \oplus \mu_2 \oplus \mu_4 \oplus \cdots & (\delta = (1,1)) \end{cases}$$

b) Let ε be the restriction of δ to $M \cap G_0$. Then

$$\mathcal{H}_{\delta\otimes\nu}\Big|_{SL(2,\mathbb{R})} \cong \mathcal{H}^{G_0}_{\varepsilon\otimes\nu};$$

here the representation on the right is the principal series for SL(2, ℝ) defined in Definition 1.3.5.

Definition 1.4.5 If $\delta \in \hat{M}$ and $\nu \in \mathbb{C}$, define the principal series $(\mathfrak{g}, K)$ module with parameters δ and ν, $X(\delta \otimes \nu)$, to be Harish-Chandra module of $\mathcal{H}_{\delta\otimes\nu}$.

We know the $\mathfrak{g}$-invariant subspaces of $\mathcal{H}_{\delta\otimes\nu}$ from Proposition 1.3.3(c) and Lemma 1.4.4(b); and the K-invariant subspaces can be read off from Lemma 1.4.4(a). So we can immediately compute the whole composition series of $\mathcal{H}_{\delta\otimes\nu}$. To express the results, some definitions are helpful.

Definition 1.4.6 Suppose $\delta \in \hat{M}$. Define a (one element) subset $A(\delta) \subseteq \hat{K}$ by

$$A(\delta) = \{\mu \in \hat{K} \mid \mu = \mu_1, \mu_0^+, \text{ or } \mu_0^-, \text{ and } \delta \text{ occurs in } \mu|_M\}$$

(so

$$A(0,0) = \{\mu_0^+\}, A(0,1) = A(1,0) = \{\mu_1\}, A(1,1) = \{\mu_0^-\}.)$$

If $\mu \in A(\delta)$, and $\nu \in \mathbb{C}$, let

$$\overline{X}(\delta \otimes \nu)(\mu) = \overline{X}(\delta \otimes \nu)$$

be the unique irreducible subquotient of $X(\delta \otimes \nu)$ containing the K-type μ; this exists by Lemma 1.4.4.

Definition 1.4.7 Fix a positive integer n; and define

$$X_d(n) = \mathrm{Ind}_{(\mathfrak{g},K_0)}^{(\mathfrak{g},K)} (X_d^{G_0}(n))$$
$$\cong \mathrm{Ind}_{(\mathfrak{g},K_0)}^{(\mathfrak{g},K)} (X_d^{G_0}(-n)),$$

the discrete series representation with parameter n. Here the representations on the right are the discrete series for SL(2, ℝ), defined in Definition 1.2.13; the induction is explained in Definition 0.3.25, and verification of the indicated isomorphism is left to the reader.

Proposition 1.4.8 Fix $\nu \in \mathbb{C}$, $\delta \in \hat{M}$, and $\mu \in A(\delta)$ (Definition 1.4.6). Let

$$\varepsilon = \delta\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = \pm 1.$$

a) If ν is not a non-zero integer of parity $-\varepsilon$, then $X(\delta\otimes\nu)$ is irreducible, and isomorphic to $\overline{X}(\delta\otimes\nu)(\mu)$.

b) If ν is a positive integer of parity $-\varepsilon$, then there is a non-split short exact sequence

$$0 \to X_d(\nu) \to X(\delta\otimes\nu) \to \overline{X}(\delta\otimes\nu)(\mu) \to 0;$$

the quotient $\overline{X}(\delta\otimes\nu)(\mu)$ is a finite dimensional module of highest weight $\nu-1$.

c) If ν is a negative integer of parity $-\varepsilon$, then there is a non-split short exact sequence

$$0 \to \overline{X}(\delta\otimes\nu)(\mu) \to X(\delta\otimes\nu) \to X_d(-\nu);$$

the submodule $\overline{X}(\delta\otimes\nu)(\mu)$ is finite dimensional, of highest weight $-\nu-1$.

The proof was sketched before Definition 1.4.6. The most important difference between this result and Proposition 1.3.3 concerns the case $\varepsilon = -1$, $\nu = 0$. In that case the principal series for SL(2, ℝ) is a direct sum of two pieces; but these two pieces are permuted by K, and the representation of $SL^{\pm}(2, \mathbb{R})$ is irreducible.

