LIE ALGEBRA COHOMOLOGY AND
THE REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

by

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ABSTRACT

We obtain a classification of the irreducible quasi-simple representations of most semisimple Lie groups, including all the classical ones. We also prove for these groups the conjecture that every irreducible representation contains some representation of a maximal compact subgroup with multiplicity one. The techniques involved are essentially algebraic, and are much simpler than those used by Langlands in his classification ([17]) of the representations for linear groups.

Let G be (say) a finite cover of a classical semisimple Lie group, and K a maximal compact subgroup. We define an ordering on $\hat{K}$ (essentially by the length of the highest weight.) Then every irreducible representation π of G has a minimal K-type $\mu \in \hat{K}$; this is just the smallest K-type occurring in $\pi|K$. To each μ we associate a cuspidal parabolic subgroup P = MAN of G, and a tempered irreducible unitary representation $\delta \in \hat{M}$, in the "limit of the discrete series." For each $\nu \in \hat{A}$, set $\pi_\mu^\nu = \operatorname{Ind}_{MAN\uparrow G} (\delta \otimes \nu \otimes 1)$. Then μ is the minimal K-type of π_μ^ν, and occurs with multiplicity one. Let $\overline{\pi}_\mu^\nu$ denote the unique irreducible subquotient of π_μ^ν containing the K-type μ. Then $\{\overline{\pi}_\mu^\nu\}$ is precisely the set of irreducible representations of G with minimal K-type μ. The only difficult part of this program is showing that there are not too many irreducible representations with minimal K-type μ. This is done as follows. One proves first that certain other K-types are smaller than μ. (This is done case-by-case, and occupies the bulk of the thesis.) If π has minimal K-type μ, it follows that these K-types do not occur in π; and one car use spectral sequence techniques to show that certain cohomology groups do not vanish. It is shown that this greatly restricts the action of $U(\mathfrak{g})^K$ on the μ-primary component of π, and hence greatly restricts π.

Thesis supervisor: Bertram Kostant
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1. Introduction

The beautiful simplicity and power of representation theory for compact Lie groups have inspired many attempts at generalization. This thesis concerns one of the simplest possibilities which has been considered, namely a classification theory for the irreducible K-finite representations of a semisimple Lie group. Such a theory, for linear groups, is provided by the work of Harish-Chandra, Langlands, and Knapp and Zuckerman ([17], [15]). Unfortunately their theory relies on rather deep analysis on the group, and the results are less than explicit from an algebraic perspective: an arbitrary representation is realized, roughly speaking, as the image of a certain integral intertwining operator between two induced representations, which are specified in terms of the asymptotic behavior of matrix coefficients of the representation.

For the classical groups, and some of the exceptional groups, this thesis provides an essentially algebraic classification theory. The basic notion is that of a minimal K-type (Definition 4.1.) Theorem 3.12 relates the existence of a nice minimal K-type for a representation to the structure of certain Lie algebra cohomology groups. According to Theorem 3.15, this in turn gives a great deal of information about the action of $U(\mathfrak{g})^K$ on the minimal K-type. In this way the "uniqueness" part of a classification theory is reduced to showing that the minimal K-type of any representation is nice (conjecture 4.2). This essentially geometric problem is dealt with in section 4 for a number of examples, including all the classical groups. In particular we prove for these examples the well known conjecture that every irreducible K-finite representation has a K-type (the minimal one) which occurs with multiplicity one. The "existence" part of the classification theory is proved along fairly standard lines (Theorem 6.2).

Not surprisingly, it turns out (for purely formal reasons) that this algebraic classification is very closely related to the analytic one. The algebraic translations of some of Langlands' results provide rather explicit cyclic vectors in certain induced representations (Corollary 6.7); this is a partial generalization of results of Kostant and Helgason for the spherical principal series, and may be of independent interest.

About two-thirds of this thesis is devoted to the proofs of Theorems 4.6 and 5.4, which proceed on a case-by-case basis. It seems reasonable to hope that these arguments can be greatly improved before too long, and they should probably be omitted on a first reading. Although the results are crucial, the proofs have no bearing on the rest of the development.

2. Preliminary results and notation

One or two general comments about the terminology are in order. This thesis might better have been called "Little tiny K-types," and it may require some effort to keep track of the various notions of little tiny. The key concepts are "π-minimal in X" (Definition 3.11); "minimal $\mathfrak{k}$-type" (Definition 4.1); and "small $\mathfrak{k}$-type" (Definition 5.3). Conceptually the distinctions among these are quite straightforward; keeping them in mind may make the overall structure of the arguments clearer. Most of the proofs below are quite easy, and one only has to worry about putting them in the right order.

The real dual of a vector space V is written V'; an asterisk denotes the complex dual. Especially in section 3, however, an asterisk may also mean "plunk," as in Λ^*V, the exterior algebra of V. X^Y denotes the centralizer of Y in X, or the subspace of X annihilated by Y, etc.

Much of the material in the present section may be found in Warner's book [25], but a number of proofs are sketched for the convenience of the reader.

Let G be a connected reductive Lie group. The Lie algebra of G will be denoted by $\mathfrak{g}_0$, with complexification $(\mathfrak{g}_0)_{\mathbb{C}} = \mathfrak{g}$, and universal enveloping algebra $U(\mathfrak{g})$.

Analogous notation for other groups (i.e. H, $\mathfrak{h}_0$, $\mathfrak{h}$, $U(\mathfrak{h})$) is followed throughout without further comment. Choose a Cartan involution θ of $\mathfrak{g}_0$, and a non-singular invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_0$. θ induces a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$; here $\mathfrak{k}_0$ is a compact Lie algebra. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}_0$: <u>notice that K need not be compact</u>.

<u>Definition 2.1</u> G (or rather the triple $(G, \theta, \langle\ ,\ \rangle)$) is said to <u>satisfy definition 2.1</u> if

i) $\mathfrak{p}_0 = \mathfrak{k}_0^{\perp}$ with respect to $\langle\ ,\ \rangle$

ii) $\langle\ ,\ \rangle|\mathfrak{k}_0$ is negative definite, and $\langle\ ,\ \rangle|\mathfrak{p}_0$ is positive definite

iii) $K \times \mathfrak{p}_0 \to G$ by $(x,y) \to x\cdot\exp(y)$ is an analytic diffeomorphism.

Henceforth G will be assumed to satisfy Definition 2.1. (Notice that this is certainly the case if G is semisimple and $\langle\ ,\ \rangle$ is the Killing form.) In particular K is a closed subgroup. By passing to a finite cover $\tilde{G}$, we can write $\tilde{G} = \tilde{K}_2\tilde{K}_1(\exp\mathfrak{p}_0^1)(\exp\mathfrak{p}_0^2)$, where $(\tilde{K}_1\exp\mathfrak{p}_0^1)$ is semisimple, and $\tilde{K}_2(\exp\mathfrak{p}_0^2)$ is central in $\tilde{G}$. One easily deduces the existence of an Iwasawa decomposition of G, and obvious generalizations of Harish-Chandra's representation theory for semisimple groups. These will be cited as they are needed.

Let $T \subseteq K$ be a Cartan subgroup of K: T is connected. Put $\mathfrak{h} = \mathfrak{g}^t = \mathfrak{k}^t + \mathfrak{p}^t = \mathfrak{t} + \mathfrak{p}^t$.

Lemma 2.2. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proof. The Lie algebra $\mathfrak{k}_0 + i\mathfrak{p}_0$ is compact, so all its subalgebras are reductive. But $\mathfrak{h} = (\mathfrak{t}_0 + i\mathfrak{p}_0^t)_{\mathbb{C}}$, and $\mathfrak{t}_0 + i\mathfrak{p}_0^t \subseteq \mathfrak{k}_0 + i\mathfrak{p}_0$; so $\mathfrak{h}$ is reductive. Since $\mathfrak{g}^{\mathfrak{h}} \subseteq \mathfrak{g}^t = \mathfrak{h}$, it remains only to check that $\mathfrak{h}$ is abelian. $[\mathfrak{h},\mathfrak{h}]$ is semisimple or zero, so it will suffice to show $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{t}$. Clearly $[\mathfrak{t},\mathfrak{t}] = [\mathfrak{t},\mathfrak{p}^t] = 0$, and $[\mathfrak{p}^t,\mathfrak{p}^t] \subseteq [\mathfrak{p},\mathfrak{p}]^t \subseteq \mathfrak{k}^t = \mathfrak{t}$. So
$[\mathfrak{h},\mathfrak{h}] = [\mathfrak{t},\mathfrak{t}] + [\mathfrak{t},\mathfrak{p}^t] + [\mathfrak{p}^t,\mathfrak{p}^t] \subseteq \mathfrak{t}$. Q.E.D.

Let $\Delta \subseteq \mathfrak{h}^*$ denote the set of non-zero roots of $\mathfrak{h}$ in $\mathfrak{g}$; it is easy to check that in fact $\Delta \subseteq i\mathfrak{t}_0' + (\mathfrak{p}_0^t)'$. The form $\langle \; , \; \rangle$ induces a nonsingular form (also written $\langle \; , \; \rangle$) on $\mathfrak{h}^*$, which is positive definite on $i\mathfrak{t}_0' + (\mathfrak{p}_0^t)'$. The decomposition $\mathfrak{h}^* = \mathfrak{t}^* + (\mathfrak{p}^t)^*$ is orthogonal; write elements of $\mathfrak{h}^*$ as (α,β) accordingly. Set

$$\langle(\alpha,\beta), (\alpha',\beta')\rangle = \langle\alpha,\alpha'\rangle_{\mathfrak{k}} + \langle\beta,\beta'\rangle_{\mathfrak{p}}$$

with obvious notation. If $\gamma = (\alpha,\beta)$, $\gamma' = (\alpha',\beta')$, it will occasionally be convenient to write $\langle\gamma,\gamma'\rangle_{\mathfrak{k}}$

instead of $\langle\alpha,\alpha'\rangle_{\mathfrak{k}}$.

Since $\mathfrak{h} = \mathfrak{g}^{\mathfrak{t}}$, every root has support on $\mathfrak{t}$: i.e. if $(\alpha,\beta) \in \Delta$, $\alpha \neq 0$. We write $\Delta = \Delta_{i\mathbb{R}} \cup \Delta_{\mathbb{C}}$, the <u>imaginary</u> and <u>complex roots</u>, where

$\Delta_{i\mathbb{R}} = \{(\alpha,\beta) \in \Delta | \beta = 0\}$, $\Delta_{\mathbb{C}} = \{(\alpha,\beta) \in \Delta | \beta \neq 0\}$.

If X is a root vector of weight (α,β), then θX is a root vector of weight $\theta(\alpha,\beta) = (\alpha,-\beta)$. So Δ is θ-invariant, $\Delta_{i\mathbb{R}}$ is precisely the set of θ-fixed roots, and the elements of $\Delta_{\mathbb{C}}$ occur in θ-conjugate pairs.

If $\mathfrak{p}^{\mathfrak{t}} \neq 0$, the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is not invariant under $\mathfrak{h}$. For this reason it will be important to understand the structure of the system of roots restricted to $\mathfrak{t}$, which we call <u>$\mathfrak{t}$-roots</u>. Suppose $(\alpha,\beta) \in \Delta$, and X is a non-zero root vector of weight (α,β), as above, so that θX is a non-zero root vector of weight $(\alpha,-\beta)$. If $\beta = 0$, then the fact that the root spaces are one dimensional forces $\theta X = cX$; and $\theta^2 = 1$ implies $c^2 = 1$, i.e. $c = \pm 1$. If $c = 1$, then $X \in \mathfrak{k}$, and we call $(\alpha,0)$ a <u>compact root</u>. If $c = -1$, $X \in \mathfrak{p}$, and $(\alpha,0)$ is a <u>noncompact</u> root. If $\beta \neq 0$, then X and θX are linearly independent, so X is neither in $\mathfrak{k}$ nor in $\mathfrak{p}$. But $X + \theta X$ is a non-zero element of $\mathfrak{k}$, and is a root vector for $\mathfrak{t}$ with weight α. Similarly $X - \theta X$ is a non-zero $\mathfrak{t}$-root vector in $\mathfrak{p}$, also of weight α. We claim that these span the space of $\mathfrak{t}$-root vectors of weight α. What must be shown is that if $(\alpha,\beta') \in \Delta$, then $\beta' = \pm \beta$. Replacing β by $-\beta$ if necessary, we may assume $<(\alpha,\beta), (\alpha,\beta')> \geq <\alpha,\alpha>_{\mathfrak{k}} > 0$. By abstract root theory, this implies that $(\alpha,\beta) - (\alpha,\beta') = (0,\beta-\beta')$ is a root, which forces $\beta = \beta'$. This proves the claim. $X + \theta X$ and $X - \theta X$ are called the <u>compact</u> and <u>noncompact</u> $\mathfrak{t}$-root vectors of weight α. Depending on which of these is under consideration, we will speak of the <u>compact root α</u> or the <u>noncompact root α</u>. One could make this quite rigorous by speaking of $\mathfrak{t}$-θ roots, i.e. the weights of $\mathfrak{t}$-invariant, θ-invariant subspaces of $\mathfrak{g}$; but no confusion should arise as long as the reader is aware of the situation.

If V is a $\mathfrak{t}$-invariant subspace of $\mathfrak{g}$, we write $\Delta_{\mathfrak{t}}(V)$ for the corresponding set of $\mathfrak{t}$-roots with multiplicities; if V is also $\mathfrak{h}$-invariant, $\Delta(V)$ will be the corresponding set of roots. When there is no possibility of confusion, the subscript $\mathfrak{t}$ in $\Delta_{\mathfrak{t}}(V)$ may be dropped to simplify the notation. If V is θ-invariant, we may refer to the compact and noncompact $\mathfrak{t}$-roots in $\Delta_{\mathfrak{t}}(V)$, in accordance with the preceding paragraph. In this case a $\mathfrak{t}$-root is called <u>imaginary</u> or <u>complex</u>

depending on whether it corresponds to an imaginary $\mathfrak{h}$-root or to a θ-conjugate pair of complex $\mathfrak{h}$-roots. This discussion may be summarized as

Proposition 2.3 The imaginary elements of $\Delta_{\mathfrak{t}}(\mathfrak{g})$ occur with multiplicity one; the complex elements occur with multiplicity two, namely one compact and one noncompact $\mathfrak{t}$-root.

Fix once and for all an algebraic ordering $\succ$ of $i\mathfrak{t}_0'$, and let $\Delta^+(\mathfrak{k}) = \Delta_{\mathfrak{t}}^+(\mathfrak{k}) = \{\alpha \in \Delta_{\mathfrak{t}}(\mathfrak{k}) \mid \alpha \succ 0\}$ be the associated system of compact positive roots. To every $\gamma \in i\mathfrak{t}_0'$ which is dominant with respect to $\Delta^+(\mathfrak{k})$, we associate a positive root system

$\Delta^+(\mathfrak{g}) = \{(\alpha,\beta) \in \Delta(\mathfrak{g}) \mid \langle\alpha\ \gamma\rangle_{\mathfrak{k}} > 0, \text{ or } \langle\alpha,\gamma\rangle_{\mathfrak{k}} = 0 \text{ and } \alpha \succ -\alpha\}.$
Then Δ^+ is a θ-invariant positive root system for $\mathfrak{h}$ in $\mathfrak{g}$, and $\Delta_{\mathfrak{t}}^+ \supseteq \Delta^+(\mathfrak{k})$. Δ^+ will be regarded as fixed in this section and the next. Put $\rho = \frac{1}{2} \sum_{(\alpha,\beta)\in\Delta^+} (\alpha,\beta)$. Since Δ^+ is θ-invariant, $\theta\cdot\rho = \rho$, i.e. $\rho \in \mathfrak{t}^*$; write $\rho = (\rho,0)$. Let $\mathfrak{n}^0 \subseteq \mathfrak{g}$ be the nilpotent subalgebra corresponding to Δ^+, and $\mathfrak{b}^0 = \mathfrak{n}^0 + \mathfrak{h}$ the associated Borel subalgebra. All of these are θ-invariant, so $\mathfrak{b}^0 = \mathfrak{b}^0 \cap \mathfrak{k} + \mathfrak{b}^0 \cap \mathfrak{p}$, etc. $\mathfrak{b}^0 \cap \mathfrak{k} = \mathfrak{n}^0 \cap \mathfrak{k} + \mathfrak{t}$ is a Borel subalgebra of $\mathfrak{k}$. The associated set of positive roots is $\Delta^+(\mathfrak{k})$.

Let $W_K = W(\mathfrak{k},\mathfrak{t})$ be the Weyl group of $\mathfrak{t}$ in $\mathfrak{k}$;

this will be referred to as the compact Weyl group. In the interest of mathematical purity, we would like to consider W_K as a subgroup of $W(\mathfrak{g}, \mathfrak{h})$. Let α be a compact $\mathfrak{t}$-root. If α is imaginary, we send the reflection $s_\alpha \in W(\mathfrak{k}, \mathfrak{t})$ to $s_{(\alpha,0)} \in W(\mathfrak{g}, \mathfrak{h})$. If α corresponds to the complex root (α,β), we know $(\alpha,\beta) - (\alpha,-\beta)$ is not a root. There are two cases:

i) $(2\alpha,0) = (\alpha,\beta) + (\alpha,-\beta)$ is a root. Send s_α to $s_{(2\alpha,0)}$.

ii) $(\alpha,\beta) + (\alpha,-\beta)$ is not a root. Then by abstract root theory, $\langle(\alpha,\beta), (\alpha,-\beta)\rangle = 0$, i.e. $\langle\alpha,\alpha\rangle_{\mathfrak{k}} = \langle\beta,\beta\rangle_{\mathfrak{p}}$. By direct computation, this implies $s_{(\alpha,\beta)}s_{(\alpha,-\beta)} = s_{(\alpha,-\beta)}s_{(\alpha,\beta)}$; their common value on $(x,y) \in \mathfrak{h}^*$ is $(x,y) - (\frac{2\langle x,\alpha\rangle_{\mathfrak{k}}}{\langle\alpha,\alpha\rangle_{\mathfrak{k}}}\alpha, \frac{2\langle y,\beta\rangle_{\mathfrak{p}}}{\langle\beta,\beta\rangle_{\mathfrak{p}}}\beta)$. Map s_α to this element of $W(\mathfrak{g}, \mathfrak{h})$. Let $\tilde{W}_K$ be the free group generated by the s_α. We have defined a map $\tilde{\phi} : \tilde{W}_K \to W(\mathfrak{g},\mathfrak{h})$; and of course there is a natural map $\psi : \tilde{W}_K \to W_K$. By inspection, these maps satisfy $\tilde{\phi}(\tilde{\sigma})|\mathfrak{t}^* = \psi(\tilde{\sigma})$. Suppose that $\psi(\tilde{\sigma}) = 1$. Since $\rho \in \mathfrak{t}^*$, the preceding observation implies $\tilde{\phi}(\tilde{\sigma})\cdot\rho = \psi(\tilde{\sigma})\cdot\rho = 1\cdot\rho = \rho$. But the only element of $W(\mathfrak{g},\mathfrak{h})$ which fixes ρ is the identity, so $\tilde{\phi}(\tilde{\sigma}) = 1$. Thus $\tilde{\phi}$ actually defines a map $\phi : W_K \to W(\mathfrak{g},\mathfrak{h})$, satisfying $\phi(\sigma)|t^* = \sigma$. Clearly ϕ is an embedding; henceforth W_K

will be freely identified as a subgroup of $W(\mathfrak{g},\mathfrak{h})$. The problem of finding a more natural way to do this is left to the reader.