Lemma 1.4.9 If $\delta = (\delta_1, \delta_2) \in \hat{M}$ (Definition 1.4.3), set $s \cdot \delta = (\delta_2, \delta_1)$.

a) $X(\delta\otimes\nu)$ and $X(\delta'\otimes\nu')$ have the same set of irreducible composition factors if and only if

$$(\delta,\nu) = (\delta',\nu') \text{ or } (s\delta',-\nu);$$

b) $\overline{X}(\delta\otimes\nu) \cong \overline{X}(\delta'\otimes\nu')$ if and only if $(\delta,\nu) = (\delta',\nu')$ or $(s\delta',-\nu)$.

Proof. The action of the Weyl group of A on $\hat{M}$ is given by s; so (a) is a special case of Theorem 4.1.4. The "if" part of (b) is immediate. For the "only if", suppose the isomorphism holds. If either module is all of the corresponding principal series, the result follows from (a); so by Proposition 1.4.8, we may assume that ν is a non-zero integer of parity $-\varepsilon$, and ν'

is a non-zero integer of parity $-\varepsilon'$. (Here $\varepsilon = \delta\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, $\varepsilon' = \delta'\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.) The hypotheses and conclusion are unaffected if we replace (δ,ν) by $(s \cdot \delta, -\nu)$, and similarly for ν'; so we may assume ν and ν' are positive. Since the restrictions to G_0 are equivalent, Proposition 1.2.14(b) gives $\nu = \nu'$, $\varepsilon = \varepsilon'$. So the composition factors of the two principal series are (by Proposition 1.4.8(b))

$$X(\delta \otimes \nu):\quad \overline{X}(\delta \otimes \nu), X_d(\nu)$$
$$X(\delta' \otimes \nu'):\quad \overline{X}(\delta' \otimes \nu'), X_d(\nu).$$

These are the same by hypothesis, so (a) gives the result we want. Q.E.D.

The real issue in this proposition is the inequivalence of $\overline{X}(\delta \otimes \nu)$ and $\overline{X}(\delta' \otimes \nu')$ when $\nu \neq 0$ and $\delta \neq \delta'$. This can also be seen more explicitly, as follows. If $A(\delta) \neq A(\delta')$, the representations do not have the same restriction to K; so the hard case is

$$\delta = (1,0),\ \delta' = (0,1);\ A(\delta) = \mu_1.$$

Choose a non-zero vector

$$w_1 \in \overline{X}(\delta \otimes \nu)$$

of weight 1 for K_0. Then

$$w_{-1} = \pi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \cdot w_1, \qquad \pi(Y)w_1 = v_{-1}$$

both have weight -1, so

$$v_{-1} = cw_{-1}.$$

The constant c is clearly an invariant of the representation; it is $\pm\frac{1}{2}\nu$, and the sign distinguishes between δ and δ'. This method is much too subtle to use in more general situations, however, and we will have to rely on Theorem 4.1.4 (which is based on a calculation of global characters).

§5. An introduction to the R group.

In this section, G will be a reductive group satisfying the hypotheses of 0.1.2, and in addition

(1.5.1)
- a) the center of G is compact
- b) every simple ideal of $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Accordingly, we write

$$\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \cdots \oplus \mathfrak{g}_0^n \oplus \mathfrak{c}_0,$$

with $\mathfrak{c}_0$ the center of $\mathfrak{g}_0$, and $\mathfrak{g}_0^i$ simple. Choose an isomorphism

$$\phi^i:\ \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_0^i;$$

if θ denotes the Cartan involution of $\mathfrak{g}_0$, we may arrange for

$$\phi^i(-{}^tX) = \theta\phi^i(X);$$

and if $\mathfrak{g}_0$ is given a θ-stable split Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{a}_0 + \mathfrak{c}_0$, we can arrange

$$\phi^i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \mathfrak{a}_0.$$

If such an $\mathfrak{h}$ is not given, we simply define

(1.5.2) $$\begin{cases} \mathfrak{a}_0 = \left[\oplus\, \phi_i\left(\mathbb{R}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\right] \\ \mathfrak{h}_0 = \mathfrak{a}_0 + \mathfrak{c}_0 . \end{cases}$$

Our goal is to understand reducibility of unitary principal series for G. As we have seen, such reducibility exists for $SL(2,\mathbb{R})$ (Proposition 1.3.3(e)), but not for $SL^{\pm}(2,\mathbb{R})$ (Proposition 1.4.8(a)). The general case of course combines these two phenomena; the R-group will provide a precise measure of this combination, and thus of the reducibility.

If our group G comes endowed with a positive root system $\Delta^+(\mathfrak{g},\mathfrak{h})$, we may assume that

(1.5.3) $\phi^i\begin{pmatrix}0&1\\0&0\end{pmatrix}$ lies in a positive root space.

If $\Delta^+(\mathfrak{g},\mathfrak{h})$ is not given, we define it so that (1.5.3) holds. We need a great deal of explicit notation. Notice first that since G is linear, ϕ^i exponentiates to

(1.5.4) $$\phi^i: \ SL(2,\mathbb{R}) \to G;$$

ϕ^i either is an isomorphism, or has kernel $\{\pm 1\}$. Set

(1.5.5)

(a) $\{\alpha_i\} = \Delta^+(\mathfrak{g},\mathfrak{h})$, $\alpha_i \in \Delta(\mathfrak{g}^i,\mathfrak{h})$

(b) $Z_i = \phi^i\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

(c) $\mathfrak{t}_0 = \bigoplus_i \langle Z_i\rangle \oplus \mathfrak{c}_0$

(d) T = centralizer of $\mathfrak{t}_0$ in G

(e) K = normalizer of $\mathfrak{t}_0$ or T in G

(f) $A = \exp(\mathfrak{a}_0)$

(g) M = centralizer of A in K

(h) $H = MA$ = centralizer of $\mathfrak{a}_0$ or $\mathfrak{h}_0$ or A in G

(i) M' = normalizer of A in K

(j) $W = W(H) = M'/M$ = Weyl group of A in G

(k) $W_K = W_K(T) = K/T$ = Weyl group of T in G

(l) $\mathfrak{n}_0 = \bigoplus_i \langle \phi^i\begin{pmatrix}0&1\\0&0\end{pmatrix}\rangle$

(m) $N = \exp \mathfrak{n}_0$

(n) $P = MAN$

(o) $\sigma_i = \exp(\frac{\pi}{2} Z_i) = \phi^i\begin{pmatrix}0&1\\-1&0\end{pmatrix} \in M'$

(p) $s_i = \sigma_i M \in W$

(q) $m_i = \sigma_i^2 = \exp(\pi Z_i) = \phi^i\begin{pmatrix}-1&0\\0&-1\end{pmatrix} \in M$

Of course K is a maximal compact subgroup, T is a compact Cartan subgroup, H is a split Cartan subgroup, and P is a Borel subgroup (and therefore a minimal parabolic subgroup).

<u>Definition 1.5.6</u> Suppose

$$\gamma = \delta \otimes \nu \in \hat{H} \quad (\delta \in \hat{M},\ \nu \in \hat{A}).$$

(Recall that H is assumed to be abelian, so that γ is one-dimensional.) Define

$$(\pi(\gamma),\ H_\gamma) = \mathrm{Ind}_P^G(\gamma),$$

the <u>principal series representation</u> with parameter γ. Thus, if we write $\rho = (\det(\mathrm{Ad}|_{\mathfrak{n}}))^{1/2} \in \hat{A}$, and $\tilde{\gamma} = \delta\otimes(\nu+\rho)$,

$$H_\gamma = \{ f:\ G \to \mathbb{C} \mid f(g\,man) = f(g)\tilde{\gamma}(ma)^{-1}, \text{ and the restriction of } f \text{ to } K \text{ lies in } L^2(K)\},$$

and $\pi(\gamma)$ is the representation by left translation. For emphasis, we may write things like $\pi^G(\gamma)$ or

$\pi(H,\gamma)$, when the groups are varying or are not clear from context. The principal series $(\mathfrak{g},K)$ module with parameter γ, $X(\gamma)$, is the Harish-Chandra module of $\pi(\gamma)$.

Lemma 1.5.7 Suppose $\gamma = \delta \otimes \nu \in \hat{H}$. Let $\pi(\gamma)$ be the principal series representation with parameter γ.

a) $\pi(\gamma)\big|_K = \mathrm{Ind}_M^K(\delta)$

b) $\pi(\gamma)\big|_{G_0} = \pi^{G_0}(\gamma\big|_{G_0})$,

the principal series representation of G_0 with parameter $\gamma\big|_{G_0} = \delta\big|_{(M\cap G_0)} \otimes \nu$.

Proof. These follow from the identifications

$$G/P \cong K/M \cong K_0/M \cap K_0 \cong G_0/P \cap G_0. \qquad \text{Q.E.D.}$$

Recall from Proposition 1.3.5 that when ν is unitary, the principal series can be reducible only when ν is zero and δ is non-trivial. So we will concentrate on this case; some aspects of the general case will be reduced to it in Chapter 4. One difference between $SL(2,\mathbb{R})$ and $SL^{\pm}(2,\mathbb{R})$ was that W acts trivially on $\hat{M}$ in the first case, but not in the second. This suggests the simplest definition of the R-group. (Another is given by Corollary 1.5.24.)