We will be quite interested in a certain special class of groups.

Definition 2.4 $\mathfrak{g}_0$ is said to be split if the real semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ is split in the usual sense, i.e. if $[\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{p}$ contains a Cartan subalgebra of $[\mathfrak{g}_0, \mathfrak{g}_0]$.

It will be useful to have another description of such $\mathfrak{g}_0$. Recall that a subset $\{\alpha_1 \ldots \alpha_n\}$ of the positive roots in an abstract root system is said to be strongly orthogonal if $\alpha_i \pm \alpha_j$ is not a root for any i, j. Let $\mathfrak{t}^*_{root}$ be the span of $\Delta_{\mathfrak{t}}$ in $\mathfrak{t}^*$.

Proposition 2.5 $\mathfrak{g}_0$ is split if and only if $\mathfrak{t}^*_{root}$ is spanned by a strongly orthogonal set of (positive) noncompact imaginary roots.

Proof. It is enough to assume $\mathfrak{g}$ is semisimple, so that $\mathfrak{t}^*_{root} = \mathfrak{t}^*$. Extend the abelian algebra $\mathfrak{p}_0^{\mathfrak{t}}$ to a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$. Then

$\mathfrak{a}_0 \subseteq \mathfrak{g}_0^{\mathfrak{p}_0^{\mathfrak{t}}} = \mathfrak{m}_0$. Clearly $\mathfrak{m}$ is reductive, and

contains $\mathfrak{h}$ as a Cartan subalgebra; and $\Delta(\mathfrak{m}) = \Delta_{i\mathbb{R}}$.
$\mathfrak{m}_0$ is an equal rank Lie algebra, i.e. $[\mathfrak{m}_0, \mathfrak{m}_0] \cap \mathfrak{k}$ contains a Cartan subalgebra of $[\mathfrak{m}_0, \mathfrak{m}_0]$.

Suppose $\mathfrak{g}_0$ is split. Then $\mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, and therefore of $\mathfrak{m}_0$; in particular $\mathfrak{m}_0$ is split. Also $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{m}_0$, so $\mathfrak{t}$ does not meet the center of $\mathfrak{m}$. It follows that $\Delta_{\mathfrak{t}}(\mathfrak{m}) = \Delta_{i\mathbb{R}}$ spans $\mathfrak{t}^*$; so $\mathfrak{t}_0$ is a compact Cartan subalgebra of the split semisimple equal rank group $[\mathfrak{m}_0, \mathfrak{m}_0]$. By a theorem of Martens (see Schmid [22]) $\mathfrak{t}^*$ is spanned by a strongly orthogonal set of positive noncompact roots of $[\mathfrak{m}_0, \mathfrak{m}_0]$, i.e. by a strongly orthogonal set of positive noncompact imaginary roots.

For the converse, this argument may be reversed - Martens' theorem is a necessary and sufficient condition. This amounts to writing down the split Cartan subalgebra corresponding to a strongly orthogonal set of noncompact imaginary roots, using a Cayley transform. Q.E.D.

Let $\mathfrak{b} \supset \mathfrak{b}^0$ be an arbitrary θ-invariant parabolic subalgebra of $\mathfrak{g}$, with nil radical $\mathfrak{n}$ and Levi subalgebra $\mathfrak{l} \supset \mathfrak{h}$; then $\mathfrak{b} = \mathfrak{l} + \mathfrak{n}$. We claim that $\mathfrak{l}$ is the complexification of a real subalgebra $\mathfrak{l}_0$. Let bar denote conjugation with respect to the real form $\mathfrak{g}_0$;

we must show $\Delta(\ell) = \overline{\Delta(\ell)}$. If $(\alpha,\beta) \in \Delta$, then $\alpha \in i\mathfrak{t}_0'$ and $\beta \in (\mathfrak{p}_0^{\mathfrak{t}})'$, so

$$\overline{(\alpha,\beta)} = (-\alpha,\beta) = -(\alpha,-\beta).$$

Hence

$$(2.6) \qquad (\alpha,\beta) \in \Delta \Rightarrow \overline{(\alpha,\beta)} = -\theta(\alpha,\beta).$$

Since $\Delta(\ell)$ is stable under -1 and θ, the assertion is proved.

Let $\mathcal{C} = \mathfrak{t}^+ + \mathfrak{a}^+ \subseteq \mathfrak{t} + \mathfrak{p}^{\mathfrak{t}}$ be the center of ℓ. Notice that $\mathfrak{t}^- = (\mathfrak{t}^+)^{\perp}$ corresponds under $\langle\, ,\, \rangle_{\mathfrak{k}}$ to $\mathfrak{t}^*_{\text{root}}(\ell)$; thus $\mathfrak{t}^- = \mathfrak{t} \cap [\ell,\ell]$.

<u>Proposition 2.7</u> $\quad$ <u>$\ell = \mathfrak{g}^{\mathfrak{t}^+}$</u>

<u>Proof</u>. Clearly $\ell \subseteq \mathfrak{g}^{\mathfrak{t}^+}$. By the general theory of parabolic subalgebras, we can find $(x,y) \in \mathcal{C}$ so that $((\alpha,\beta))(x,y) \geq 0$ whenever $(\alpha,\beta) \in \Delta^+$, with equality precisely for $(\alpha,\beta) \in \Delta(\ell) \cap \Delta^+$. Now suppose $(\alpha,\beta) \in \Delta^+ - \Delta(\ell)$. Since this set is θ-invariant, $(\alpha,-\beta) \in \Delta^+ - \Delta(\ell)$. Thus $((\alpha,\pm\beta))(x,y) > 0$, which forces $\alpha(x) > 0$. Hence the corresponding root vector $X^{(\alpha,\beta)}$ does not commute with x; so $X^{(\alpha,\beta)} \notin \mathfrak{g}^{\mathfrak{t}^+}$. This proves $\ell \supseteq \mathfrak{g}^{\mathfrak{t}^+}$. $\qquad$ Q.E.D.

Set $L = G^{\mathfrak{t}^+}$. Then L is a closed subgroup of G, with Lie algebra $\mathfrak{l}_0$. The decomposition $G = K \cdot \exp \mathfrak{p}_0$ is invariant under conjugation by $\exp(\mathfrak{t}_0^+)$, so $L = K^{\mathfrak{t}^+} \cdot (\exp \mathfrak{p}_0^{\mathfrak{t}^+})$. That $K^{\mathfrak{t}^+}$ is connected is a standard fact for K compact, easily generalized to the present situation. Hence L satisfies definition 2.1 with respect to the restriction of $< , >$ and θ.

Put $R = \dim \mathfrak{n} \cap \mathfrak{p}$, $S = \dim \mathfrak{n} \cap \mathfrak{k}$. If $V \subseteq \mathfrak{g}$ is a $\mathfrak{t}$-invariant subspace, with $A = \Delta_{\mathfrak{t}}(V)$, write

$$\rho(V) = \rho(A) = \frac{1}{2} \sum_{\alpha \varepsilon A} \alpha \ \varepsilon \ \mathfrak{t}^*$$

We use the same notation for $\mathfrak{h}$-invariant subspaces and Δ; in other words, if V is an $\mathfrak{h}$-invariant subspace, $\rho(V) \ \varepsilon \ \mathfrak{h}^*$. An unqualified ρ will always mean $\rho(\mathfrak{n}^0) = \rho(\Delta^+)$. We also define $\rho_c = \rho(\mathfrak{n}^0 \cap \mathfrak{k}) = \rho(\Delta^+(\mathfrak{k}))$.

Finally, we need a little representation theory. For details see Warner [25]. Finite dimensional irreducible modules for K, $\mathfrak{k}$, $L \cap K$, T, etc., will be freely identified with their highest weights (with respect to $\mathfrak{t}$ and an appropriate part of $\mathfrak{n}^0 \cap \mathfrak{k}$.) The set of all equivalence classes of such modules will be written $\hat{K}$, $\hat{\mathfrak{k}}$, etc. Let V be a complete locally convex space, and $\pi : G \to \text{Aut } V$ a continuous irreducible representation. V has a dense subspace V^∞ of differentiable vectors, on which $U(\mathfrak{g})$ acts. Suppose the center of G acts by scalars on V (e.g. if V is a Banach space and π is topologically completely irreducible (TCI)). Then V^∞ decomposes nicely into K-primary components V_γ^∞, $\gamma \in \hat{K}$. Let $\mathfrak{z}(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ denote the center of $U(\mathfrak{g})$. π is said to admit a <u>central character</u> $\chi_\pi : \mathfrak{z}(\mathfrak{g}) \to \mathbf{C}$ if every $z \in \mathfrak{z}(\mathfrak{g})$ acts on V^∞ by a scalar $\chi_\pi(z)$ (e.g. if π is TCI as above). In this case each V_γ^∞ is finite dimensional, and the $U(\mathfrak{g})$ module $V_K = \sum_{\gamma\in\hat{K}} V_\gamma^\infty$ is algebraically irreducible. Even if π is not irreducible,

<u>Definition 2.8</u> π is called <u>admissible</u> if V^∞ decomposes into finite dimensional K-primary components V_γ^∞; the $U(\mathfrak{g})$ module $V_K = \sum_{\gamma\in\hat{K}} V_\gamma^\infty$ is called the <u>Harish-Chandra module</u> of π.

In case π is admissible, V_K is algebraically irreducible iff V is topologically irreducible. Both V_K and V have (possibly infinite) composition series; the subquotients are admissible and irreducible. Two such representations are called <u>infinitesimally equivalent</u> if their Harish-Chandra modules are algebraically equivalent. (This is the same as Naimark equivalence or, in the unitary case, unitary equivalence.) Let $U^{\mathfrak{k}}$ be the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$. This ring preserves the $\mathfrak{k}$-primary decomposition of V_K, so V_γ^∞ becomes a $U^{\mathfrak{k}}$ module for each $\gamma \in \hat{\mathfrak{k}}$.

<u>Theorem 2.9</u> (Harish-Chandra) <u>If π is admissible and irreducible, and $V_\gamma^\infty \neq 0$, then the $U^{\mathfrak{k}}$ module structure of V_γ^∞ determines π up to infinitesimal equivalence.</u>

Of course all of this works for L as well as for G.

3. Lie algebra cohomology

Most of the basic results on Lie algebra cohomology and spectral sequences used here may be found in Cartan-Eilenberg [2]. The proofs of Theorems 3.3 and 3.5 were inspired by a paper of Casselman and Osborne ([4]); Allan Cooper suggested that their result (a weak form of Theorem 3.3) could be formulated along the more natural lines adopted here.

Write $U(\mathfrak{g}) = [U(\mathfrak{l}) \otimes U(\bar{\mathfrak{n}})] \oplus \mathfrak{n}\, U(\mathfrak{g})$, and let $\tilde{\xi} : U(\mathfrak{g}) \to U(\mathfrak{l}) \otimes U(\bar{\mathfrak{n}})$ be the corresponding projection. The idea of the proof of Proposition 2.7 shows that if $u \in U(\mathfrak{g})^{\mathfrak{t}}$, then $\tilde{\xi}(u) \in U(\mathfrak{l})^{\mathfrak{t}}$; notice that this is not quite standard since $\mathfrak{t}$ is not a full Cartan subalgebra of $\mathfrak{g}$. But the usual argument does show that $\tilde{\xi}|U(\mathfrak{g})^{t}$ is a homomorphism, and that

$$
\begin{aligned}
&\tilde{\xi} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{l}) \\
(3.1) \qquad &\tilde{\xi} : U(\mathfrak{g})^{\mathfrak{k}} \to U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}} .
\end{aligned}
$$

Now $R + S = \dim(\mathfrak{n} \cap \mathfrak{p}) + \dim(\mathfrak{n} \cap \mathfrak{k}) = \dim \mathfrak{n}$, so $\Lambda^{R+S}\mathfrak{n}$ is a one dimensional $\mathfrak{l}$ module. For $m \in \mathfrak{l}$, let $\phi(m)$ denote the scalar by which m acts on $\Lambda^{R+S}\mathfrak{n}$. Then $m \to m + \phi(m)$ defines a Lie algebra homomorphism of $\mathfrak{l}$ into $U(\mathfrak{l})$, which extends to an algebra automorphism $\tilde{\phi}$ of

$U(\ell)$. Define $\xi = \tilde{\phi} \circ \tilde{\xi}$; then

$$\xi : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\ell)$$

(3.2)

$$\xi : U(\mathfrak{g})^{k} \to U(\ell)^{\ell \cap k}$$

Suppose X is a $\mathfrak{g}$-module. Write $H^i(\mathfrak{n},X)$ for the ith Lie algebra cohomology group of X with respect to the subalgebra $\mathfrak{n}$ of $\mathfrak{g}$. Recall that there is an operator

$$d : \mathrm{Hom}\,(\Lambda^i \mathfrak{n}, X) \to \mathrm{Hom}\,(\Lambda^{i+1} \mathfrak{n}, X),$$

the <u>coboundary operator of Lie algebra cohomology</u>, and that $H^i(\mathfrak{n},X)$ is defined to be the ith cohomology group of the resulting cochain complex. If $\mathfrak{s} \subseteq \mathfrak{g}$ normalizes $\mathfrak{n}$, then $H^i(\mathfrak{n},X)$ has the structure of an $\mathfrak{s}$ module.

$\mathrm{Hom}\,(\Lambda^* \mathfrak{n}, X)$ can be made into a $\mathfrak{Z}(\mathfrak{g})$ module, by acting only on the second factor. This action clearly commutes with d (because $\mathfrak{Z}(\mathfrak{g})$ centralizes $\mathfrak{n}$) so that $H^*(\mathfrak{n},X)$ is a $\mathfrak{Z}(\mathfrak{g})$ module. In the case of $H^0(\mathfrak{n},X)$, this is just the fact that $X^{\mathfrak{n}}$ is invariant under $\mathfrak{Z}(\mathfrak{g})$. One knows that the action of $\mathfrak{Z}(\mathfrak{g})$ on $X^{\mathfrak{n}}$ factors through a certain homomorphism of $\mathfrak{Z}(\mathfrak{g})$ into $\mathfrak{Z}(\ell)$ - this observation is the key to the computation of the central characters of the finite dimensional representations of $\mathfrak{g}$. The following theorem is a fairly straightforward generalization to arbitrary cohomology; of course the proof is by "dimension shifting."

Theorem 3.3 *If $z \in \mathfrak{z}(\mathfrak{g})$ and $\omega \in H^i(\mathfrak{n}, X)$, then*

$$z \cdot \omega = \xi(z) \cdot \omega$$

Here the action on the left side of the equation is that just defined, and the action on the right is the usual $\mathfrak{l}$-module structure.

Proof. Proceed by downward induction on i, starting with the case $i = R + S = \dim \mathfrak{n}$. It is immediate from the definition that

$$H^{R+S}(\mathfrak{n}, X) \cong \operatorname{Hom}(\Lambda^{R+S}\mathfrak{n}, X/\mathfrak{n}X) \cong (\Lambda^{R+S}\mathfrak{n})^* \otimes X/\mathfrak{n}X.$$

Since $\mathfrak{l}$ normalizes $\mathfrak{n}$, $\mathfrak{n}X$ is invariant under $\mathfrak{l}$. Thus both $(\Lambda^{R+S}\mathfrak{n})^*$ and $X/\mathfrak{n}X$ are $\mathfrak{l}$ modules. As an $\mathfrak{l}$ module, $H^{R+S}(\mathfrak{n}, X)$ is their tensor product. The action of $\mathfrak{z}(\mathfrak{g})$ on $H^{R+S}(\mathfrak{n}, X)$ is on the $X/\mathfrak{n}X$ factor alone.

If $x \in X$, write $\bar{\bar{x}}$ for the corresponding element of $X/\mathfrak{n}X$. Suppose that $N \in (\Lambda^{R+S}\mathfrak{n})^*$, $x \in X$, and that $\omega = N \otimes \bar{\bar{x}} \in H^{R+S}(\mathfrak{n}, X)$. Then $z \cdot \omega = N \otimes \overline{\overline{z \cdot x}}$. Also $z \equiv \tilde{\xi}(z) \bmod \mathfrak{n}U(\mathfrak{g})$, and $\mathfrak{n}U(\mathfrak{g}) \cdot x \subseteq \mathfrak{n}X$; so $z \cdot \omega = N \otimes \overline{\overline{\tilde{\xi}(z) \cdot x}}$. On the other hand, suppose $m \in \mathfrak{l}$. Recall that m acts on $\Lambda^{R+S}\mathfrak{n}$ by the scalar $\phi(m)$. Since $N \in (\Lambda^{R+S}(\mathfrak{n}))^*$, $m \cdot N = -\phi(m)N$. By the definition of a tensor product of $\mathfrak{l}$ modules,

$$\tilde{\phi}(m) \cdot (N \otimes \bar{\bar{x}}) = (m + \phi(m)) \cdot (N \otimes \bar{\bar{x}})$$
$$= (m \cdot N) \otimes \bar{\bar{x}} + N \otimes \overline{\overline{m \cdot x}} + \phi(m)(N \otimes \bar{\bar{x}})$$
$$= -\phi(m)(N \otimes \bar{\bar{x}}) + N \otimes \overline{\overline{m \cdot x}} + \phi(m)(N \otimes \bar{\bar{x}})$$
$$= N \otimes \overline{\overline{mx}} .$$

Since $\tilde{\phi}$ is an automorphism of $U(\ell)$, this implies

(3.4) for all $y \in U(\ell)$, $\tilde{\phi}(y) \cdot (N \otimes \bar{\bar{x}}) = N \otimes \overline{\overline{y \cdot x}}$.

In particular, $z \cdot \omega = N \otimes \overline{\overline{\tilde{\xi}(z) \cdot x}} = \tilde{\phi}(\tilde{\xi}(z)) \cdot (N \otimes \bar{\bar{x}}) = \xi(z) \cdot \omega$. This establishes the result for $i = R + S$.

Suppose that the theorem holds for some $i + 1 \leq R + S$. By choosing a set of generators for X, one can find a free $U(\mathfrak{g})$ module F and a submodule F_0 so that $0 \to F_0 \to F \to X \to 0$ is exact. There is then a long exact sequence

$$\cdots \to H^i(\mathfrak{n}, F) \to H^i(\mathfrak{n}, X) \overset{\delta}{\to} H^{i+1}(\mathfrak{n}, F_0) \to \cdots$$

Now $U(\mathfrak{g})$ is a free $U(\mathfrak{n})$ module, so F is also free as a $U(\mathfrak{n})$ module. It follows that $H^j(\mathfrak{n}, F) = 0$ for $j < \dim \mathfrak{n} = R + S$. Since $i + 1 \leq R + S$, $H^i(\mathfrak{n}, F) = 0$. Thus δ is injective. By the inductive hypothesis, $z \cdot \delta(\omega) = \xi(z) \cdot \delta(\omega)$. It is standard that δ commutes with the action of $U(\ell)$, so that $\xi(z) \cdot \delta(\omega) = \delta(\xi(z) \cdot \omega)$. We have observed that the action of $\mathfrak{Z}(\mathfrak{g})$ commutes with d on the cochain level; it follows easily that $z \cdot \delta(\omega) = \delta(z \cdot \omega)$. So $\delta(z \cdot \omega) = \delta(\xi(z) \cdot \omega)$. Since δ is injective, $z \cdot \omega = \xi(z) \cdot \omega$. Q.E.D.