Definition 1.5.8 (Knapp-Stein [26]). Suppose $\delta \in \hat{M}$, and $\delta(m_i) = -1$ for all i (cf. (1.5.5)(q)). Define

$$M'_\delta = \{x \in M' \mid x \cdot \delta \cong \delta\} \supseteq M$$

$$W_\delta = R_\delta = M'_\delta/M \subseteq W;$$

these are the stabilizer of δ in the Weyl group and the R-group for δ (which coincide in this case, but not in general).

(Recall that M' acts on $\hat{M}$ by

$$(x \cdot \delta)(m) = \delta(x^{-1}mx) \qquad (\delta \in \hat{M},\ m \in M);$$

this makes sense because M' normalizes M. Clearly M acts trivially, so this lifts to an action of W on $\hat{M}$.)

Since W is a product of copies of $\mathbb{Z}/2\mathbb{Z}$, the R-group is as well; so the dual group $\hat{R}_\delta$ is defined.

Proposition 1.5.9 (Knapp-Stein [26]). Suppose $\delta \in \hat{M}$, and

$$\delta(m_i) = -1, \text{ all } i \quad (\text{cf. } (1.5.5(q)))\ ;$$

and suppose $\nu \in \hat{A}$ is trivial. Then the unitary principal series representation $\pi(\delta \otimes \nu)$ decomposes as a direct sum of $|R_\delta|$ distinct irreducible representations (Definition 1.5.8). There is a natural simply transitive action of the dual group $\hat{R}_\delta$ on the set of constituents of $\pi(\delta \otimes \nu)$.

Knapp and Stein's proof works by studying analytically defined intertwining operators on $\pi(\delta \otimes \nu)$. We will use a more algebraic approach, which is designed to study the K-types of the constituents at the same time. This requires a little preparation.

Definition 1.5.10 Suppose $\delta \in \hat{M}$, and $\delta(m_i) = -1$ for all i. The set of fine K-types containing δ is defined to be

$$A(\delta) = \{\mu \in \hat{K} \mid \mu(\sqrt{-1}\, Z_i) \text{ has eigenvalues } \pm 1 \text{ for all } i, \text{ and } \delta \text{ occurs in } \mu|_M \}.$$

(If $G = SL(2, \mathbb{R})$, $A(\delta) = \{\pm 1\}$; if $G = SL^{\pm}(2, \mathbb{R})$, $A(\delta) = \{\mu_1\}$ (Definition 1.4.6).)

Lemma 1.5.11 In the setting of Proposition 1.5.9, suppose

$$\vec{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n .$$

Define

$$X_{\vec{i}} = \{v \in X(\delta \otimes \nu) \mid \pi(Z_j)v = [\sqrt{-1}\, i_j]v\} .$$

Then $X_{\vec{i}}$ is zero unless all i_j are odd, in which case it has dimension one. If $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n$, set

$$X^{\varepsilon} = \bigoplus_{\substack{\vec{i} \in \mathbb{Z}^n \\ \operatorname{sgn} i_j = \varepsilon_j}} X_{\vec{i}}$$

If $S \subseteq (\mathbb{Z}/2\mathbb{Z})^n$, set

$$X^S = \bigoplus_{\varepsilon \in S} X^{\varepsilon}.$$

Then each X^S is a $\mathfrak{g}$-invariant subspace of $X(\delta \otimes 1)$, and every $\mathfrak{g}$-invariant subspace is of this form.

Proof. This follows from Lemma 1.5.7(b) and Proposition 1.3.3(b). Q.E.D.

Corollary 1.5.12 In the setting of Lemma 1.5.11, set

$$X_f = \bigoplus_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n} X_{\varepsilon};$$

here we identify $(\mathbb{Z}/2\mathbb{Z})$ with $\{\pm 1\}$, and f stands for "fine". Then X_f is K-invariant, and

a) $$X_f\big|_K \cong \bigoplus_{\mu \in A(\delta)} \mu .$$

Identify the representation space V_μ of μ with a subspace of X_f, and define $S(\mu) \subseteq (\mathbb{Z}/2\mathbb{Z})^n$ by

$$V_\mu = \bigoplus_{\varepsilon \in S(\mu)} X_{\varepsilon}.$$

Then

$$X(\delta \otimes 1)(\mu) = X^{S(\mu)} \subseteq X(\delta \otimes 1)$$

is K-stable and $\mathfrak{g}$-stable; it is the unique irreducible constituent of $X(\delta \otimes 1)$ containing the K-type μ. Thus

$$X(\delta \otimes 1) = \bigoplus_{\mu \in A(\delta)} X(\delta \otimes 1)(\mu)$$

is the decomposition of $X(\delta \otimes 1)$ into irreducible components.

Proof. The element Z_i (cf. 1.5.5(b)) is specified up to sign by the fact that it lies in $\mathfrak{g}_0^i \cap \mathfrak{k}_0$, and its length in the Killing form. These properties are preserved by Ad(K); so if $k \in K$, we can define $q_i(k) = \pm 1$ by

(1.5.13) $$\operatorname{Ad}(k)(Z_i) = q_i(k) Z_i .$$

In particular, this makes it clear that X_f is K-invariant. More precisely, define

(1.5.14) $$q\colon K \to (\mathbb{Z}/2\mathbb{Z})^n, \quad q(k) = (q_1(k), \ldots, q_n(k)).$$

Then

(1.5.15) $$\pi(k) X_{\varepsilon} = X_{q(k)\varepsilon} .$$

Formula (a) of the corollary is clear from Lemma 1.5.7(a) and Definition 1.5.10. The other statements follow easily. Q.E.D.

Corollary 1.5.16 In the setting of Corollary 1.5.12 and (1.5.14), the sets $S(\mu)$ are the cosets of $q(K)$ in $(\mathbb{Z}/2\mathbb{Z})^n$. Thus the quotient group $(\mathbb{Z}/2\mathbb{Z})^n/q(K)$ acts in a natural, simply transitive way on the set of constituents of $X(\delta \otimes \nu)$: if

$$\bar{\varepsilon} = \varepsilon + q(K) \in (\mathbb{Z}/2\mathbb{Z})^n/q(K),$$

then

$$\bar{\varepsilon} \cdot X^{S(\mu)} = X^{\varepsilon S(\mu)}.$$

To complete the proof of Proposition 1.5.9, we need only make an identification

(1.5.17) $$\hat{R}_\delta \cong (\mathbb{Z}/2\mathbb{Z})^n/q(K).$$

To do this, set

$$P(n) = \{\text{subsets of } \{1,2,\ldots,n\}\}.$$

Regard $P(n)$ as the dual group of $(\mathbb{Z}/2\mathbb{Z})^n$ by the pairing

(1.5.18) $$\varepsilon(A) = \prod_{i\in A} \varepsilon_i \qquad (A \in P(n),\ \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n).$$

Now identify

$$W \cong P(n)$$

by the map

$$A \to \prod_{i\in A} s_i \qquad (A \in P(n),\ s_i \text{ as in } 1.5.5(p)).$$

The dual group of $R_\delta \cong W_\delta$ is then naturally isomorphic to the quotient

$$(\mathbb{Z}/2\mathbb{Z})^n/\mathrm{Ann}(W_\delta);$$

here

$$\mathrm{Ann}(W_\delta) = \{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n \mid \text{for all } A \in W_\delta,\ \prod_{i\in A} \varepsilon_i = 1\}.$$

So what must be shown is that

(1.5.19) $$\mathrm{Ann}(W_\delta) = q(K) \subseteq (\mathbb{Z}/2\mathbb{Z})^n.$$

To prove this, notice first that the kernel of q is (almost by definition) the centralizer of $\mathfrak{t}_0$ in K; so

(1.5.20) the map q of (1.5.14) defines an injection $\bar{q}$: $W(K/T) \to (\mathbb{Z}/2\mathbb{Z})^n$.

For us the main point is that q is trivial on $K_0 = T_0$, the identity component of K. Since M meets every component of G, and hence of K, it follows that

(1.5.21) $$q(K) = q(M).$$

Lemma 1.5.22 Suppose $A \in P(n)$; let

$$w = \prod_{i\in A} s_i$$

be the corresponding element of W. Then if $m \in M$, and $\delta(m_i) = -1$ for all i,

$$(w \cdot \delta)(m) = (q(m)[A])\delta(m) \qquad \text{(notation (1.5.14), (1.5.18));}$$
$$= [\prod_{i\in A} q_i(m)]\delta(m).$$