Notice that the argument was entirely formal: the definitions of this section are much deeper than the theorems.

The next result is a little more subtle. Hom $(\Lambda^* \mathfrak{n} \cap \mathfrak{k}, X)$ can be made into a $U(\mathfrak{g})^{\mathfrak{k}}$ module, by acting only on the second factor. This action commutes with d, so that $H^*(\mathfrak{n} \cap \mathfrak{k}, X)$ is a $U(\mathfrak{g})^{\mathfrak{k}}$ module. In many cases $H^*(\mathfrak{n} \cap \mathfrak{k}, X)$ is quite large, and this action will reflect all of the $U(\mathfrak{g})^{\mathfrak{k}}$ structure of X; so one cannot hope to compute it as easily as the $\mathfrak{z}(\mathfrak{g})$ action. The appropriate rabbit is the following.

The decomposition $\Lambda^* \mathfrak{n} \cong [\Lambda^*(\mathfrak{n} \cap \mathfrak{k})] \otimes [\Lambda^*(\mathfrak{n} \cap \mathfrak{p})]$ induces projections $\pi_i : \Lambda^{i+R} \mathfrak{n} \to \Lambda^i(\mathfrak{n} \cap \mathfrak{k}) \otimes \Lambda^R(\mathfrak{n} \cap \mathfrak{k})$, and thus extensions

Hom $(\Lambda^i(\mathfrak{n} \cap \mathfrak{k}), X) \otimes [\Lambda^R(\mathfrak{n} \cap \mathfrak{p})]^* \to$ Hom $(\Lambda^{i+R} \mathfrak{n}, X)$. (Recall that $R = \dim \mathfrak{n} \cap \mathfrak{p}$.) It is not difficult to see that these extensions intertwine $d \otimes 1$ and d (i.e. they define a map of cochain complexes.) This will be proved in the course of establishing Theorem 3.14, but the reader can easily provide a direct argument. We therefore have maps

$$\pi_i : H^i(\mathfrak{n} \cap \mathfrak{k}, X) \otimes [\Lambda^R(\mathfrak{n} \cap \mathfrak{p})]^* \to H^{R+i}(\mathfrak{n}, X)$$

which preserve the $\mathfrak{l} \cap \mathfrak{k}$ module structure.

$H^*(\mathfrak{n}, X)$ tends to be relatively small, so that π_i in general has a large kernel. Theorem 3.5 computes the $U(\mathfrak{g})^{\mathfrak{k}}$ action modulo this kernel; the rest of section 3 is devoted to finding conditions for π_i to be non-trivial.

Theorem 3.5 If $u \in U(\mathfrak{g})^{\mathfrak{k}}$, $\omega \in H^i(\mathfrak{n} \cap \mathfrak{k}, X)$, and $P \in [\Lambda^R(\mathfrak{n} \cap \mathfrak{p})]^*$, then $\pi_i((u \cdot \omega) \otimes P) = \xi(u) \cdot \pi_i(\omega \otimes P)$

Proof. Proceed by downward induction on i, starting with the case $i = \dim \mathfrak{n} \cap \mathfrak{k} = S$. Then

$$H^S(\mathfrak{n} \cap \mathfrak{k}, X) \cong (\Lambda^S(\mathfrak{n} \cap \mathfrak{k}))^* \otimes X/(\mathfrak{n} \cap \mathfrak{k})X$$

$$H^{R+S}(\mathfrak{n}, X) \cong (\Lambda^{R+S}(\mathfrak{n}))^* \otimes X/\mathfrak{n}X \cong (\Lambda^S(\mathfrak{n} \cap \mathfrak{k}))^* \otimes (\Lambda^R(\mathfrak{n} \cap \mathfrak{p}))^* \otimes X/\mathfrak{n}X.$$

With respect to these identifications,

$\pi_S : (\Lambda^S(\mathfrak{n} \cap \mathfrak{k}))^* \otimes X/(\mathfrak{n} \cap \mathfrak{k})X \otimes (\Lambda^R(\mathfrak{n} \cap \mathfrak{p}))^*$

$\to (\Lambda^S(\mathfrak{n} \cap \mathfrak{k}))^* \otimes (\Lambda^R(\mathfrak{n} \cap \mathfrak{p}))^* \otimes X/\mathfrak{n}X$ is just the obvious map induced by the projection $X/(\mathfrak{n} \cap \mathfrak{k})X \to X/\mathfrak{n}X$. The action of $U(\mathfrak{g})^{\mathfrak{k}}$ on $H^S(\mathfrak{n} \cap \mathfrak{k}, X)$ is on the $X/(\mathfrak{n} \cap \mathfrak{k})X$ factor only.

If $x \in X$, write $\bar{x} \in X/(\mathfrak{n} \cap \mathfrak{k})X$, $\bar{\bar{x}} \in X/\mathfrak{n}X$. Suppose $\omega = Q \otimes x \in H^S(\mathfrak{n} \cap \mathfrak{k}, X)$, with $Q \in (\Lambda^S(\mathfrak{n} \cap \mathfrak{k}))^*$, $x \in X$. Then $u \cdot \omega = Q \otimes \overline{u \cdot x}$, so $\pi_S((u \cdot \omega) \otimes P) = (Q \otimes P) \otimes \overline{\overline{u \cdot x}}$. Now $u \equiv \tilde{\xi}(u) \bmod \mathfrak{n}U(\mathfrak{g})$, and $\mathfrak{n}U(\mathfrak{g}) \cdot x \subseteq \mathfrak{n}X$; so

$\pi_S((u \cdot \omega) \otimes P) = (Q \otimes P) \otimes \overline{\overline{\tilde{\xi}(u) \cdot x}}$. By (3.4),

$Q \otimes P \otimes \overline{\overline{\tilde{\xi}(u) \cdot x}} = \tilde{\phi}(\tilde{\xi}(u)) \cdot (Q \otimes P \otimes \overline{\overline{x}}) = \xi(u) \cdot (\pi_S(\omega \otimes P))$.

This proves the result when $i = S$.

Suppose then that the theorem holds for some $i + 1 \leq S$. Choose an exact sequence of $\mathfrak{g}$ modules $0 \to F_0 \to F \to X \to 0$, with F a free $U(\mathfrak{g})$ module. The two long exact sequences in cohomology give

$$\begin{array}{ccccc} H^i(\mathfrak{n}\cap\mathfrak{k},F)\otimes(\Lambda^R(\mathfrak{n}\cap\mathfrak{p}))^* & \to & H^i(\mathfrak{n}\cap\mathfrak{k},X)\otimes(\Lambda^R(\mathfrak{n}\cap\mathfrak{p}))^* & \xrightarrow{\delta_1\otimes 1} & H^{i+1}(\mathfrak{n}\cap\mathfrak{k},F_0)\otimes(\Lambda^R(\mathfrak{n}\cap\mathfrak{p}))^* \\ \downarrow\pi_i & & \downarrow\pi_i & & \downarrow\pi_{i+1} \\ H^{R+i}(\mathfrak{n},F) & \to & H^{R+i}(\mathfrak{n},X) & \xrightarrow{\delta_2} & H^{R+i+1}(\mathfrak{n},F_0) \end{array}$$

Since $i < S$, $H^{R+i}(\mathfrak{n},F) = 0$; so δ_2 is injective. By the inductive hypothesis,

$\pi_{i+1}((u \cdot \delta_1(\omega)) \otimes P) = \xi(u) \cdot \pi_{i+1}(\delta_1(\omega) \otimes P)$. We have observed that the action of $U^{\mathfrak{k}}$ commutes with d on the cochain level; it is immediate that $u \cdot \delta_1(\omega) = \delta_1(u \cdot \omega)$. The maps π_* intertwine d and $d \otimes 1$ on the cochain level; so $\pi_{i+1}(\delta_1 \otimes 1) = \delta_2\pi_i$. Thus $\delta_2(\pi_i((u \cdot \omega) \otimes P) = \xi(u) \cdot \delta_2(\pi_i(\omega \otimes P))$. Finally, it is standard that the action of $\mathfrak{l}$ commutes with δ_2, so that $\delta_2(\pi_i((u \cdot \omega) \otimes P) = \delta_2(\xi(u) \cdot \pi_i(\omega \otimes P))$. Since δ_2 is injective, $\pi_i((u \cdot \omega) \otimes P) = \xi(u) \cdot \pi_i(\omega \otimes P)$. Q.E.D.

Henceforth it will be assumed that X is a

Harish-Chandra module, i.e. that there is a $\mathfrak{k}$-invariant decomposition $X = \sum_{\gamma \in \hat{\mathfrak{k}}} X(\gamma)$. Here $X(\gamma)$ is a finite dimensional γ-primary semisimple $\mathfrak{k}$-module. Recall that we are identifying $\mathfrak{k}$-modules with their highest $\mathfrak{t}$-weights. Suppose then that $\gamma \in \hat{\mathfrak{k}}$, and let F be an irreducible $\mathfrak{k}$-module in the equivalence class γ. Write F^{γ} for the highest weight space of F, i.e. the subspace of F annihilated by $\mathfrak{n}^{o} \cap \mathfrak{k}$; F^{γ} is one dimensional and has $\mathfrak{t}$-weight γ. Similarly, define $X^{\gamma} \subseteq X(\gamma)$ to be the subspace of X of $\mathfrak{t}$-weight γ which is annihilated by $\mathfrak{n}^{o} \cap \mathfrak{k}$. Then the dimension of X^{γ} is precisely the multiplicity of γ in X. X^{γ} is stable under $U(\mathfrak{g})^{\mathfrak{k}}$, and the corresponding action completely determines the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on $X(\gamma)$. We make a similar convention for $\mathfrak{l} \cap \mathfrak{k}$ and $\mathfrak{l}$.

Now $\mathfrak{n}$ and X are semisimple $\mathfrak{l} \cap \mathfrak{k}$ modules, and it follows easily that $H^{*}(\mathfrak{n},X)$ is as well. The maps π_i may be restricted to the various highest weight spaces with respect to $\mathfrak{l} \cap \mathfrak{k}$: write

$$\pi_i^{\gamma} : [H^i(\mathfrak{n} \cap \mathfrak{k}, X) \otimes (\Lambda^R(\mathfrak{n} \cap \mathfrak{p}))^*]^{\gamma - 2\rho(\mathfrak{n} \cap \mathfrak{p})} \to H^{i+R}(\mathfrak{n},X)^{\gamma - 2\rho(\mathfrak{n} \cap \mathfrak{p})}.$$

Since $\Lambda^R(\mathfrak{n} \cap \mathfrak{p})$ is one dimensional and has $\mathfrak{t}$-weight $2\rho(\mathfrak{n} \cap \mathfrak{p})$, this may be rewritten as

$$(3.6) \qquad \pi_i^{\gamma} : H^i(\mathfrak{n} \cap \mathfrak{k}, X)^{\gamma} \otimes (\Lambda^R(\mathfrak{n} \cap \mathfrak{p}))^* \to H^{i+R}(\mathfrak{n},X)^{\gamma - 2\rho(\mathfrak{n} \cap \mathfrak{p})}.$$

To study these maps, we need some information about $H^i(\mathfrak{n} \cap \mathfrak{k}, X)$, which (as the reader might guess) requires a little more notation. If $\sigma \in W_K$, set $\Delta_\sigma^+(\mathfrak{k}) = \{\alpha \in \Delta^+(\mathfrak{k}) | \sigma^{-1}(\alpha) \in \Delta^-(\mathfrak{k})\}$. (Clearly this refers to $\mathfrak{t}$-roots; it should be rather obvious why the subscript $\mathfrak{t}$ in $\Delta_\mathfrak{t}$ has been omitted.) As is well known, and easy to check,

$$(3.7) \qquad 2\rho(\Delta_\sigma^+(\mathfrak{k})) = \sum_{\alpha \in \Delta_\sigma^+(\mathfrak{k})} \alpha = \rho_c - \sigma \cdot \rho_c .$$

Define $\ell(\sigma) = |\Delta_\sigma^+(\mathfrak{k})|$ (the number of elements of $\Delta_\sigma^+(\mathfrak{k})$), $W_K^1 = \{\sigma \in W_K | \Delta_\sigma^+(\mathfrak{k}) \subseteq \Delta(\mathfrak{n} \cap \mathfrak{k})\}$, $W_K^1(j) = \{\sigma \in W_K^1 | \ell(\sigma) = j\}$. Suppose $\gamma \in \hat{\mathfrak{k}}$, with $F \in \gamma$ an irreducible $\mathfrak{k}$-module.

<u>Theorem 3.8</u> (Kostant [16]) <u>$H^j(\mathfrak{n} \cap \mathfrak{k}, F)^\mu = 0$ unless $\mu = \sigma(\gamma+\rho_c) - \rho_c$ for some $\sigma \in W_K^1(j)$. In that case the space is one dimensional, i.e. the $\mathfrak{l} \cap \mathfrak{k}$-type μ occurs with multiplicity one in $H^j(\mathfrak{n} \cap \mathfrak{k}, F)$</u>.

The maps π_i^γ can now be studied by standard spectral sequence techniques. Consider the graded $\mathfrak{l}$ module $A^* = \mathrm{Hom}\,(\Lambda^* \mathfrak{n}, X)$, with differential d (the coboundary operator of Lie algebra cohomology.) Pick a sequence $\{V_a\}_{a=0}^N$ of $\mathfrak{l} \cap \mathfrak{k}$-invariant subspaces of $\Lambda^*(\mathfrak{n} \cap \mathfrak{p})$ so that

1) $V_0 = \Lambda^0(\mathfrak{n} \cap \mathfrak{p}) = \mathbb{C}$

2) $V_a \subseteq \Lambda^{r(a)}(\mathfrak{n} \cap \mathfrak{p})$; and $a \leq a' \Rightarrow r(a) \leq r(a')$

3) $(\mathfrak{n} \cap \mathfrak{k}) \cdot V_a \subseteq \langle V_0, \ldots V_{a-1} \rangle$.

This is always possible: choosing $x \in \mathfrak{t}^+ = (\text{cent}\,\mathfrak{l}) \cap \mathfrak{t}$ as in the proof of Lemma 2.7, we see that if $E \subseteq \Lambda^*(\mathfrak{n} \cap \mathfrak{p})$ is $\mathfrak{l} \cap \mathfrak{k}$ irreducible, and x acts on E by the scalar c, then x acts on $(\mathfrak{n} \cap \mathfrak{k}) \cdot E$ with eigenvalues strictly greater than c. Using the eigenspaces of x in $\Lambda^*(\mathfrak{n} \cap \mathfrak{p})$, we may therefore arrange (3).

Filter A^* by $A^n_a = \{f \in A^n | f(q^{n-r} \wedge p^r) = 0$ whenever $q^{n-r} \in \Lambda^{n-r}(\mathfrak{n} \cap \mathfrak{k})$ and $p^r \in V_{a'}$ for some $a' < a\}$. Then $A_0 = A$, $A_1 = \{f | f = 0 \text{ on } \Lambda^*(\mathfrak{n} \cap \mathfrak{k})\}$, etc. There are $\mathfrak{l} \cap \mathfrak{k}$ module maps

$$\tau^n_a : A^n_a \to \text{Hom}\,(\Lambda^{n-r(a)}(\mathfrak{n} \cap \mathfrak{k}), X) \otimes V^*_a .$$

For $\text{Hom}\,(\Lambda^{n-r(a)}(\mathfrak{n} \cap \mathfrak{k}), X) \otimes V^*_a = \text{Hom}\,([\Lambda^{n-r(a)}(\mathfrak{n} \cap \mathfrak{k})] \otimes V_a, X)$; and τ^n_a is just the natural restriction map corresponding to the inclusion $[\Lambda^{n-r(a)}(\mathfrak{n} \cap \mathfrak{k})] \otimes V_a \hookrightarrow \Lambda^n \mathfrak{n}$. Then

1) $f \in A^n_a$ iff $\tau^n_{a'}(f) = 0$ for all $a' < a$; in particular, if $f \in A^n_a$, then $f \in A^n_{a+1}$ iff $\tau^n_a(f) = 0$.

2) If $f \in A^n_a$, then $\tau^{n+1}_a(df) = (d_{(\mathfrak{n} \cap \mathfrak{k})} \otimes 1)(\tau^n_a(f))$; and $\tau^{n+1}_{a'}(df) = 0$ for $a' < a$.

Statement 1) follows immediately from the definitions. For 2), suppose $f \in A^n_a$. Fix

$a' \leq a$, $Q = q_0\wedge\ldots\wedge q_{n-r(a')} \in \Lambda^{n-r(a')+1}(\mathfrak{m}\cap\mathfrak{k})$,

and $P = p_{n-r(a')+1}\wedge\ldots\wedge p_n \in V_{a'}$. (An element of $V_{a'}$ is actually a linear combination of such terms; since the argument would be unchanged in general, we take P of this form to simplify the notation.) By definition, $\tau^{n+1}_{a'}(df)$ is a map from $\Lambda^{n+1-r(a')}(\mathfrak{m}\cap\mathfrak{k}) \otimes V_{a'}$ to X; $\tau^{n+1}_{a}(df)(Q \otimes P) = df(Q\wedge P)$. Recall that in general,

$$df(y_0\wedge\ldots\wedge y_n) =$$
$$\sum_{i=0}^{n} (-1)^i y_i\cdot f(y_0\wedge\ldots\hat{y}_i\ldots\wedge y_n) + \sum_{0\leq i<j\leq n}(-1)^{i+j} f([y_i,y_j]\wedge y_0\wedge\ldots\hat{y}_i\ldots\hat{y}_j\ldots)$$

(Here the circumflex $\hat{}$ indicates that the argument is to be omitted.) Thus

$$\text{i)}\quad df(Q\wedge P) = \sum_{i=0}^{n-r(a')} (-1)^i q_i\cdot f(q_0\wedge\ldots\hat{q}_i\ldots q_{n-r(a')}\wedge P)$$

$$\text{ii)}\quad + \sum_{i=n-r(a')+1}^{n} (-1)^i p_i\cdot f(Q\wedge p_{n-r(a')+1}\ldots\hat{p}_i\ldots\wedge p_n)$$

$$\text{iii)}\quad + \sum_{i<j\leq n-r(a')} (-1)^{i+j} f([q_i,q_j]\wedge q_0\ldots\hat{q}_i\ldots\hat{q}_j\ldots\wedge q_{n-r(a')}\wedge P)$$

$$\text{iv)}\quad + \sum_{i=1}^{n-r(a')} (-1)^i \sum_{j=n-r(a')+1}^{n} (-1)^{j+n-r(a')} f(q_0\wedge\ldots\hat{q}_i\ldots\wedge q_{n-r(a')}\wedge [q_i,p_j]\wedge p_{n-r(a')+1}\cdots\hat{p}_j\ldots\wedge p_n)$$

$$\text{v)}\quad + \sum_{n-r(a')+1\leq i<j} (-1)^{i+j} f([p_i,p_j]\wedge Q\wedge p_{n-r(a')}\wedge\ldots\hat{p}_i\ldots\hat{p}_j\ldots\wedge p_n).$$

Since $f \in A_a^n$, $f(X \wedge Y) = 0$ if $X \in \Lambda^{\ell}(\mathfrak{n} \cap \mathfrak{k})$, for some $\ell > n - r(a)$. Since $r(a') \leq r(a)$, every term in lines (ii) and (v) is of this form. Line (iv) may be rewritten as

$$\sum_{i=1}^{n} \pm f(q_0 \ldots \hat{q}_i \ldots q_{n-r(a')} \wedge (q_i \cdot P)).$$

By assumption (3) in the definition of the V_a, $q_i \cdot P \in \langle V_0, \ldots V_{a'-1} \rangle \subseteq \langle V_0, \ldots V_{a-1} \rangle$. Since $f \in A_a^n$, $f(X \wedge Y) = 0$ whenever $X \in \Lambda^*(\mathfrak{n} \cap \mathfrak{k})$ and $Y \in \langle V_0, \ldots, V_{a-1} \rangle$. Thus line (iv) is zero. If $a' < a$, $P \in \langle V_0 \ldots V_{a-1} \rangle$, so lines (i) and (iii) vanish. This proves the second statement of claim (2). Finally, if $a = a'$, lines (i) and (iii) are precisely $(d_{\mathfrak{n} \cap \mathfrak{k}} \otimes 1)(\tau_a^n(f))(Q \otimes P)$. This proves (2).