Proof. We first compute $w^{-1}mw$. By (1.5.13) and (1.5.5)(o),

$$m\sigma_i m^{-1} = \mathrm{Ad}(m) \cdot \exp(\tfrac{\pi}{2} Z_i)$$
$$= \exp\left(\tfrac{\pi}{2} \left(\mathrm{Ad}(m) \cdot Z_i\right)\right)$$
$$= \begin{cases} \sigma_i \text{ if } q_i(m) = 1 \\ \sigma_i^{-1} \text{ if } q_i(m) = -1 . \end{cases}$$

Now $\sigma_i^2 = m_i$ (cf. (1.5.5)(q)), and $m_i^2 = 1$; so

$$\sigma_i^{-1} = \sigma_i m_i .$$

The preceding computation therefore gives

$$\sigma_i^{-1} m \sigma_i = \begin{cases} m \text{ if } q_i(m) = 1 \\ mm_i \text{ if } q_i(m) = -1. \end{cases}$$

So

(1.5.23) $$w^{-1}mw = (m)\Big(\prod_{\substack{i \in A \\ q_i(m)=-1}} m_i\Big).$$

Finally, we calculate

$$(w \cdot \delta)(m) = \delta(w^{-1}mw)$$
$$= \delta\Big(m \cdot \prod_{\substack{i \in A \\ q_i(m)=-1}} m_i\Big)$$
$$= \delta(m) \prod_{i \in A} q_i(m)$$

since $\delta(m_i) = -1$. Q.E.D.

As an immediate corollary, we see that

$$W_\delta = \mathrm{Ann}(q(M)) = \mathrm{Ann}(q(K))$$

(the last equality being (1.5.21)); and (1.5.19) follows. The proof of Proposition 1.5.9 is now complete.

Corollary 1.5.24 In the setting of Proposition 1.5.9,

$$|R_\delta| = 2^n / |W_K(T)|;$$

here n is the number of simple factors of G. More precisely, there is a natural isomorphism

$$\hat{R}_\delta \cong W(\mathfrak{g}, \mathfrak{t}) / W_K(T).$$

Proof. This follows from (1.5.20), (1.5.19), and (1.5.17). Q.E.D.

For a later application we will want another description of the representations $X(\delta \otimes \nu)(\mu)$.

Lemma 1.5.25 Define

$$K^\# = T$$
$$G^\# = TG_0 = K^\# AN$$
$$M^\# = M \cap T$$
$$P^\# = M^\# AN$$

In the setting of Proposition 1.5.9, define

$$\delta^\# = \delta|_{M^\#}.$$

Then

a) $$\pi_G(\delta \otimes \nu)\big|_{G\#} \cong \pi_{G\#}(\delta^\# \otimes \nu);$$

b) Given $\mu \in A(\delta)$, define $S(\mu)$ as in Corollary 1.5.12. If $\varepsilon \in S(\mu)$, let $\mu_\varepsilon \in \hat{T}$ be the weight of T on X_ε (notation (1.5.11)). Then

$$X(\delta \otimes 1)(\mu)\big|_{(\mathfrak{g}, K^\#)} = \bigoplus_{\varepsilon \in S(\mu)} X(\delta^\# \otimes 1)(\mu_\varepsilon);$$

c) If $\mu \in A(\delta)$, and $\varepsilon \in S(\mu)$, then

$$X(\delta \otimes 1)(\mu) = \mathrm{Ind}_{\mathfrak{g},K^{\#}}^{\mathfrak{g},K} (X(\delta^{\#} \otimes 1)(\mu_\varepsilon)) .$$

Proof. Part (a) follows from the identifications

$$G/P \cong G_0/P \cap G_0 \cong G^{\#}/P^{\#}$$

just as in Lemma 1.5.7. Part (b) is clear from the description of $X(\delta \otimes 1)(\mu)$ given in Corollary 1.5.12. Part (c) follows from (b) by Lemma 0.3.26 (Frobenius reciprocity).

Q.E.D.

Proposition 1.5.26 In the setting of Proposition 1.5.9, fix $\mu \in A(\delta)$ and $\varepsilon \in S(\mu)$ (cf. (1.5.12)). For each i between 1 and n (that is, for each simple factor of $\mathfrak{g}$), recall the map

$$\phi^i : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}_0 .$$

Define U_i to be either $\phi^i(X)$ or $\phi^i(Y)$ (notation (1.2.1)) in $\mathfrak{g}$, making the choice so that

$$[Z_i, U_i] = 2\sqrt{-1}\, \varepsilon_i\, U_i$$

(notation (1.5.5)(b); compare (1.2.1)). Put

$$\mathfrak{u} = \oplus\, \mathbb{C} \cdot U_i \qquad \mathfrak{g} = \mathfrak{t} + \mathfrak{u}$$