From (1) and (2) one deduces immediately

(3) $dA_a \subseteq A_a$

(4) τ_a^n induces a map from A_a^n / A_{a+1}^n to $\mathrm{Hom}\,(\Lambda^{n-r(a)}(\mathfrak{n} \cap \mathfrak{k}), X) \otimes V_a^*$ which is an isomorphism of differential $\mathfrak{l} \cap \mathfrak{k}$ modules.

The spectral sequence of the filtration A_a now gives

<u>Theorem 3.9</u> <u>There is a spectral sequence $E_t^{ab} \Rightarrow H^{a+b}(\mathfrak{n}, X)$. $E_1^{ab} \cong H^{a+b-r(a)}(\mathfrak{n} \cap \mathfrak{k}, X) \otimes V_a^*$; here $V_a \subseteq \Lambda^{r(a)} \mathfrak{n} \cap \mathfrak{p}$. The differential d_t has bidegree $(t, 1-t)$ and is an $\mathfrak{l} \cap \mathfrak{k}$ module map.</u>

Corollary 3.10 $H^i(\mathfrak{n},X)$ is a Harish-Chandra module for $\mathfrak{l}$.

The corollary follows rather easily from Kostant's theorem 3.8; in fact E_1^{ab} already has the $\mathfrak{l}\cap\mathfrak{k}$ structure of a Harish-Chandra module.

An apology may be in order for the proof of the preceding theorem. A slightly stronger version (i.e. smaller E_1^{ab} terms) can be read off from the famous Hochschild-Serre spectral sequence ([10]). However, I know of no applications of the improvement, and the grubbier approach adopted above may indicate what's going on a little more clearly.

Fix a $\mathfrak{k}$-type μ of X with $X(\mu) \neq 0$.

Definition 3.11 μ is $\mathfrak{n}$-minimal in X if whenever $\gamma \in \hat{\mathfrak{k}}$, $J \geq 1$, and $F \in \gamma$ is an irreducible $\mathfrak{k}$ module, then

$$(H^{J-1}(\mathfrak{n}\cap\mathfrak{k},F) \otimes [\Lambda^{R-J}(\mathfrak{n}\cap\mathfrak{p})]^*)^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})} \neq 0 \text{ implies } X(\gamma) = 0.$$

Thus the condition is that certain $\mathfrak{k}$-types γ should not occur in X. A more computable formulation is given by Proposition 4.8.

Theorem 3.12 Suppose μ is $\mathfrak{n}$-minimal in X. Then

$$\pi_0^{\mu} : H^0(\mathfrak{n}\cap\mathfrak{k},X)^{\mu} \otimes [\Lambda^R(\mathfrak{n}\cap\mathfrak{p})]^* \to H^R(\mathfrak{n},X)^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})}$$

is injective.

Proof. Let N be the index with $V_N = \Lambda^R(\mathfrak{n}\cap\mathfrak{p})$; of course $r(N) = R$. Then $A_{N+1} = 0$, and it is easy to see that

A_N^{R+n} is precisely the image of π_n. In fact,

$$\pi_n : \mathrm{Hom}\ (\Lambda^n(\mathfrak{n}\cap\mathfrak{k}),X) \otimes V_N^* \to \mathrm{Hom}\ (\Lambda^{R+n}\mathfrak{n},X)$$

is the inverse of τ_N^{R+n}. (We can now settle a point left open earlier. Recall that we proved

$\tau_N^{R+n+1}(df) = (d_{\mathfrak{n}\cap\mathfrak{k}} \otimes 1)(\tau_N^{R+n}(f))$ for $f \in A_N^{R+n}$ = image of π_n.

It follows that $d(\pi_n(h)) = \pi_{n+1}(d \otimes 1(h))$ for $h \in \mathrm{Hom}\ (\Lambda^*(\mathfrak{n}\cap\mathfrak{k}),X) \otimes V_N^*$; this was used in the proof of Theorem 3.5.) Thus the assertion of the present theorem is that the term $(E_1^{N,R-N})^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})}$ of the spectral sequence in Theorem 3.9 "persists to E_∞." Since the bidegre of d_t is $(t,1-t)$, it suffices to check that $(E_1^{ab})^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})} = 0$ whenever $a+b = R-1$. In this case,

$$(E_1^{ab})^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})} = (H^{R-1-r(a)}(\mathfrak{n}\cap\mathfrak{k},X) \otimes V_a^*)^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})}$$

$$\subseteq \sum_{\gamma\in\hat{\mathfrak{k}}} (H^{J-1}(\mathfrak{n}\cap\mathfrak{k},X(\gamma)) \otimes [\Lambda^{R-J}(\mathfrak{n}\cap\mathfrak{p})]^*)^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})};$$

here we put $J = R-r(a)$. This vanishes by Definition 3.14.

Q.E.D.

Because of the symmetry of $H^*(\mathfrak{n}\cap\mathfrak{k},X)$ with respect to W_K^1 (cf. Theorem 3.8) one can actually get considerably more information about $H^*(\mathfrak{n},X)$ from the assumption that μ is $\mathfrak{n}$-minimal. This is probably important for relating $H^*(\mathfrak{n},X)$ to the global structure of X; but that problem will not be pursued here.

We want to study the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on $X(\mu)$ in case μ is $\mathfrak{n}$-minimal. By definition of the action of $U^{\mathfrak{k}}$ on $\mathfrak{n}\cap\mathfrak{k}$-cohomology, this is the same as studying $X^{\mu} = H^0(\mathfrak{n}\cap\mathfrak{k},X)^{\mu}$ as a $U^{\mathfrak{k}}$ module.

Combining Theorems 3.5 and 3.12, we see that <u>this $U^{\mathfrak{k}}$ action factors through the homomorphism ξ.</u> This can be improved a little, using Corollary 3.10. Let $\mathcal{J}_{\mu} \subseteq U(\mathfrak{k})$ be the kernel of the representation μ, and put $I_{\mu} = I_{\mu}(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{k}} \cap [U(\mathfrak{g})\cdot\mathcal{J}_{\mu}]$. Then we have the following more precise version of Theorem 2.9.

<u>Proposition 3.13</u> <u>I_{μ} is a two-sided ideal in $U^{\mathfrak{k}}$, and the action of $U^{\mathfrak{k}}$ on Y^{μ} factors through $U^{\mathfrak{k}} \to U^{\mathfrak{k}}/I_{\mu}$ for any Harish-Chandra module Y. This establishes a one to one correspondence between irreducible Harish-Chandra modules Y with $Y(\mu) \neq 0$ and irreducible $U(\mathfrak{g})^{\mathfrak{k}}/I_{\mu}$ modules.</u>

For a proof, see Lepowsky-McCollum [2]. We will apply this result to $\mathfrak{l}$, and the Harish-Chandra module $H^*(\mathfrak{n},X)$. Let

$$(3.14) \qquad \xi^{\mu} : U(\mathfrak{g})^{\mathfrak{k}} \to U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}}/I_{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})}(\mathfrak{l})$$

be the composition of ξ with the quotient map. Notice that $2\rho(\mathfrak{n}\cap\mathfrak{p})$ is the weight of a one dimensional $\mathfrak{l}\cap\mathfrak{k}$ module, so that $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is $\mathfrak{l}\cap\mathfrak{k}$ dominant integral. By 3.5, 3.12, and 3.13, we have proved

<u>Theorem 3.15</u> <u>Suppose μ is $\mathfrak{n}$-minimal in X. Then the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on X^{μ} factors through ξ^{μ}.</u>

The consequences of this theorem are dealt with in detail in section 4, but the following example indicates its usefulness and limitations quite clearly. For details, the reader may consult Schmid [23]. Suppose that $\mathfrak{g}_0$ is semisimple and equal rank (so that $\mathfrak{t} = \mathfrak{h}$) and that G has finite center. Let $\Lambda \subseteq i\mathfrak{t}_0'$ denote the lattice of differentials of characters of T. An element λ of $\Lambda+\rho$ is said to be <u>non-singular</u> if $\langle\lambda,\alpha\rangle \neq 0$ for every $\alpha \in \Delta$. Let $(\Lambda+\rho)'$ be the set of nonsingular elements of $\Lambda+\rho$. Harish-Chandra has shown that the discrete series of G is parametrized by the $\mathfrak{n}^{o}\cap\mathfrak{k}$ dominant weights in $(\Lambda+\rho)'$. Let λ be such a weight, and X the Harish-Chandra module of the corresponding discrete series representation. Let Δ^{+} be the positive root system associated to λ (defined in §2) and let $\mathfrak{b} = \mathfrak{b}^{o}$. Put $\mu = \lambda-\rho_c + \rho(\mathfrak{n}^{o}\cap\mathfrak{p})$ (which is the highest weight of a K-type).

<u>Theorem 3.16</u> (Schmid) <u>With notation as in the preceding paragraph, μ is $\mathfrak{n}^{o}$ minimal in X, and occurs with multiplicity</u>

one. Suppose μ is $\mathfrak{n}^0$ minimal in the irreducible Harish-Chandra module Y; then Y is equivalent to X.

Proof. Schmid has proved that μ is $\mathfrak{n}^0$ minimal in X (in particular that $X(\mu) \neq 0$.) In the present case, $\mathfrak{l} = \mathfrak{t} = \mathfrak{l} \cap \mathfrak{k}$; so $U(\mathfrak{l})^{\mathfrak{l} \cap \mathfrak{k}} = U(\mathfrak{t})$. $I_{\mu-2\rho(\mathfrak{n}^0 \cap \mathfrak{p})} = \mathcal{J}_{\mu-2\rho(\mathfrak{n}^0 \cap \mathfrak{p})}$ is just the kernel of the character $\mu-2\rho(\mathfrak{n}^0 \cap \mathfrak{p}) : U(\mathfrak{t}) \to \mathbb{C}$. Thus $U(\mathfrak{l})^{\mathfrak{l} \cap \mathfrak{k}} / I_{\mu-2\rho(\mathfrak{n}^0 \cap \mathfrak{p})}(\mathfrak{l}) \cong \mathbb{C}$, and $\xi^\mu : U(\mathfrak{g})^{\mathfrak{k}} \to \mathbb{C}$ is a character of $U(\mathfrak{g})^{\mathfrak{k}}$. By Theorem 3.15 and Proposition 3.13, X^μ and Y^μ are irreducible modules for $U(\mathfrak{g})^{\mathfrak{k}}/\ker \xi^\mu \cong \mathbb{C}$; so obviously they are equivalent and one dimensional. So μ occurs with multiplicity one in X; and the second part of Proposition 3.13 gives $X \cong Y$. Q.E.D.

Notice that the proof actually exhibits the action of $U^{\mathfrak{k}}$ on X^μ rather explicitly. This character has also been computed (in a slightly different form) by Enright and Varadarajan ([7]).

Schmid's proof that μ is $\mathfrak{n}^0$ minimal in X, while basically analytic, invokes a much stronger form of the preceding result. With a little more work, however, the results of this section can be applied in his argument (cf. the proof of Lemma 6.3), giving a complete proof of Theorem 3.16 independent of his (rather difficult) algebraic computations: we omit the details. In any case one doe <u>not</u> get his estimates for the multiplicities of other K types in X.

4. The Classification Problem

We would like to produce a classification theory for Harish-Chandra modules, based on the results of section 3. Recall the algebraic ordering $\prec$ of $i\mathfrak{t}_0'$. Define a second ordering $<$ on $i\mathfrak{t}_0'$ by $\gamma_1 < \gamma_2$ if

$\langle\gamma_1 + 2\rho_c, \gamma_1 + 2\rho_c\rangle < \langle\gamma_2 + 2\rho_c, \gamma_2 + 2\rho_c\rangle$ or

$\langle\gamma_1 + 2\rho_c, \gamma_1 + 2\rho_c\rangle = \langle\gamma_2 + 2\rho_c, \gamma_2 + 2\rho_c\rangle$ and

$\gamma_1 \prec \gamma_2$. Put $||\gamma|| = \langle\gamma + 2\rho_c, \gamma + 2\rho_c\rangle$.

Let X be an irreducible Harish-Chandra module for $\mathfrak{g}$. To avoid some minor complications, assume that $\gamma \in i\mathfrak{t}_0'$ whenever $X(\gamma) \neq 0$, i.e. that X is a unitarizable $\mathfrak{k}$-module. <u>$\hat{\mathfrak{k}}$ will henceforth refer only to γ of this sort.</u> (In case X is the Harish-Chandra module of a representation and K is compact, this condition is automatic.) The irreducibility of X implies that $\{\gamma | X(\gamma) \neq 0\}$ is contained in a certain translate of the lattice of $\mathfrak{t}$-roots in $i\mathfrak{t}_0'$. Hence the minimum of the positive real numbers $\{||\gamma|| \mid X(\gamma) \neq 0\}$ is attained on a finite non-empty set of γ. The following definition therefore makes sense.

<u>Definition 4.1</u> The <u>minimal $\mathfrak{k}$-type</u> μ of X is the unique minimal element of $\{\gamma | X(\gamma) \neq 0\}$ with respect to $<$.

μ is essentially the $\mathfrak{k}$-type with $||\mu||$ minimal. The main reason for using $||\mu||$ instead of the "compact Casimir" $\langle\mu+\rho_c, \mu+\rho_c\rangle$ is that Definition 4.1 produces Schmid's "lowest type" ([23]) for discrete series representations; and one can (with some effort) produce examples in which the $\mu+\rho_c$ definition does not.

To solve the classification problem, it is clearly enought to describe the set of Harish-Chandra modules with a fixed minimal $\mathfrak{k}$-type μ. To apply the machinery of section 3, we need some θ-invariant parabolics. Let Δ^+ be the positive root system associated to $\mu+2\rho_c$; recall that if $\alpha \in \Delta_{\mathfrak{t}}(\mathfrak{g})$, $\alpha \in \Delta^+$ iff $\langle\alpha,\mu+2\rho_c\rangle > 0$ or $\langle\alpha,\mu+2\rho_c\rangle = 0$ and $\alpha \succ -\alpha$. Thus we have a θ-invariant Borel subalgebra $\mathfrak{b}^0 \supseteq \mathfrak{h}$ as in section 2, with nil radical $\mathfrak{n}^0$; $\Delta(\mathfrak{n}^0) = \Delta^+$. It remains to define the parabolic $\mathfrak{b} \supseteq \mathfrak{b}^0$. It is not clear how to do this in general, but examples (cf. Theorem 4.6 and Corollary 4.16) support

Conjecture 4.2 *The θ-invariant subalgebra $\mathfrak{b} \supseteq \mathfrak{b}^0$ may be chosen in such a way that*

i) *whenever μ is the minimal $\mathfrak{k}$-type of the Harish-Chandra module X, then the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on X^{μ} factors through the homomorphism ξ^{μ} (Definition 3.14)*

ii) *the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{b}$ is split (Definition 2.4)*

iii) the $\mathfrak{l}\cap\mathfrak{k}$ -type of highest weight $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is small

iv) if $\mu'-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is a small $\mathfrak{l}\cap\mathfrak{k}$ -type associated to $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$, and μ' is dominant for $\mathfrak{k}$, then $\mu \preceq \mu'$

v) $\mu+2\rho_c-\rho$ is dominant with respect to the positive imaginary roots supported on $\mathfrak{t}^+$

For the definitions of "small" and "associated", see section 5. Conditions iv) and v) are needed only to prove the existence of the representations described by Theorem 4.5.

The philosophy is this. Recall that $\mathfrak{t}^+ + \mathfrak{a}^+$ is the center of $\mathfrak{l}$. Let $\tilde{\mathfrak{a}}_0 \subseteq [\mathfrak{l},\mathfrak{l}] \cap \mathfrak{p}_0$ be a Cartan subalgebra of $[\mathfrak{l}_0,\mathfrak{l}_0]$ (which exists because $\mathfrak{l}_0$ is split). Set $\mathfrak{a} = \mathfrak{a}(\mu) = \mathfrak{a}^+ + \tilde{\mathfrak{a}}$; then $\mathfrak{t}^+ + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{l}$, and hence also of $\mathfrak{g}$. Harish-Chandra often works with the reductive group $G^{\mathfrak{a}}$ (which will appear in the realizations of Section 6). The present development is concentrating on $L = G^{\mathfrak{t}^+}$. According to, for instance, Kostant's quantization theory, the really fundamental objects are the Cartan subgroups $G^{\mathfrak{t}^+ + \mathfrak{a}}$.
The problem with dealing directly with these is that (in Kostant's language) the associated polarizations are

usually neither real (the split principal series case) nor purely complex (the discrete series case.) In Harish-Chandra's point of view, the discrete series are taken as fundamental building blocks, and one works essentially with real polarizations (i.e. induced representations). Here we start with the principal series for split groups, and use purely complex polarizations (i.e. $\mathfrak{n}$ -cohomology).

When Conjecture 4.2 holds, the "uniqueness" part of the classification problem reduces to a study of the representations of a split group containing a small K-type (see Theorem 4.5 below). These representations can be described almost completely, using Harish-Chandra's subquotient theorem (Theorem 5.2). The result is

<u>Theorem 4.3.</u> <u>Let $G = KAN$ be an Iwasawa decomposition of a split reductive group, and let $\mu \in \hat{K}$ be small. Then there is an analytic family $\{\pi_\mu^\nu\}_{\nu\in\hat{A}}$ of admissible representations of G (namely a certain principal series) so that</u>

i) <u>π_μ^ν contains μ exactly once; and the other small K-types in π_μ^ν are precisely those associated to μ.</u>

ii) <u>π_μ^ν is irreducible for almost all ν, and has a finite composition series in general.</u>

iii) π_μ^ν and $\pi_\mu^{\nu'}$ have equivalent composition series iff $\nu = \sigma \circ \nu'$ for some σ in a certain subgroup W_μ of the Weyl group of A in G.

By i), π_μ^ν has a unique irreducible subquotient $\overline{\pi_\mu^\nu}$ containing the K-type μ. Then

iv) The representations $\{\overline{\pi_\mu^\nu}\}$ exhaust the infinitesimal equivalence classes of admissible irreducible representations of G containing μ.

Since the only proof available for i) is very long, this theorem will be proved in section 5.

Corollary 4.4 An irreducible Harish-Chandra module for a split group contains $\mathfrak{k}$-types from at most one family of associate small $\mathfrak{k}$-types. If such a family exists, the minimal $\mathfrak{k}$-type belongs to it, and each element occurs with multiplicity one or zero.

Suppose that Conjecture 4.2 holds for the $\mathfrak{k}$-type μ; choose $\mathfrak{b}$ accordingly. Recall that $\mathfrak{a}_0 = \mathfrak{a}(\mu)_0 = \tilde{\mathfrak{a}}_0 + \mathfrak{a}_0^+$ is a maximal abelian subalgebra of $\mathfrak{l} \cap \mathfrak{p}_0$. Apply Theorem 4.3 to the split reductive group L, and the small $\mathfrak{l} \cap \mathfrak{k}$ type $\mu - 2\rho(\mathfrak{n} \cap \mathfrak{p})$. We may choose the subgroup $A(\mu)$ from the Iwasawa decomposition of L to have Lie algebra $\mathfrak{a}(\mu)_0$. Suppose $\nu \in \mathfrak{a}(\mu)^* \cong \widehat{A(\mu)}$. By 4.3 (i), the highest weight space of the $\mathfrak{l}\cap\mathfrak{k}$ type $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ in $\overline{\pi_\mu^\nu}$ is a one dimensional module for $R_\mu = U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}}/I_{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})}(\mathfrak{l})$. Let $\chi_\nu : R_\mu \to \mathbb{C}$ be the corresponding homomorphism. Recall the map $\xi^\mu : U(\mathfrak{g})^{\mathfrak{k}} \to R_\mu$ (3.14); define characters $\xi_\nu^\mu : U(\mathfrak{g})^{\mathfrak{k}} \to \mathbb{C}$ by $\xi_\nu^\mu = \chi_\nu \cdot \xi^\mu$. Then

Theorem 4.5 Let X be an irreducible Harish-Chandra module with minimal $\mathfrak{k}$-type μ, and suppose that Conjecture 4.2 (i-iii) holds for μ. With notation as in the preceding paragraph, there is a $\nu \in \mathfrak{a}(\mu)^*$ so that the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on X^μ is given by the character ξ_ν^μ. In particular, μ occurs with multiplicity one.

Proof. By 4.2 (i) and Proposition 3.13, X^μ is an irreducible module for $\xi^\mu(U^{\mathfrak{k}}) \subseteq R_\mu$. Let $Z \subseteq R_\mu$ be the image of the center of $U(\mathfrak{l})$ under the quotient map $U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}} \to R_\mu$; since $\xi^\mu|\mathfrak{z}(\mathfrak{g})$ is essentially a Harish-Chandra homomorphism, $\xi^\mu(\mathfrak{z}(\mathfrak{g})) \to Z$ is an integral ring extension. By Corollary 5.5, R_μ is commutative, and $Z \to R_\mu$ is an integral extension. Hence $\xi^\mu(U^{\mathfrak{k}}) \subseteq R_\mu$ is an integral extension; so X^μ lifts to an irreducible module for R_μ. By Proposition 3.13 again, this lifted module is just the action of R_μ in the $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$-type of some irreducible Harish-Chandra module for $\mathfrak{l}$. By Theorem 4.3 (iv), R_μ must therefore act on X^μ by the scalars χ_ν, some

$\nu \in \mathfrak{a}(\mu)^*$. But this is precisely the conclusion of the theorem. Q.E.D.

It should be remarked that the ring theory here is all avoidable. We will prove Conjecture 4.2 (i) by exhibiting explicitly certain cohomology groups on which ℓ acts; so the necessary liftings appear automatically (although not uniquely, just as above). In some cases the construction of the cohomology is a little subtle, however; and it seemed best to make Conjecture 4.2 as simple as possible, despite the resulting uglinees of the preceding proof.

The existence theorem corresponding to 4.5 is Theorem 6.2.

By a straightforward reduction, it suffices to prove Conjecture 4.2 for G simply connected and $\mathfrak{g}_0$ a simple Lie algebra. A case-by-case attack is therefore not utterly ridiculous. Let $\tilde{G}$ denote the universal covering group of G.

<u>Theorem 4.6</u> <u>Conjecture 4.2 holds in the following cases</u>

a) G has only one conjugacy class of Cartan subgroups

b) $G = \widetilde{SL}(n,R)$

c) $G = \widetilde{SU}(p,q)$

d) $G = \widetilde{SP}(p,q)$

e) $G = \widetilde{SP}(n,R)$

f) $G = \widetilde{SO}^*(2n)$

g) $G = \widetilde{SO}(p,q)$

h) G = split form of G_2

i) G = rank one form of F_4

For every case but g), we will actually prove the following stronger versions of parts of 4.2:

4.2 (i)': Whenever μ is the minimal $\mathfrak{k}$-type of the Harish-Chandra module X, then μ is $\mathfrak{n}$-minimal in X. (Definition 3.14)

4.2 (iii-iv)' $\mu - 2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal (Definition 5.3)

That 4.2 (i)' => 4.2 (i) is precisely Theorem 3.15. That 4.2 (iii-iv)' => 4.2 (iii) and 4.2 (iv) is trivial from the definitions (see section 5). We will in general say little about 4.2 (v) - the reader may observe that in each case much stronger conditions will be proved in the course of establishing 4.2 (i).

We will say that the $\mathfrak{k}$-type μ is $\mathfrak{n}$-minimal if whenever μ is the minimal $\mathfrak{k}$-type of an irreducible Harish-Chandra module X, then μ is $\mathfrak{n}$-minimal in X. The first step toward proving Theorem 4.6 is to give some computable conditions for μ to be $\mathfrak{n}$-minimal, for a fixed θ-invariant $\mathfrak{b} \supseteq \mathfrak{b}^0$.

Definition 4.7 The weight $\phi \in \mathfrak{t}^*$ is said to give rise to J-cohomology larger than μ if there is a $\mathfrak{k}$-type $\gamma \geq \mu$ so that if F is an irreducible $\mathfrak{k}$ module of highest weight γ, then $H^J(\mathfrak{n}\cap\mathfrak{k}, F)^{\phi} \neq 0$.

Write $\Delta(\mathfrak{n}\cap\mathfrak{p}) = \{\beta_1, \ldots, \beta_R\}$, $\Delta(\mathfrak{n}\cap\mathfrak{k}) = \{\alpha_1 \ldots \alpha_S\}$ (the noncompact and compact $\mathfrak{t}$-roots in $\mathfrak{n}$, respectively).

Proposition 4.8 Suppose that for any non-empty subset $\{\beta_{i_1} \ldots \beta_{i_J}\}$ of $\Delta(\mathfrak{n}\cap\mathfrak{p})$, the weight $\mu-\beta_{i_1} \ldots -\beta_{i_J}$ does not give rise to J-1 cohomology larger than μ. Then μ is $\mathfrak{n}$-minimal.

Proof. Suppose that F is an irreducible $\mathfrak{k}$ module of highest weight γ, and that
$(H^{J-1}(\mathfrak{n}\cap\mathfrak{k},F) \otimes \{\Lambda^{R-J}(\mathfrak{n}\cap\mathfrak{p})]^*)^{\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})} \neq 0$.
By the definition of $\mathfrak{n}$-minimal, together with Definitions 4.1 and 3.11, what we must show is that $\gamma < \mu$. By a standard fact about highest weights in a tensor product, there are weights ϕ and ψ such that $H^{J-1}(\mathfrak{n}\cap\mathfrak{k},F)^{\phi} \neq 0$, ψ is a weight of $[\Lambda^{R-J}(\mathfrak{n}\cap\mathfrak{p})]^*$, and $\phi+\psi = \mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$. Thus $\psi = -(\beta_{\ell_1}+\ldots+\beta_{\ell_{R-J}})$ for some R-J element subset of $\Delta(\mathfrak{n}\cap\mathfrak{p})$. Recall that

$$2\rho(\mathfrak{n}\cap\mathfrak{p}) = \sum_{\beta\in\Delta(\mathfrak{n}\cap\mathfrak{p})} \beta = \sum_{i=1}^{R} \beta_i .$$

It is immediate that $\phi = \mu-\beta_{i_1} \ldots -\beta_{i_J}$, where $\{\beta_{i_J}\}$ is the complement of $\{\beta_{\ell_i}\}$ in $\Delta(\mathfrak{n}\cap\mathfrak{p})$. Now $H^{J-1}(\mathfrak{n}\cap\mathfrak{k},F)^{\phi} \neq 0$; but by hypothesis, ϕ does not give rise to J-1 cohomology larger than μ. By Definition 4.7, this forces $\gamma < \mu$. Q.E.D.

Suppose then that $J \geq 1$, $\{\beta_{i_j}\}_{j=1}^{J} \subseteq \Delta(\mathfrak{n}\cap\mathfrak{p})$, and that $\mu-\beta_{i_1} \ldots -\beta_{i_J}$ gives rise to J-1 cohomology larger than μ.

Let γ be the corresponding $\mathfrak{k}$-type. To get some restrictions on the $\{\beta_{i_j}\}$, we want to compute $||\mu||-||\gamma||$; since $\gamma \geq \mu$, this number must be non-positive. By Kostant's theorem 3.8, there is a $\sigma \in W_K^1(J-1)$ with

$$\mu - \beta_{i_1} \cdots - \beta_{i_J} = \sigma(\gamma+\rho_c) - \rho_c. \tag{4.9}$$

Thus

$$\gamma+2\rho_c = \sigma^{-1}[(\mu+2\rho_c)-\beta_{i_1}\cdots-\beta_{i_J}-(\rho_c-\sigma\cdot\rho_c)].$$

Since σ^{-1} is an isometry, it follows that

$$||\gamma|| = \langle(\mu+2\rho_c)-\Sigma\beta_{i_j}-(\rho_c-\sigma\cdot\rho_c),\ (\mu+2\rho_c)-\Sigma\beta_{i_j}-(\rho_c-\sigma\cdot\rho_c)\rangle_{\mathfrak{k}}.$$

By a short computation, this may be written as

$$||\mu||-||\gamma||=2\langle\mu+2\rho_c,\Sigma\beta_{i_j}+\rho_c-\sigma\cdot\rho_c\rangle - \langle\Sigma\beta_{i_j}+\rho_c-\sigma\cdot\rho_c,\Sigma\beta_{i_j}+\rho_c-\sigma\cdot\rho_c\rangle$$

By (3.7), $\rho_c-\sigma\cdot\rho_c=2\rho(\Delta_\sigma^+(\mathfrak{k}))$. Since $\sigma \in W_K^1(J-1)$, $\Delta_\sigma^+(\mathfrak{k})$ is a $J-1$ element subset of $\Delta(\mathfrak{n}\cap\mathfrak{k})$, say $\{\alpha_{i_1}\cdots\alpha_{i_{J-1}}\}$.

So $||\mu||-||\gamma||=2\langle\mu+2\rho_c,\Sigma\beta_{i_j}+\Sigma\alpha_{i_j}\rangle_{\mathfrak{k}}-\langle\Sigma\beta_{i_j}+\Sigma\alpha_{i_j},\Sigma\beta_{i_j}+\Sigma\alpha_{i_j}\rangle_{\mathfrak{k}}$

$$||\mu||-||\gamma||=2\langle\mu+2\rho_c-\rho,\Sigma\beta_{i_j}+\Sigma\alpha_{i_j}\rangle_{\mathfrak{k}}+\langle2\rho-(\Sigma\beta_{i_j}+\Sigma\alpha_{i_j}),\Sigma\beta_{i_j}+\Sigma\alpha_{i_j}\rangle_{\mathfrak{k}}$$

<u>Definition 4.10</u> $\lambda = \lambda(\mu) = \mu + 2\rho_c - \rho.$

For an equal rank semisimple group, (4.10) is the equation relating Harish-Chandra's discrete series parameter λ and the discrete series' "lowest" $\mathfrak{k}$-type μ (Schmid [23]). This observation, which came out of some very helpful discussions with H. Hecht, was the original motivation for Definition 4.1.

Set $B = \{\alpha_{i_1} \ldots \alpha_{i_{J-1}}\} \cup \{\beta_{i_1} \ldots \beta_{i_J}\} \subseteq \Delta(\mathfrak{n})$. The last equation may now be written as

$$(4.11) \quad ||\mu||-||\gamma|| = 2\langle\lambda(\mu),\ \Sigma\beta_{i_j} + \Sigma\alpha_{i_j}\rangle_{\mathfrak{k}} + \langle 2\rho(B^c),\ 2\rho(B)\rangle_{\mathfrak{k}} .$$

Here $B^c = \Delta_{\mathfrak{t}}(\mathfrak{n}^0) - B$, so that $\rho(B^c) + \rho(B) = \rho$. Suppose we can show that the right side of (4.11) is always positive (or zero, with $\gamma \prec \mu$). Then Proposition 4.8 will imply that μ is $\mathfrak{n}$-minimal. Now $\mathfrak{n} \subseteq \mathfrak{n}^0$, and $\lambda+\rho = \mu+2\rho_c$ is dominant with respect to $\mathfrak{n}^0$ (by the choice of $\mathfrak{n}^0$). So $\langle\lambda+\rho,\beta_i\rangle$ and $\langle\lambda+\rho,\alpha_i\rangle$ are non-negative for all i; it follows that the first term of 4.11 is never very negative. The second term is easy to handle, as we shall see.

If $\tau \in W(\mathfrak{g},\mathfrak{h})$, set $\Delta^+_\tau(\mathfrak{g}) = \{\alpha \in \Delta^+ | \tau^{-1}\alpha \in \Delta^-\}$. Then

$$(4.12) \quad 2\rho(\Delta^+_\tau(\mathfrak{g})) = \sum_{\alpha\in\Delta^+_\tau(\mathfrak{g})} \alpha = \rho - \tau\cdot\rho$$

Because it preserves Δ, it is easy to see that the Cartan involution θ defines an automorphism $\tau \to \theta\tau\theta$ of $W(\mathfrak{g},\mathfrak{h})$. We claim that $\tau = \theta\tau\theta \iff \tau\cdot\rho \in \mathfrak{t}^* \iff \Delta^+_\tau(\mathfrak{g})$ is θ-invariant. For suppose that $\Delta^+_\tau(\mathfrak{g})$ is θ invariant. Then clearly $2\rho(\Delta^+_\tau(\mathfrak{g}))$ is θ-invariant, i.e. is in $\mathfrak{t}^*$. Hence $\rho-\tau\cdot\rho \in \mathfrak{t}^*$; since $\rho \in \mathfrak{t}^*$, this implies $\tau\cdot\rho \in \mathfrak{t}^*$. If $\tau\cdot\rho \in \mathfrak{t}^*$, then $\theta(\tau\cdot\rho) = \tau\cdot\rho$; since $\theta\rho = \rho$, $(\theta\tau\theta)\cdot\rho = \tau\cdot\rho$, i.e. $\tau^{-1}(\theta\tau\theta)\cdot\rho = \rho$. The only element of $W(\mathfrak{g},\mathfrak{h})$ fixing ρ is the identity, so $\tau = \theta\tau\theta$. Finally, if τ commutes with θ, it is clear that $\Delta^+_\tau(\mathfrak{g})$ is θ-invariant.

<u>Lemma 4.13</u> <u>Suppose $B \subseteq \Delta^+$, $B^c = \Delta^+ - B$. Then $\langle 2\rho(B), 2\rho(B^c)\rangle_{\mathfrak{k}} \geq 0$. Equality holds iff $B = \Delta^+_\tau(\mathfrak{g})$ for some $\tau \in W(\mathfrak{g},\mathfrak{h})$ which commutes with θ.</u>

<u>Proof</u>. (Kostant) By a short computation,

$$\langle 2\rho(B), 2\rho(B^c)\rangle_{\mathfrak{k}} = \langle \rho,\rho\rangle_{\mathfrak{k}} - \langle \rho-2\rho(B), \rho-2\rho(B)\rangle_{\mathfrak{k}} .$$

Consider the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight ρ. Its weights (with respect to $\mathfrak{h}$) are precisely the various $\rho-2\rho(B)$, $B \subseteq \Delta^+$. By the Cartan-Weyl theory, each weight of a finite dimensional representation is at most as long as the highest one, with equality precisely for the extremal weights. Thus $\langle \rho,\rho\rangle - \langle \rho-2\rho(B), \rho-2\rho(B)\rangle \geq 0$, with equality iff

$\rho-2\rho(B) = \tau\cdot\rho$ for some $\tau \in W(\mathfrak{g},\mathfrak{h})$. One knows that $\rho-2\rho(B) = \tau\cdot\rho$ iff $B = \Delta_\tau^+(\mathfrak{g})$.

$\rho \in \mathfrak{t}^*$, so $\langle\rho,\rho\rangle = \langle\rho,\rho\rangle_{\mathfrak{k}}$. $\rho-2\rho(B)$ is a sum of roots, so $\langle\rho-2\rho(B),\rho-2\rho(B)\rangle_{\mathfrak{p}} \geq 0$, with equality iff $\rho-2\rho(B) \in \mathfrak{t}^*$. Thus $\langle\rho-2\rho(B),\rho-2\rho(B)\rangle \geq \langle\rho-2\rho(B),\rho-2\rho(B)\rangle_{\mathfrak{k}}$, with equality iff $\rho-2\rho(B) \in \mathfrak{t}^*$. Assembling these inequalities,

$\langle 2\rho(B),2\rho(B^c)\rangle_{\mathfrak{k}} = \langle\rho,\rho\rangle_{\mathfrak{k}} - \langle\rho-2\rho(B),\rho-2\rho(B)\rangle_{\mathfrak{k}} \geq 0,$

with equality iff $B = \Delta_\tau^+(\mathfrak{g})$, and $\tau\cdot\rho \in \mathfrak{t}^*$. Q.E.D.

Definition 4.14 A noncompact imaginary positive root β is bad if there is a $\sigma \in W_K^1$ such that

a) $\Delta_\sigma^+(\mathfrak{k})$ consists entirely of complex roots

b) λ is singular with respect to β and each $\alpha \in \Delta_\sigma^+(\mathfrak{k})$

c) if the noncompact imaginary roots in $\Delta_\sigma^+(\ )$ are $\phi_1 \ldots \phi_\ell$, then $\sigma^{-1}(\phi_1+\ldots+\phi_\ell-\beta) \succeq 0$ (in the ordering $\mathcal{S}$ of $i\mathfrak{t}_0'$.)

Proposition 4.15 Suppose $\lambda(\mu)$ satisfies

i) $\lambda(\mu)$ is dominant with respect to $\mathfrak{n}$

ii) $\langle\alpha,\lambda(\mu)\rangle_{\mathfrak{k}} > 0$ for $\alpha \in \Delta_{i\mathbb{R}}(\mathfrak{n}\cap\mathfrak{k})$

iii) $\Delta_{i\mathbb{R}}(\mathfrak{n}\cap\mathfrak{p})$ contains no bad roots.