Make $\mathbb{C}_{\mu_\varepsilon}$, the representation space of μ_ε (see (1.5.25)(b)) a $\mathfrak{g}$ module by making $\mathfrak{u}$ act trivially (and $\mathfrak{t}$ by the differential of μ_ε). Make

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}), \mathbb{C}_{\mu_\varepsilon})$$

a $U(\mathfrak{g})$ module by acting on the right in the first factor, and a T module by the adjoint action on $U(\mathfrak{g})$ and the action on $\mathbb{C}_{\mu_\varepsilon}$; and write

$$\mathrm{pro}_{\mathfrak{g},T}^{\mathfrak{g},T}(\mathbb{C}_{\mu_\varepsilon})$$

for the T-finite part of this module. (This definition is explained more carefully at (6.1.21).) Then, in the notation of Lemma 1.5.25,

$$X(\delta^{\#} \otimes 1)(\mu_\varepsilon) \cong \mathrm{pro}_{\mathfrak{g},T}^{\mathfrak{g},T}(\mathbb{C}_{\mu_\varepsilon})$$
$$X(\delta \otimes 1)(\mu) \cong \mathrm{Ind}_{\mathfrak{g},T}^{\mathfrak{g},K}(\mathrm{pro}_{\mathfrak{g},T}^{\mathfrak{g},T}(\mathbb{C}_{\mu_\varepsilon})) .$$

Proof. Write $\bar{\mathfrak{u}}$ for the algebra defined by $-\varepsilon$ as in the statement of the proposition. Clearly

$$\mathfrak{g} = \mathfrak{u} + \mathfrak{t} + \bar{\mathfrak{u}},$$

so by Poincaré-Birkhoff-Witt,

$$U(\mathfrak{g}) \cong U(\mathfrak{u}) \otimes U(\mathfrak{t}) \otimes U(\bar{\mathfrak{u}})$$
$$= U(\mathfrak{g}) \otimes U(\bar{\mathfrak{u}}) .$$

So

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}), \mathbb{C}_{\mu_\varepsilon}) \cong \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes U(\bar{\mathfrak{u}}), \mathbb{C}_{\mu_\varepsilon})$$
$$\cong \mathrm{Hom}_{\mathbb{C}}(U(\bar{\mathfrak{u}}), \mathbb{C}_{\mu_\varepsilon})$$
$$\cong \mathbb{C}_{\mu_\varepsilon} \otimes [S(\bar{\mathfrak{u}})]^* .$$

This is an isomorphism of T modules, using the adjoint action of T on $\bar{\mathfrak{u}}$. Comparing this formula with Lemma 1.5.11, one checks easily that

(*) $$\mathrm{pro}_{\mathfrak{g},T}^{\mathfrak{g},T}(\mathbb{C}_{\mu_\varepsilon})\big|_T \cong X(\delta^\# \otimes 1)(\mu_\varepsilon)\big|_T.$$

So to prove that these modules are isomorphic, we only have to find a non-zero $(\mathfrak{g},T)$-module map

$$\phi: \; X(\delta^\# \otimes 1)(\mu_\varepsilon) \to \mathrm{Hom}_{\mathfrak{g}}\big(U(\mathfrak{g}),\mathbb{C}_{\mu_\varepsilon}\big);$$

ϕ will then be injective since $X(\delta^\# \otimes 1)(\mu_\varepsilon)$ is irreducible, and surjective by (*). To find such a map ϕ, first choose a non-zero T-module map

$$f: \; X(\delta^\# \otimes 1)(\mu_\varepsilon) \to \mathbb{C}_{\mu_\varepsilon}.$$

By the choice of $\mathfrak{g}$, if α is a root of T in $\mathfrak{u}$, then the weight $\mu_\varepsilon - \alpha$ of T does not occur in $X(\delta^\# \otimes 1)(\mu_\varepsilon)$. It follows that

$$f(z \cdot x) = 0 \qquad (x \in X(\delta^\# \otimes 1)(\mu_\varepsilon),\; z \in \mathfrak{u}).$$

So f is actually a $\mathfrak{g}$-module map. Now to define ϕ, we have to specify for each x in the $(\mathfrak{g},T)$ module, a map $\phi(x)$ from $U(\mathfrak{g})$ to $\mathbb{C}_{\mu_\varepsilon}$. We define

$$\phi(x)(u) = f(u \cdot x) \qquad (x \in X(\delta^\# \otimes 1)(\mu_\varepsilon),\; u \in U(\mathfrak{g})).$$