Then μ is $\mathfrak{n}$-minimal.

Proof. By Lemma 4.13 and hypothesis i), both terms in 4.11 are non-negative. If one is positive we are done; so suppose both terms are zero (so that $||\gamma|| = ||\mu||$.) By Lemma 4.13, there is a θ-invariant $\tau \in W(\mathfrak{g},\mathfrak{h})$ such that $B = \Delta^+_\tau(\mathfrak{g})$. Since the first term of (4.11) is zero, $\langle\lambda(\mu),\alpha\rangle_{\mathfrak{k}} = 0$ whenever $\alpha \in B$. By hypothesis (ii), it follows that $\alpha_{i_1} \ldots \alpha_{i_{J-1}}$ are all in $\Delta_{\mathbb{C}}(\mathfrak{n}\cap\mathfrak{k})$. Since B is θ-invariant, it must contain the J-1 corresponding noncompact complex roots (cf. the discussion preceding Proposition 2.3). So we may assume $\alpha_{i_j} = \beta_{i_j}$ for $j = 1\ldots J-1$. The remaining β_{i_J} is necessarily θ-fixed, i.e. $\beta_{i_J} \in \Delta_{i\mathbb{R}}(\mathfrak{n}\cap\mathfrak{p})$. By (4.9),

$$\gamma = \sigma^{-1}((\mu-\beta_{i_J}) - \sum_{j=1}^{J-1} \beta_{i_j}+\rho_c)-\rho_c = \sigma^{-1}((\mu-\beta_{i_J})-\Sigma\alpha_{i_j}+\rho_c)-\rho_c.$$

By (3.7), $\sigma^{-1}(-\Sigma\alpha_{i_j}+\rho_c)-\rho_c = \sigma^{-1}(-\rho_c+\sigma\cdot\rho_c+\rho_c)-\rho_c = 0$.

So $\gamma = \sigma^{-1}\mu - \sigma^{-1}\beta_{i_J}$.

Since $\langle\lambda,\alpha\rangle_{\mathfrak{k}} = 0$ for all $\alpha \in \Delta^+_\sigma(\mathfrak{k})$, $\sigma\cdot\lambda = \lambda$; i.e. $\mu+2\rho_c-\rho = \sigma(\mu+2\rho_c-\rho)$. Rearranging, and applying (3.7) and (4.12), we get

(*) $\mu-\sigma\cdot\mu = \rho-\sigma\cdot\rho-2(\rho_c-\sigma\cdot\rho_c) = 2\rho(\Delta_\sigma^+(\mathfrak{g})) - 4\rho(\Delta_\sigma^+(\mathfrak{k}))$.

Now $\Delta_\sigma^+(\mathfrak{g})$ contains $\Delta_\sigma^+(\mathfrak{k})$ and the corresponding non-compact complex roots, together with (say) $\phi_1\ldots\phi_\ell$. Clearly $2\rho(\Delta_\sigma^+(\mathfrak{g})) = 4\rho(\Delta_\sigma^+(\mathfrak{k})) + \phi_1+\ldots+\phi_\ell$. Thus (*) becomes

(**) $$\mu - \sigma\cdot\mu = \phi_1 + \ldots + \phi_\ell.$$

Every compact root in $\Delta_\sigma^+(\mathfrak{g})$ is already in $\Delta_\sigma^+(\mathfrak{k})$; it follows that the ϕ_i are all noncompact imaginary roots. By (**)

$$\begin{aligned}\gamma = \sigma^{-1}\mu - \sigma^{-1}\beta_{i_J} &= \mu + \sigma^{-1}((\mu-\sigma\mu)-\beta_{i_J}) \\ &= \mu + \sigma^{-1}(\phi_1+\ldots+\phi_\ell-\beta_{i_J}).\end{aligned}$$

What we need for $\mathfrak{n}$-minimality is $\mu > \gamma$; since $||\gamma|| = ||\mu||$, this amounts to $\mu-\gamma \succ 0$. Since β_{i_J} is not a bad root, $\mu-\gamma = -\sigma^{-1}(\phi_1+\ldots+\phi_\ell-\beta_{i_J}) \succ 0$. Q.E.D.

Corollary 4.16 Suppose $\lambda(\mu)$ satisfies

i) $\lambda(\mu)$ is $\mathfrak{n}^0$ dominant

ii) $\lambda(\mu)$ is non-singular with respect to $\Delta_{i\mathbb{R}}(\mathfrak{n}^0\cap\mathfrak{k})$

iii) $\Delta_{i\mathbb{R}}(\mathfrak{n}^0\cap\mathfrak{p})$ contains no bad roots (Definition 4.14; in particular if $\lambda(\mu)$ is non-singular with respect to $\Delta_{i\mathbb{R}}(\mathfrak{n}^0\cap\mathfrak{p})$.)

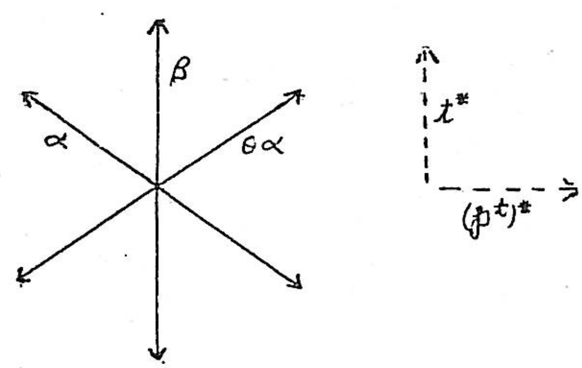
Then μ is $\mathfrak{n}^0$ minimal. Thus Conjecture 4.2 holds for such μ, with $\mathfrak{l} = \mathfrak{h}$, and $\mathfrak{a}(\mu)_0 = \mathfrak{p}_0^{\mathfrak{t}}$.

Proof. By Proposition 4.15, μ is $\mathfrak{n}^0$ minimal. $\mathfrak{l} = \mathfrak{h}$ is abelian, and therefore split, with every $\mathfrak{l} \cap \mathfrak{k}$ type principal series minimal. Q.E.D.

It is fairly clear that the conditions of 4.16 hold for "most" μ, i.e. when μ is sufficiently far from every root wall in $i\mathfrak{t}_0'$. In that case Enright ([6]) has in fact proved Theorem 4.5 (that the representations with minimal type μ are more or less parametrized by $(\mathfrak{p}_0^{\mathfrak{t}})^*$) using a much stronger non-singularity condition on μ. The condition given here seems to be more nearly precise; this should be clear from the considerations of section 6.

An example may be enlightening.

Let $G = SL(3,\mathbb{R})$. The $\mathfrak{h}$-roots of $\mathfrak{g}$ are pictured below. Here $\mathfrak{t}$ is one dimensional (the vertical direction) and $\mathfrak{p}^{\mathfrak{t}}$ is one dimensional (the horizontal).



The complex roots (i.e. those with support on $\mathfrak{p}^{t}$) are $\{\pm\alpha, \pm\theta\alpha\}$; β is imaginary and noncompact. There is one compact positive root, namely the compact $\mathfrak{t}$-root corresponding to the complex root α; we call this α also, as a $\mathfrak{t}$-root. Then the noncompact $\mathfrak{t}$-roots are α and $\beta = 2\alpha$. (All of this is easily checked by finding the

$$\text{eigenvectors of } T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \subseteq K = SO(3)$$

in $s\ell(3,\mathbb{R})$.) A $\mathfrak{t}$-weight $r\alpha$ is dominant integral with respect to $\Delta^{+}(\mathfrak{k}) = \{\alpha\}$ iff r is non-negative and $2r \in \mathbb{Z}$. So let $\mu = \frac{n}{2}\alpha$; the associated positive root system is necessarily $\{\alpha, \theta\alpha, \beta\} = \Delta(\mathfrak{n}^{0})$. We want conditions for μ to be $\mathfrak{n}^{0}$ minimal. $2\rho_c = \alpha$, and $\rho = \frac{1}{2}(\alpha+\theta\alpha+\beta) = \beta = 2\alpha$ as a $\mathfrak{t}$-root. So $\lambda = \mu+2\rho_c-\rho = (\frac{n}{2}+1-2)\alpha = (\frac{n-2}{2})\alpha$. If $n > 2$, λ is dominant and non-singular with respect to $\mathfrak{n}^{0}$. By Corollary 4.15, μ is $\mathfrak{n}^{0}$ minimal; since $\dim \mathfrak{p}_0^{t} = 1$, there is at most a one (complex) parameter family of representations of G with minimal $\mathfrak{k}$-type μ. (It is not hard to see that these may be realized essentially as a principal series associated to a maximal cuspidal parabolic subgroup of G.) If $n = 0$, 1, or 2 (i.e. μ is the 1, 2, or 3 dimensional representation of SO(3)) then μ is in fact principal series minimal (see Table 5.8). Theorem 4.3 describes the 2 parameter family of representations of G with minimal $\mathfrak{k}$-type μ in those cases; in particular, there are too many representations for μ to have been $\mathfrak{n}^0$ minimal. Let us see why Corollary 4.16 does not apply to the case $\mu = \alpha$ (i.e. $n = 2$, the 3 dimensional representation of K.) Here $\lambda = \frac{n-2}{2}\alpha = 0$, so λ is dominant with respect to $\mathfrak{n}^0$. $\Delta_{i\mathbb{R}}(\mathfrak{n}^0 \cap \mathfrak{k})$ is empty, since there are no imaginary compact roots; so the second condition of (4.16) is vacuously satisfied. Hence the third condition must fail, which means that β must be a bad root in the sense of Definition 4.14. This can be seen directly as follows. Set $\sigma = -1 \in W_K$. Clearly $\Delta^+_\sigma(\mathfrak{k}) = \{\alpha\}$, and α is indeed a complex root; and $\Delta^+_\sigma(\mathfrak{g}) = \Delta^+ = \{\alpha, \theta\alpha, \beta\}$. As $\mathfrak{t}$-roots, $\Delta^+_\sigma(\mathfrak{g})$ therefore consists precisely of $\Delta^+_\sigma(\mathfrak{k})$, the corresponding noncompact root, and $\beta = \phi_1$; so $\sigma^{-1}(\phi_1 - \beta) = 0$, and β is a bad root. Finally, the problem can be seen from the perspective of Proposition 4.8. Consider the two noncompact roots α and $\beta = 2\alpha$. Then

$$\mu-\alpha-\beta = \mu-3\alpha = \alpha-3\alpha = -2\alpha = \sigma(\alpha+\rho_c)-\rho_c = \sigma(\mu+\rho_c)-\rho_c.$$

Thus the $\mathfrak{k}$-type μ has 1 cohomology of weight $\mu-\alpha-\beta$, which prevents μ from being $\mathfrak{n}^0$ minimal.

So far we have classified the irreducible admissible representations of a rank two group, explained a mysterious definition, and illustrated the structure theory of section 2 in one fell swoop. Still more mileage can be gotten out of this example, however. One might hope that all discussion of higher cohomology is irrelevant. There are a number of results about the uniqueness of the discrete series (see for example [23]) which assert that if a representation contains a $\mathfrak{k}$-type μ, but does not contain $\mu-\beta$ for any noncompact positive root β, then the representation is equivalent to a certain discrete series representation; or in general that there is at most a ($\dim \mathfrak{p}_0^t$) parameter family of such representations. All of these results require strong non-singularity hypotheses on μ, however. In the present case ($G = SL(3,R)$, $\mu=\alpha$) the minimality of μ in X implies that $\mu-\alpha = 0$ cannot occur in X; and $\mu-\beta = \mu-2\alpha = -\alpha$ is not even dominant. Yet there are a great many representations with minimal $\mathfrak{k}$-type μ (more than a $\dim \mathfrak{p}_0^t = 1$ parameter family.) To explain this, one must consider the fact that $\mu-\alpha-\beta$ gives rise to 1-cohomology which <u>can</u> occur in X.

<u>Proof of 4.6(a)</u>. Since $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras, $\mathfrak{p}_0^t$ must be a maximal abelian subalgebra of $\mathfrak{p}_0$. Thus $(\mathfrak{g}^{\mathfrak{p}_0^t}) \cap \mathfrak{p}_0 = \mathfrak{p}_0^t$. Now $\mathfrak{g}^{\mathfrak{p}^t} \supseteq \mathfrak{h}$, and the roots of $\mathfrak{h}$ in $\mathfrak{g}^{\mathfrak{p}_0^t}$ are obviously the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$. So $\Delta_{iR}(\mathfrak{n}^\circ \cap \mathfrak{p})$ is empty. Since

$\Delta_{\mathbb{C}}(\mathfrak{n}^0 \cap \mathfrak{p}) = \Delta_{\mathbb{C}}(\mathfrak{n}^0 \cap \mathfrak{k})$ by Proposition 2.3, it follows that every $\mathfrak{t}$-root is equal to a compact $\mathfrak{t}$-root. In particular μ is dominant with respect to $\mathfrak{n}^0$. To prove that the conditions of Corollary 4.16 hold, it is therefore enough to show that $2\rho_c - \rho$ is dominant with respect to $\mathfrak{n}^0$ and non-singular with respect to $\Delta_{i\mathbb{R}}(\mathfrak{n}_0 \cap \mathfrak{k})$.

$$2\rho_c - \rho = \rho_c - \rho(\mathfrak{n}^0 \cap \mathfrak{p}) = \rho(\Delta_{i\mathbb{R}}(\mathfrak{n}^0 \cap \mathfrak{k})) + \rho(\Delta_{\mathbb{C}}(\mathfrak{n}^0 \cap \mathfrak{k})) - \rho(\Delta_{\mathbb{C}}(\mathfrak{n}^0 \cap \mathfrak{p}))$$

$$= \rho(\Delta_{i\mathbb{R}}(\mathfrak{n}^0 \cap \mathfrak{k})) \text{ by the preceding remarks.}$$

$\Delta_{i\mathbb{R}}(\mathfrak{k})$ is a root system in its own right, with $\Delta_{i\mathbb{R}}(\mathfrak{n}^0 \cap \mathfrak{k})$ as positive roots. It follows immediately that $2\rho_c - \rho$ is dominant and non-singular with respect to $\Delta_{i\mathbb{R}}(\mathfrak{n}^0 \cap \mathfrak{k})$. Let (α,β) be a simple positive root of Δ. We want to show $\langle 2\rho_c, \alpha\rangle_{\mathfrak{k}} - \langle \rho, \alpha\rangle_{\mathfrak{k}} \geq 0$. If $\beta = 0$, this has just been proved; so assume $\beta \neq 0$. Then $(\alpha,\beta) - (\alpha,-\beta) = (0,2\beta)$ is not a root. We claim that $(\alpha,\beta) + (\alpha,-\beta) = (2\alpha,0)$ is not a root. Let X be a non-zero root vector of weight (α,β); put $x_{\mathfrak{k}} = X + \theta X$, $x_{\mathfrak{p}} = X - \theta X$. Then $x_{\mathfrak{k}} \in \mathfrak{k}$, $x_{\mathfrak{p}} \in \mathfrak{p}$, $x_{\mathfrak{k}} + x_{\mathfrak{p}} = 2X$, and $x_{\mathfrak{k}} - x_{\mathfrak{p}} = 2\theta X$. If $(2\alpha,0)$ were a root, $[X,\theta X]$ would be a non-zero root vector for $(2\alpha,0)$. But $[X,\theta X] = \frac{1}{4}[x_{\mathfrak{k}}+x_{\mathfrak{p}}, x_{\mathfrak{k}}-x_{\mathfrak{p}}] = -\frac{1}{2}[x_{\mathfrak{k}}, x_{\mathfrak{p}}] \in \mathfrak{p}$; so $(2\alpha,0)$ would be an imaginary noncompact root. No such roots exist, which proves the claim. By abstract root theory, $\langle(\alpha,\beta),(\alpha,-\beta)\rangle = 0$, i.e. $\langle\alpha,\alpha\rangle_{\mathfrak{k}} = \langle\beta,\beta\rangle_{\mathfrak{p}}$, or

$<(\alpha,\beta),(\alpha,\beta)> = 2<\alpha,\alpha>_{\mathfrak{k}}$. Since (α,β) is simple,

$$1 = \frac{2<\rho,(\alpha,\beta)>}{<(\alpha,\beta),(\alpha,\beta)>} = \frac{<\rho,\alpha>_{\mathfrak{k}}}{<\alpha,\alpha>_{\mathfrak{k}}},$$ i.e. $<\rho,\alpha>_{\mathfrak{k}} = <\alpha,\alpha>_{\mathfrak{k}}$. Since α is necessarily a positive compact root, $<2\rho_c,\alpha>_{\mathfrak{k}} \geq <\alpha,\alpha>_{\mathfrak{k}}$; so $<2\rho_c,\alpha>_{\mathfrak{k}} - <\rho,\alpha>_{\mathfrak{k}} \geq 0$. This completes the proof that $2\rho_c-\rho$ is $\mathfrak{n}^0$-dominant, and 4.6(a) follows.

<u>Proof of 4.6(b)</u> Here $K = \widetilde{SO}(n)$, the universal cover of SO(n). We omit the famous case n = 2 (which is of course well understood; in any case it will appear later as SU(1,1)). Thus K = Spin (n), a two-sheeted cover of SO(n). For definiteness assume n is even; henceforth it will be written as 2n. On the SO(2n) level, a torus is

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_n & \sin\theta_n \\ & & & -\sin\theta_n & \cos\theta_n \end{pmatrix}$$

Take the θ_i as coordinates in $\mathfrak{t}$; in the dual coordinates, $i\mathfrak{t}_0'$ is then identified with $\mathbb{R}^n$. After normalization, $< , >_{\mathfrak{k}}$ is the usual inner product on $\mathbb{R}^n$.

A straightforward computation produces the weight vectors of $\mathfrak{t}$ in $s\ell(2n,R)$. With respect to the usual basis $\varepsilon_1 \ldots \varepsilon_n$ of $\mathbb{R}^n$, the compact roots are $q_{ij}^{\pm\pm} = \pm\varepsilon_i \pm \varepsilon_j$, $1 \le i < j \le n$. The noncompact roots are $p_{ij}^{\pm\pm} = \pm\varepsilon_i \pm \varepsilon_j$, and $p_i^{\pm} = \pm 2\varepsilon_i$. $\mathfrak{p}^{t}$ has dimension n-1. Clearly the complex roots are the $q_{ij}^{\pm}$ and $p_{ij}^{\pm\pm}$; for according to Proposition 2.3, a $\mathfrak{t}$-root occurs with multiplicity two iff it is complex. The imaginary roots are the $p_i^{\pm}$.

Order $i\mathfrak{t}_0' \cong \mathbb{R}^n$ lexicographically. Then the compact roots are the $q_{ij}^{+\pm}$, so that

$2\rho_c = (2(n-1), 2(n-2), \ldots 2, 0)$ by an easy computation. A weight $(a_1 \ldots a_n)$ is dominant integral iff (a_i) is decreasing, $a_{n-1} + a_n \ge 0$, and either all $a_i \in \mathbb{Z}$ or all $a_i \in \mathbb{Z} + \frac{1}{2}$. Suppose $\mu = (\mu_1 \ldots \mu_n)$ is a $\mathfrak{k}$-type, i.e. μ is dominant integral. Then $\mu+2\rho_c = (\mu_1+2(n-1), \ldots \mu_{n-1}+2, \mu_n)$. Recall the definition of $\mathfrak{b}^0$. Clearly $\langle p_{ij}^{+\pm}, \mu+2\rho_c\rangle = \langle q_{ij}^{+\pm}, \mu+2\rho_c\rangle > 0$; so $q_{ij}^{+\pm} \in \Delta(\mathfrak{n}^0)$. Since (μ_i) is decreasing and $\mu_{n-1}+\mu_n \ge 0$, $\mu_i \ge 0$ for $i < n$. Thus $\langle p_i^{+}, \mu+2\rho_c\rangle > 0$ for $i < n$. If

$\mu_n \neq 0$, $\langle p_n^{\mathrm{sgn}\,\mu_n}, \mu+2\rho_c\rangle > 0$. If $\mu_n = 0$, $\langle p_n^+, \mu+2\rho_c\rangle = 0$, and $p_n^+ > p_n^-$ (recall that $i\mathfrak{t}_0'$ is ordered lexicographically.) Define sgn (0) = + ; then we have shown that

$$\Delta_{\mathfrak{t}}(\mathfrak{b}^0 \cap \mathfrak{p}) = \{p_{ij}^{++}_{--}, p_i^{\mathrm{sgn}\,\mu_i}\}$$

One computes immediately that $\rho = 2\rho_c + (1,1\ldots 1,\mathrm{sgn}\,\mu_n)$, so that

(4.17) $\mu+2\rho_c-\rho = \lambda(\mu) = (\mu_1-1,\mu_2-1,\ldots\mu_{n-1}-1,\mu_n-\mathrm{sgn}\,\mu_n)$.

If $\mu_n = 1$, set $s = n$; otherwise let s denote the largest index with $|\mu_s| > 1$. Let $\mathfrak{l} \supset \mathfrak{h}$ be the θ-invariant subalgebra corresponding to the $\mathfrak{t}$-roots perpendicular to $\varepsilon_1\ldots\varepsilon_s$;

$$\Delta_{\mathfrak{t}}(\mathfrak{l}) = \{q_{ij}^{++}_{--}, p_{i\,j}^{++}_{--}, p_i^{+}_{-}|i,j > s\}$$

Let $\mathfrak{n}$ be the span of the root vectors corresponding to the remaining positive roots;

(4.18) $\Delta_{\mathfrak{t}}(\mathfrak{n}) = \{q_{ij}^{++}_{--}, p_{ij}^{++}_{--}, p_i^{\mathrm{sgn}\,\mu_i}|i \le s\}$.

Then $\mathfrak{b} = \mathfrak{l} \oplus \mathfrak{n} \supseteq \mathfrak{b}^0$ is a θ-invariant parabolic. The span $\mathfrak{t}^*_{\mathrm{root}}(\mathfrak{l})$ of the $\mathfrak{t}$-roots of $\mathfrak{l}$ in $\mathfrak{t}^*$ is clearly $\langle\varepsilon_{s+1}\ldots\varepsilon_n\rangle$; and $\{p_i^{\mathrm{sgn}\,\mu_i}|i > s\}$ is a strongly orthogonal spanning set of noncompact imaginary roots for

$\mathfrak{t}^*_{root}(\mathfrak{l})$. By Proposition 2.5, $\mathfrak{l}_0$ is split; in fact it is easy to see that the semisimple ideal of $\mathfrak{l}_0$ is isomorphic to $s\ell(2(n-s),R)$.

We check first that $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal for $\mathfrak{l}$. If $s = n$, then $\mathfrak{l} = \mathfrak{h}$ is abelian, and there is nothing to prove. Suppose then that $s < n$. From (4.18), we deduce

$$\Delta_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{p}) = \{p_{ij}^{++},\ p_i^{+} | i \leq s\}.$$

Thus $2\rho(\mathfrak{n}\cap\mathfrak{p})$ vanishes on the last $n-s$ coordinates, and $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p}) = (\ldots\mu_{s+1},\ldots\mu_n)$. By the definition of s, $|\mu_{s+1}| \leq 1$; since μ is dominant integral, it follows that the only possibilities for $(\mu_{s+1}\ldots\mu_n)$ are $(1,\ldots 1,0,\ldots 0)$ (with at least one zero), $(1,1\ldots 1,-1)$, and $(\frac{1}{2},\ldots\frac{1}{2},\pm\frac{1}{2})$. Consulting Table 5.8, we see that these are all principal series minimal for the semisimple ideal $s\ell(2(n-s),R)$ of $\mathfrak{l}_0$; and it is immediate that $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal for $\mathfrak{l}$.

To show that μ is $\mathfrak{n}$-minimal, we apply Proposition 4.15. Since $\Delta_{iR}(\mathfrak{k})$ is empty, condition ii) will always be vacuously satisfied. Suppose first that $s = n$ and $\mu_n \geq 1$; thus $\mathfrak{b} = \mathfrak{b}^0$. Since (μ_i) is decreasing,

$\mu_i \geq 1$ for all i. It is clear from (4.17) that μ is dominant with respect to $\mathfrak{n}_0$. Suppose a noncompact positive imaginary root p_r^+ is bad. Notice that since $i\mathfrak{t}_0'$ is ordered lexicographically, every p_i^+ is $\succ 0$. Suppose $\sigma \in W_K$ is as in Definition 4.14. One knows that W_K operates on $i\mathfrak{t}_0' \tilde{=} \mathbb{R}^n$ by permutation and changing an even number of signs. It follows easily that $\Delta_\sigma^+(\mathfrak{g})$ contains on even number of p_j^+, say $\{\phi_1 \ldots \phi_{2\ell}\}$. There are now two cases. If $\sigma^{-1}p_r^+ \in \Delta^-$, then $p_r^+ \in \Delta_\sigma^+(\mathfrak{g})$, say $p_r^+ = \phi_1$. Then $\sigma^{-1}(\phi_1 + \ldots + \phi_{2\ell} - p_r^+) = \sigma^{-1}(\phi_2 + \ldots + \phi_{2\ell})$. Every $\sigma^{-1}\phi_i$ is a negative noncompact imaginary root, i.e. some p_j^-. Thus their sum is $\prec 0$, contradicting c) of Definition 4.14. If, on the other hand, $\sigma^{-1}p_r^+ \in \Delta^+$, then $\sigma^{-1}(-p_r^+)$ and every $\sigma^{-1}\phi_i$ is a negative noncompact imaginary root; so again $\sigma^{-1}(\phi_1 + \ldots + \phi_{2\ell} - p_r^+) \prec 0$, contradicting 4.14 c). So there are no bad roots in this case; by Proposition 4.15, μ is $\mathfrak{n}_0$ minimal.

In every other case, $|\mu_s| > 1$, so that $|\mu_i| > 1$ for $i \leq s$. By (4.17), $\lambda(\mu)_i \neq 0$ for $i < s$, and $\operatorname{sgn} \lambda(\mu)_i = \operatorname{sgn} \mu_i$. We have seen that $\Delta_{i\mathbb{R}}(\mathfrak{n} \cap \mathfrak{g}) = \{p_i^{\operatorname{sgn} \mu_i} | i \leq s\}$ (see (4.18)), so $\lambda(\mu)$ is dominant and non-singular with respect to $\Delta_{i\mathbb{R}}(\mathfrak{n} \cap \mathfrak{p})$. In particular there are no bad roots. We claim that $\lambda(\mu)$ is $\mathfrak{n}$ dominant. By (4.18), it remains only to show this for the roots $\{p_{ij}^{++} \mid i \leq s\}$; recall that $p_{ij}^{++} = q_{ij}^{++}$ as $\mathfrak{t}$-roots. Consider for example p_{ij}^{++}. Since $i < j \leq n$, μ_i is positive; and we have seen $|\mu_i| > 1$. Thus $\mu_i > 1$. Also

$$\langle p_{ij}^{++}, \lambda \rangle = \langle \varepsilon_i + \varepsilon_j, \lambda \rangle = \lambda_i + \lambda_j = \mu_i + \mu_j - 1 - \operatorname{sgn} \mu_j.$$

If $\mu_j < 0$, $\langle p_{ij}^{++}, \lambda \rangle = \mu_i + \mu_j \geq 0$ since μ is dominant. Suppose then that $\mu_j \geq 0$, so that $\langle p_{ij}^{++}, \lambda \rangle = \mu_i + \mu_j - 2$. If $\mu_i \in \mathbb{Z}$, then $\mu_i > 1$ implies $\mu_i \geq 2$; so $\mu_i + \mu_j - 2 \geq 0$. If $\mu_i \in \mathbb{Z} + \frac{1}{2}$, then $\mu_i \geq \frac{3}{2}$, and also $\mu_j \in \mathbb{Z} + \frac{1}{2}$; so $\mu_j \geq \frac{1}{2}$, and $\mu_i + \mu_j - 2 \geq 0$. In every case $\langle p_{ij}^{++}, \lambda \rangle \geq 0$. A similar argument can be given for p_{ij}^{+-}. Thus μ is indeed $\mathfrak{n}$ dominant, and so $\mathfrak{n}$ minimal by Proposition 4.15. This proves 4.6 (b) for $\widetilde{SL}(2n,\mathbb{R})$. $\widetilde{SL}(2n+1,\mathbb{R})$ is similar but considerably less messy; so the details are left to the reader.

To continue, we need a refinement of 4.11. Let $\nu \in i\mathfrak{t}_0'$; then by 4.11, with notation as in that situation, (4.19) $||\mu||-||\gamma|| = 2<\lambda(\mu)+\frac{1}{2}v, \Sigma\beta_{i_j} + \Sigma\alpha_{i_j}> + <2\rho(B^c)-v, 2\rho(B)>$.

Lemma 4.20 Suppose $v = \sum_{\delta\in\Delta^+(\ell)} c_\delta \delta$, $0 \leq c_\delta \leq 1$.

Then $<2\rho(B^c)-v, 2\rho(B)> \geq 0$. Equality holds iff

i) $B = \Delta_\tau^+(\mathfrak{g})$ for some $\tau \in W(\mathfrak{g}, \mathfrak{h})$ commuting with θ.

ii) $<\delta, \rho-\tau\cdot\rho> = 0$ for all $\delta \in \Delta^+(\ell)$ with $c_\delta \neq 0$.

Proof. By taking appropriate convex combinations, it is enough to do this in case $c_\delta = 0$ or 1; so that $v = \Sigma\delta_i$ for some subset $\{\delta_i\}$ of $\Delta^+(\ell)$. Define

$C_0 = \{\delta \in \Delta^+(\ell) | \delta \neq \delta_i$, any i, and $<2\rho(B),\delta>_{\mathfrak{k}} > 0\}$

$C_1 = \{\delta_i | <2\rho(B),\delta_i>_{\mathfrak{k}} > 0\}$.

Considering the positive root system for ℓ defined by $2\rho(B)$, it is easy to see that $C = C_0 \cup C_1 = \Delta_{\tau_1}^+(\ell)$, for some $\tau_1 \in W(\ell, \mathfrak{h})$ commuting with θ. By Lemma 4.13,

(*) $<2\rho(\Delta^+(\ell))-2\rho(C_0)-2\rho(C_1), 2\rho(C_0)+2\rho(C_1)>_{\mathfrak{k}} = 0$.

Since $\mathfrak{n}$ is invariant under ℓ, $<\rho(n),\delta>_{\mathfrak{k}} = 0$ for all $\delta \in \Delta^+(\ell)$. Put $C_2 = \{\delta_i\} - C_1$; so $2\rho(C_2) + 2\rho(C_1) = v$. Applying these observations, we compute

$<2\rho(B^c)-v, 2\rho(B)>_{\mathfrak{k}} = <2\rho-2\rho(B)-2\rho(C_1)-2\rho(C_2), 2\rho(B)>_{\mathfrak{k}}$

$= <2\rho-2\rho(B)-2\rho(C_0)-2\rho(C_1), 2\rho(B)>_{\mathfrak{k}} + <2\rho(C_0)-2\rho(C_2), 2\rho(B)>_{\mathfrak{k}}$

$= <2\rho-2\rho(B\cup C_0\cup C_1), 2\rho(B\cup C_0\cup C_1)>_{\mathfrak{k}} - <2\rho-2\rho(B\cup C_0\cup C_1), 2\rho(C_0\cup C_1)>_{\mathfrak{k}}$

$+ <2\rho(C_0)-2\rho(C_2), 2\rho(B)>$. Call the first term (I), and write $2\rho = 2\rho(\mathfrak{n}) + 2\rho(\Delta^+(\mathfrak{l}))$. Then

$<2\rho(B^c)-v, 2\rho(B)> = (I) - <2\rho(\mathfrak{n}), 2\rho(C_0\cup C_1)>_{\mathfrak{k}} + <2\rho(B), 2\rho(C_0\cup C_1)>_{\mathfrak{k}}$

$-<2\rho(\Delta^+(\mathfrak{l}))-2\rho(C_0)-2\rho(C_1), 2\rho(C_0)+2\rho(C_1)>_{\mathfrak{k}} + <2\rho(B), 2\rho(C_0)-2\rho(C_2)>_{\mathfrak{k}}$

$= (I) + <2\rho(B), 4\rho(C_0)+2\rho(C_1)-2\rho(C_2)>_{\mathfrak{k}}$ by (*) and the remarks following it. By Lemma 4.13, (I) ≥ 0; and the second term is non-negative by the definition of C_0, C_1, and C_2. This proves the desired inequality. In case of equality, necessarily C_0 and C_1 are empty, (I) = 0, and $-<2\rho(B), 2\rho(C_2)>_{\mathfrak{k}} = -<2\rho(B), \Sigma\delta_i>_{\mathfrak{k}} = 0$. By Lemma 4.13, this is equivalent to the conditions stated (recall that $\rho-\tau\cdot\rho = 2\rho(\Delta^+_\tau(\mathfrak{g}))$.) Q.E.D.

<u>By choosing $v = \Sigma c_\delta \delta$ appropriately, it will always be possible to arrange for $\lambda(\mu) + \frac{1}{2}v = \tilde{\lambda}(\mu, v\mathfrak{b}) = \tilde{\lambda}$ to be dominant with respect to $\mathfrak{n}^0$ in the cases we consider.</u> (The reader may easily see how this would be done for $s\ell(n,\mathbb{R})$.) This is the strong form of conjecture 4.2(v)

which was alluded to after the statement of Theorem 4.5.

It is easy to check whether $\tilde{\lambda}$ is $\mathcal{N}^0$ dominant: for let $\delta_1 \dots \delta_m$ be the simple roots in $\mathcal{N}_0$. Since $2\frac{\langle\delta_i,\rho\rangle_k}{\langle\delta_i,\delta_i\rangle} = 1$, $\tilde{\lambda}$ will be dominant iff

$$\langle\mu+2\rho_c+\tfrac{1}{2}\nu,\delta_i\rangle_k \geq \frac{\langle\delta_i,\delta_i\rangle}{2}$$

for all i. Furthermore $\tilde{\lambda}$ is singular precisely with respect to those simple roots for which equality holds.

To illustrate the methods, let $G = SU(p,q)$ $(p \leq q)$ and put $\mu = 0$, the trivial representation of K. K is $S(U(p) \times U(q))$, and has a torus consisting of the diagonal matrices

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & & & & & 0 \\ & \ddots & & & & & \\ & & e^{i\theta_p} & & & & \\ & & & e^{i\phi_1} & & & \\ & & & & \ddots & & \\ 0 & & & & & & e^{i\phi_a} \end{pmatrix} \; \Sigma\theta_i + \Sigma\phi_j = 0 \right\}.$$

$\mathfrak{t}_0$ may then be identified with the subspace

$\{[(\theta_1 \ldots \theta_p), (\phi_1 \ldots \phi_q)] \mid \Sigma\theta_i + \Sigma\phi_j = 0\}$ of $\mathbb{R}^p \oplus \mathbb{R}^q$;

similarly $i\mathfrak{t}_0'$ is identified with the same space, with the usual inner product for $\langle\ ,\ \rangle_{\mathfrak{k}}$. G is an equal rank group, so $\mathfrak{p}^{\mathfrak{t}} = 0$. Identifying $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ with $s\ell(p+q,\mathbb{C})$, one checks easily that the roots of $\mathfrak{t}$ in $\mathfrak{g}$ are the following (with respect to the basis $\{\varepsilon_i^1, \varepsilon_j^2 \mid 1 \leq i \leq p,\ 1 \leq j \leq q\}$ of $\mathbb{R}^p \oplus \mathbb{R}^q$):

compact:	$e_{i,i'} = \varepsilon_i^1 - \varepsilon_{i'}^1$,	$1 \leq i,\ i' \leq p,\ i \neq i'$
	$f_{j,j'} = \varepsilon_j^2 - \varepsilon_{j'}^2$,	$1 \leq j,\ j' \leq q,\ j \neq j'$
noncompact:	$g_{ij}^{\pm} = \pm(\varepsilon_i^1 - \varepsilon_j^2)$	$1 \leq i \leq p,\ 1 \leq j \leq q$.

We will sometimes refer to ε_i^1 as the <u>i-coordinates</u> and ε_j^2 as the <u>j-coordinates</u>. Order $i\mathfrak{t}_0' \subseteq \mathbb{R}^p \oplus \mathbb{R}^q$ lexicographically. Then $\Delta^+(\mathfrak{k}) = \{e_{i,i'}, f_{j,j'} \mid i<i', j<j'\}$, and one computes easily that

$$2\rho_c = [(p-1, p-3, \ldots p-2i+1, \ldots -(p-1)), (q-1, \ldots q-2j+1, \ldots -(q-1))]$$

Since $\mu = 0$, $\mu + 2\rho_c = 2\rho_c$. By the definition of $\mathfrak{b}^0$,

$$\Delta(\mathfrak{n}^0 \cap \mathfrak{p}) = \{g_{ij}^+ \mid p-2i+1 \geq q-2j+1\} \cup \{g_{ij}^- \mid p-2i+1 < q-2j+1\}$$

$$= \{g_{ij}^+ \mid j-i \geq \tfrac{1}{2}(q-p)\} \cup \{g_{ij}^- \mid j-i < \tfrac{1}{2}(q-p)\}.$$

Assume for definiteness that $q-p$ is even, so that $\frac{q-p}{2}$ is a non-negative integer. The simple roots of Δ^+ are precisely those which cannot be expressed as a sum of two other positive roots. Using this, one can check that the simple roots are

$$\{f_{j,j+1} \mid 1 \leq j \leq \tfrac{q-p}{2} - 1, \tfrac{q+p}{2} \leq i < q\} \cup \{g^+_{i,\frac{q-p}{2}+i}, g^-_{i,\frac{q-p}{2}+i-1} \mid 1 \leq i \leq p\}$$

Let $\mathfrak{b} \supseteq \mathfrak{b}^0$ be the (θ-invariant) parabolic corresponding to the simple noncompact roots $\{g^+_{i,\frac{q-p}{2}+i}\}$. Since these are strongly orthogonal, the semisimple ideal of ℓ_0 is a product of p copies of $s\ell(2,R)$; so ℓ_0 is split. We leave to the reader the straightforward verification that $\mu - 2\rho(\mathfrak{n} \cap \mathfrak{p})$ is principal series minimal for ℓ_0.