To see that $\phi(x)$ is a $\mathfrak{g}$-module map, suppose $q \in \mathfrak{g}$. Then

$$\begin{aligned}\phi(x)(qu) &= f(qu \cdot x)\\ &= f(q \cdot (ux))\\ &= q \cdot f(u \cdot x) \quad (\text{since } f \text{ is a } \mathfrak{g}\text{-module map})\\ &= q \cdot [\phi(x)(u)],\end{aligned}$$

as required. To check that ϕ is a $\mathfrak{g}$ module map, suppose $g \in \mathfrak{g}$. Then

$$\begin{aligned}[\phi(gx)](u) &= f(u \cdot (gx))\\ &= f(ug \cdot x)\\ &= \phi(x)(ug)\\ &= [g \cdot \phi(x)](u).\end{aligned}$$

So

$$\phi(gx) = g \cdot \phi(x),$$

as required. Similarly one sees that ϕ is a T module map. This proves the first claim of the proposition. The second follows from Lemma 1.5.25(c). Q.E.D.

The reader may have noticed that the key point in the construction of the map ϕ was the non-vanishing of

$$X/\mathfrak{u} \cdot X$$

(with $X = X(\delta^\# \otimes 1)(\mu_\varepsilon)$). This group is nearly the same as

$$H^n(\mathfrak{u},X).$$

In Chapter 6, we will generalize this construction, characterizing $(\mathfrak{g},K)$ modules in terms of certain cohomology groups

$$H^R(\mathfrak{u},X)$$

(no longer with $R = \dim \mathfrak{u}$).

<u>Example 1.5.27</u> Let $G = SO(2,2)$; this group has two connected components, and satisfies the hypotheses (1.5.1). We can regard G as the group of 4×4 matrices of determinant one, preserving the indefinite

form

$$||x|| = x_1x_2 + x_3x_4.$$

We can take for A the group of matrices with diagonal entries (x, x^{-1}, y, y^{-1}) (with x and y positive). The group M has order 4, and is generated by two elements

$$m_+ = \mathrm{diag}(-1,-1,1,1)$$
$$m_- = \mathrm{diag}(1,1,-1,-1);$$

here "diag" is the diagonal matrix with indicated entries. The two elements m_i coming from simple $SL(2,\mathbb{R})$ factors coincide, and are equal to m_+m_-. (They would be distinct in the corresponding spin group.) The Weyl group of A has two generators. These act on diagonal elements by

$$w_1(x,x^{-1},y,y^{-1}) = (x^{-1},x,y^{-1},y)$$
$$w_2(x,x^{-1},y,y^{-1}) = (y,y^{-1},x,x^{-1}).$$

Thus w_1 acts trivially on M, and w_2 interchanges m_+ and m_-. (The simple reflections in W are w_2 and w_1w_2). M has exactly two characters satisfying $\delta(m_i) = -1$; these are δ_+ and δ_-, defined by

$$\delta_+(m_+) = -1,\ \delta_+(m_-) = 1$$
$$\delta_-(m_+) = 1,\ \delta_-(m_-) = -1.$$

Clearly

$$w_1 \cdot \delta_\pm = \delta_\pm$$
$$w_2 \cdot \delta_\pm = \delta_\mp .$$

So if δ is either δ_+ or δ_-, $R_\delta = W_\delta$ is the two element group $\{1, w_1\}$. It can be shown (compare Lemma 4.3.46) that the R-group can always be made trivial by adding certain outer automorphisms to the group. (This does not apply to the more general R-groups to be defined for non-principal series representations.) A typical example is the passage from $SL(2,\mathbb{R})$ to $SL^{\pm}(2,\mathbb{R})$. In this case, one might guess that the right group is $O(2,2)$. It is not, however. For one thing, the automorphism of the Lie algebra defined by permuting the first two coordinates (which lies in $O(2,2)$) is not inner for the complexification -- it permutes the two simple factors. So $O(2,2)$ does not belong to the category of groups we are considering. (In any case, it can be shown that the unitary principal series for $O(2,2)$ can be reducible.) The right element to add to the group is

$$\tilde{m} = \mathrm{diag}(1,-1,1,-1);$$

that is, we should consider the group of matrices preserving the square of the quadratic form, and having determinant one. It is easy to check that

$$w_1 \cdot \tilde{m} = \tilde{m}(m_+m_-) ,$$

and therefore that w_1 cannot fix either extension of δ_+ (or of δ_-) to the group generated by M and $\tilde{m}$.