Set $v = \sum_{i=1}^{p} g^+_{i,\frac{q-p}{2}+i}$. Then by direct computation

$$\mu+2\rho_c + \tfrac{1}{2}v = [(p-\tfrac{1}{2},\ldots p-2i+\tfrac{3}{2},\ldots -(p-\tfrac{3}{2})),(q-1,\ldots q-2(\tfrac{q-p}{2})+1,$$
$$q-2(\tfrac{q-p}{2}+1)+\tfrac{1}{2},\ldots q-2j+\tfrac{1}{2}\ldots, q-2(\tfrac{q+p}{2})+\tfrac{1}{2},$$
$$q-2(\tfrac{q+p}{2}+1)+1,\ldots -(q-1))].$$

From this one computes

$$\langle f_{j,j+1}, \mu+2\rho_c+\tfrac{1}{2}v\rangle_{\mathfrak{k}} = 2 \qquad 1 \le j \le \tfrac{q-p}{2} - 1, \ \tfrac{q+p}{2} < j < q$$

$$\langle f_{\frac{q+p}{2},\frac{q+p}{2}+1}, \ \mu+2\rho_c + \tfrac{1}{2}v\rangle_{\mathfrak{k}} = \tfrac{3}{2}$$

$$\langle g^{+}_{i,\frac{q-p}{2}+i}, \mu+2\rho_c+\tfrac{1}{2}v\rangle_{\mathfrak{k}} = p-2i+\tfrac{3}{2} - (q-2(\tfrac{q-p}{2}+i) + \tfrac{1}{2})$$

$$= 1 \qquad 1 \le i \le p$$

$$\langle g^{-}_{i,\frac{q-p}{2}+i-1}, \mu+2\rho_c+\tfrac{1}{2}v\rangle_{\mathfrak{k}} = q-2(\tfrac{q-p}{2}+i-1)+\tfrac{1}{2} - (p-2i+\tfrac{3}{2})$$

$$= 1 \qquad 2 \le i \le p$$

$$\langle g^{-}_{i,\frac{q-p}{2}}, \mu+2\rho_c+\tfrac{1}{2}v\rangle_{\mathfrak{k}} = (p+1)-(p - \tfrac{1}{2}) = \tfrac{3}{2}.$$

Since all roots have length 2, the remarks following Lemma 4.20 imply that $\tilde{\lambda}$ will be $\mathfrak{n}^0$ dominant if $\langle\mu+2\rho_c+\tfrac{1}{2}v,\delta\rangle \ge 1$ for each simple root δ; this has just been verified.

With the usual notation, suppose now that $\mu - \sum_{j=1}^{J} \beta_{i_j}$ gives rise to J-1 cohomology larger than μ; recall that $\{\alpha_{i_j}\}_{j=1}^{J-1}$ are the corresponding compact roots, and $B = \{\alpha_{i_j}\} \cup \{\beta_{i_j}\}$. By (4.19) and Lemma 4.20, there is a $\tau \in W(\mathfrak{g},\mathfrak{h})$ with $B = \Delta^{+}_{\tau}(\mathfrak{g})$; also $||\mu|| = ||\gamma||$. Since $\mu = 0$, it follows that $\gamma = 0$; so by (4.9) and (3.7)

$$-\Sigma\beta_{i_j} = \mu-\Sigma\beta_{i_j} = \sigma(\gamma+\rho_c)-\rho_c = \sigma\cdot\rho_c-\rho_c = -\Sigma\alpha_{i_j}.$$

so $\Sigma\beta_{i_j} = \Sigma\alpha_{i_j}$; since $\rho-\tau\cdot\rho = \Sigma\beta_{i_j} + \Sigma\alpha_{i_j}$, it follows that

(*) every coordinate of $\rho-\tau\cdot\rho$ is even.

Lemma 4.20 and (4.19) also imply that every root in $\Delta^+_\tau(\mathfrak{g})$ is a sum of the simple roots annihilating $\tilde{\lambda}$, namely $\{g^+_{i,\frac{q-p}{2}+i} \mid 1 \leq i \leq p\} \cup \{g^-_{i,\frac{q-p}{2}+i-1} \mid 2 \leq i \leq p\}$. Since $W(\mathfrak{g},\mathfrak{h})$ is the permutation group on the coordinates, it is not difficult to deduce that τ acts only on the i coordinates, and the j coordinates between $\frac{q-p}{2}+1$ and $\frac{q+p}{2}$. Write $\rho = [(\rho^1_1,\rho^1_2,\ldots,\rho^1_p),(\rho^2_1,\ldots\rho^2_q)]$; say $\rho^1_1 = n$. (Actually, $n = \frac{2p-1}{2}$, but this is unimportant.) Since $g^+_{i,\frac{q-p}{2}+1}$ and $g^-_{i,\frac{q-p}{2}+i-1}$ are simple roots of length 2, $1 = \langle\rho,g^+_{i,\frac{q-p}{2}+i}\rangle = \langle\rho,\varepsilon^1_i-\varepsilon^2_{\frac{q-p}{2}+i}\rangle = \rho^1_i-\rho^2_{\frac{q-p}{2}+i}$, and $1 = \rho^2_{\frac{q-p}{2}+i} - \rho^1_{i+1}$.

We deduce immediately that $\rho^1_i = n-2i+2$, $\rho_{\frac{q-p}{2}+1} = n-2i+1$ for $1 \leq i \leq p$. (Henceforth such computations will often be left to the reader.) By (*), τ preserves the i coordinates and the j coordinates separately; say $\tau = [\sigma_1, \sigma_2]$ with obvious notation. We claim that $\sigma_1(i) = \sigma_2(\frac{q-p}{2}+i) - \frac{q-p}{2}$ for $1 \le i \le p$. If not, a finite set argument shows that necessarily $\sigma_1(i) > \sigma_2(\frac{q-p}{2} + i) - \frac{q-p}{2}$ for some i. Then

$$<\tau^{-1}(g^+_{i,\frac{q-p}{2}+i}),\rho>_{\mathfrak{k}} = <g^+_{i,\frac{q-p}{2}},\tau\cdot\rho> = \rho^1_{\sigma_1\cdot i} - \rho^2_{\sigma_2\cdot(\frac{q-p}{2}+i)} < 0$$

by the computation of ρ. So $\tau^{-1}\cdot(g^+_{i,\frac{q-p}{2}+i})$ would be a negative root, i.e. $g^+_{i,\frac{q-p}{2}+i} \in \Delta^+_\tau(\mathfrak{g}) \subseteq \Delta(\mathfrak{n})$. But $g^+_{i,\frac{q-p}{2}+i}$ is a root of $\mathfrak{l}$; a contradiction. So $\sigma_1(i) = \sigma_2(\frac{q-p}{2}+i) - \frac{q-p}{2}$. We deduce that, as automorphisms of $i\mathfrak{t}_0'$, $\det[\sigma_1, 1] = \det[1,\sigma_2]$, so that $\det \tau = (\det[\sigma_1,1])^2$. Now if $w \in W(\mathfrak{g},\mathfrak{h})$, one knows that $\det w = (-1)^{\ell(w)}$; so it follows that $\ell(\tau)$ is even. On the other hand, $\ell(\tau) = |B| = J+J-1 = 2J-1$; a contradiction. So γ cannot exist, and μ is $\mathfrak{n}$-minimal, proving Conjecture 4.2 in this case.

By Theorem 4.5, we have proved that SU(p,q) has at most a p-parameter family of spherical representations. This is of course less than earthshaking; the point was to indicate the idea of the following computations in a relatively simple situation. The reader may consider himself warned.

Proof of 4.6 (c) This is the case $\mathfrak{g}_0 = su(p,q)$, $p \le q$, $\mu = [(\mu_1^1, \ldots \mu_p^1), (\mu_1^2, \ldots \mu_q^2)]$ an arbitrary $\mathfrak{k}$-type; we retain the notation of the preceding example. Since μ is dominant integral, both sequences (μ_i^1) and (μ_j^2) decrease by integers. Set $\mu + 2\rho_c = a = [(a_1^1, \ldots a_p^1), (a_1^2, \ldots a_q^2)]$; then the positive root system defined by a satisfies

$$\Delta(\mathfrak{n}^o \cap \mathfrak{p}) = \{g_{ij}^+ \mid a_i^1 - a_j^2 \ge 0\} \cup \{g_{ij}^- \mid a_i^1 - a_j^2 < 0\}.$$

Define $a_0^1 = a_0^2 = \infty$, $a_{p+1}^1 = a_{q+1}^2 = -\infty$ (formally). For $1 \le i \le p$, define $j(i)$, $0 \le j(i) \le q$, so that $a_{j(i)}^2 > a_i^1 \ge a_{j(i)+1}^2$. The index i is called upper transitional if $a_{i-1}^1 \ge a_{j(i)}^2$, and lower transitional if $a_{j(i)+1}^2 > a_{i+1}^1$. It is called upper critical if $0 < a_{j(i)}^2 - a_i^1 \le 1$, and lower critical if $0 \le a_i^1 - a_{j(i)+1}^2 < 1$. The simple roots in Δ^+ are then

$$\{e_{i,i+1} \mid i \text{ is not lower transitional and } i+1 \text{ is not upper transitional}\} \cup$$

$$(*) \quad \{f_{j,j+1} \mid j \text{ is not } j(i) \text{ for any } i\} \cup \{g_{i,j(i)}^- \mid i \text{ is upper transitional}\}$$

$$\cup \{g_{i,j(i)+1}^+ \mid i \text{ is lower transitional}\}.$$

By the integrality of μ, $a_i^1 - a_j^2 \equiv \varepsilon \bmod \mathbb{Z}$, for some <u>fixed</u> $\varepsilon \in [0,1)$. It follows that i is upper critical iff $a_{j(i)}^2 - a_i^1 = 1 - \varepsilon$, and i is lower critical iff $a_i^1 - a_{j(i)+1}^2 = \varepsilon$. Within the i coordinates or the j coordinates, ρ_c necessarily decreases by 1; so $a_i^1 - a_{i+1}^1 \geq 2$, and $a_j^2 - a_{j+1}^2 \geq 2$. One deduces easily that every upper (respectively lower) critical index is also upper (lower) transitional. We can therefore let $\mathfrak{l}$ be the parabolic corresponding to the simple roots $\{g_{i,j(i)}^- \mid i \text{ is upper critical}\} \cup \{g_{i,j(i)+1}^+ \mid i \text{ is lower critical}\}$. We claim that these are strongly orthogonal; since they are simple, it suffices to show that the sum of two of them is never a root. For pairs of the same sign this is obvious; so suppose $g_{i,j(i)}^- + g_{i',j(i')+1}^+$ is a root. By the description of the root system, either $i = i'$, or $j(i) = j(i')+1$. If $i = i'$, i is both upper and lower critical, so that

$$a_{j(i)}^2 - a_{j(i)+1}^2 = (a_{j(i)}^2 - a_i^1) + (a_i^1 - a_{j(i)+1}^2) = 1-\varepsilon+\varepsilon = 1,$$

which is impossible since (a_j^2) decreases by at least 2. Similarly one can rule out $j(i) = j(i') + 1$, proving the claim. Thus the semisimple ideal of $\mathfrak{l}_0$ is a product of copies of $\mathfrak{sl}(2,\mathbb{R})$, and is therefore split. Since $\Lambda^{R+S}\mathfrak{n}$

is a one dimensional $\mathfrak{l}$-module of weight $2\rho(\mathfrak{n}\cap\mathfrak{p}) + 2\rho(\mathfrak{n}\cap\mathfrak{k})$, $\mu - 2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal iff $\mu + 2\rho(\mathfrak{n}\cap\mathfrak{k})$ is. Here $\mathfrak{n}\cap\mathfrak{k} = \mathfrak{n}^{\circ}\cap\mathfrak{k}$, so $2\rho(\mathfrak{n}\cap\mathfrak{k}) = 2\rho_c$; so we must show that $a = \mu+2\rho_c$ is principal series minimal for $\mathfrak{l}$. This is immediate from the characterization of critical indices, and the description of principal series minimal types in Table 5.8; details are left to the reader. It remains to show that μ is $\mathfrak{n}$ minimal.

$$\text{Set } v = \sum_{\substack{i \text{ upper} \\ \text{critical}}} \varepsilon\cdot g^{-}_{i,j(i)} + \sum_{\substack{i \text{ lower} \\ \text{critical}}} (1-\varepsilon) g^{+}_{i,j(i)+1}$$

We claim that $\tilde{\lambda} = \mu + 2\rho_c + \frac{1}{2}v - \rho$ is $\mathfrak{n}^0$-dominant. All roots have length 2; so according to the method outlined after Lemma 4.20, we must show that

$\langle\mu + 2\rho_c + \frac{1}{2}v, \delta\rangle \geq 1$ whenever δ is a simple root.

Consider first a simple root $e_{i,i+1}$. We know from (*) that i is not lower critical, and $i+1$ is not upper critical. Using the definition of v, this implies that

$$-\varepsilon \leq \langle v, \varepsilon^1_i\rangle \leq 0 \qquad 0 \leq \langle v, \varepsilon^1_{i+1}\rangle \leq 1-\varepsilon; \quad \text{so}$$

$$\begin{aligned}\langle\mu + 2\rho_c + \tfrac{1}{2}v, e_{i,i+1}\rangle &= \langle\mu + 2\rho_c + \tfrac{1}{2}v, \varepsilon^1_i - \varepsilon^1_{i+1}\rangle \\ &\geq \langle\mu + 2\rho_c, \varepsilon_i - \varepsilon_{i+1}\rangle + \tfrac{1}{2}(-\varepsilon - (1-\varepsilon)) \\ &\geq 2 - \tfrac{1}{2} = \tfrac{3}{2}.\end{aligned}$$

Similarly, if $f_{j,j+1}$ is a simple root, $\langle\mu + 2\rho_c + \frac{1}{2}v, f_{j,j+1}\rangle \geq \frac{3}{2}$. Suppose $g^-_{i,j(i)}$ is a simple root not in $\Delta(\ell)$. Then i is not upper critical, so $a^2_{j(i)} - a^1_i > 1 - \varepsilon$; by integrality, this forces $a^2_{j(i)} - a^1_i \geq 2 - \varepsilon$. Now $\langle v,\varepsilon^1_i\rangle$ and $\langle v,\varepsilon^2_{j(i)}\rangle$ can be non-zero only if i is lower critical or $j(i) = j(i') + 1$ and i' is lower critical; in any case

$$0 \leq \langle v,\varepsilon^1_i\rangle \leq 1-\varepsilon,\ 0 \geq \langle v,\varepsilon^2_{j(i)}\rangle \geq -(1-\varepsilon).\quad \text{Thus}$$

$$\begin{aligned}\langle\mu+2\rho_c+\tfrac{1}{2}v,\ g^-_{i,j(i)}\rangle_{k} &\geq a^2_{j(i)}-a^1_i + \tfrac{1}{2}(-(1-\varepsilon) - (1-\varepsilon))\\ &\geq 2 - \varepsilon - (1-\varepsilon) = 1.\end{aligned}$$

Suppose that equality holds; then $a^2_{j(i)} - a^1_i = 2 - \varepsilon$; i is lower critical, so $a^1_i - a^2_{j(i)+1} = \varepsilon$; and $j(i) = j(i') + 1$, with $a^1_{i'} - a^2_{j(i)} = \varepsilon$. These imply $a^1_{i'} - a^1_i = 2$, so that $i' = i - 1$, and that $a^2_{j(i)} - a^2_{j(i)+1} = 2$. Since $a = \mu + 2\rho_c$, it follows that μ is singular with respect to $e_{i-1,i}$ and $f_{j(i),j(i)+1}$. Also $v^1_{i-1} = 1-\varepsilon = v^1_i$, $v_{j(i)} = -(1-\varepsilon) = v_{j(i)+1}$, so v is singular with respect to these compact roots. To summarize: if $\langle\mu+2\rho_c+\frac{1}{2}v,g^-_{i,j(i)}\rangle_{k} = 1$, then $j(i-1)+1 = j(i)$, and the noncompact roots $g^+_{i-1,j(i)}$ and $g^+_{i,j(i)+1}$ are in $\Delta(\ell)$. μ and v are singular with respect to the compact roots $e_{i-1,i}$ and $f_{j(i),j(i)+1}$.

Suppose next that $g^+_{i,j(i)+1}$ is a simple root not in ℓ. If $\varepsilon \neq 0$, we argue exactly as above, and get identical results. If $\varepsilon = 0$, we still get $\langle\mu+2\rho_c+\frac{1}{2}\nu, g^+_{i,j(i)+1}\rangle_k \geq 1$. Suppose that equality holds. We cannot deduce that $g^-_{i,j(i)}$, say, is a root of ℓ; but suppose $\langle\mu+2\rho_c+\frac{1}{2}\nu, g^-_{i,j(i)}\rangle_k = 1$. In this case $g^-_{i,j(i)}$ is necessarily in $\Delta(\ell)$; for otherwise the preceding paragraph would imply that $g^+_{i,j(i)+1}$ is a root of ℓ, which we have assumed is not the case. Exactly as before, we deduce that μ and ν are singular with respect to the compact root $f_{j(i),j(i)+1}$. Similarly, if $\langle\mu+2\rho_c+\frac{1}{2}\nu, g^-_{i+1,j(i)+1}\rangle_k = 1$, then $j(i+1) = j(i) + 1$, $g^-_{i+1,j(i)+1}$ is a root of ℓ, and μ and ν are singular with respect to $e_{i,i+1}$.

Finally, it is very easy to check that $\langle\mu+2\rho_c+\frac{1}{2}\nu, \delta\rangle = 1$ if δ is a simple root in $\Delta(\ell)$. This proves that $\tilde{\lambda}$ is η^0-dominant.

Suppose then that $\mu - \sum_{j=1}^{J} \beta_{ij}$ gives rise to $J-1$ cohomology greater than μ, with all notation as usual.

By (4.19) and Lemma 4.20, $||\mu|| = ||\gamma||$, all α_{i_j} and β_{i_j} are sums of simple roots annihilating $\tilde{\lambda}$, and $B = \Delta_\tau^+(\mathfrak{g})$ for some θ-invariant $\tau \in W(\mathfrak{g},\mathfrak{h})$. Let $\tilde{\ell}_0 \supseteq \ell$ be the subalgebra of $\mathfrak{g}_0$ corresponding to the roots which annihilate $\tilde{\lambda}$. Then the simple roots in $\Delta(\tilde{\ell}_0)$ are certain noncompact simple roots $g_{ij}^{\pm}$ which are described above (namely those whose inner product with $\mu+2\rho_c+\frac{1}{2}\nu$ is 1.)

There is a natural notion of adjacency among the simple roots of $\tilde{\mathfrak{g}} = s\ell(n,C)$; two simple roots δ_1,δ_2 are adjacent iff $\delta_1+\delta_2$ is a root. Call the sequence $\delta_1 \ldots \delta_r$ of simple roots <u>contiguous</u> if the δ_i are distinct, and δ_i is adjacent to δ_{i+1} for $1 \leq i \leq r-1$. Then $\delta_1 + \ldots + \delta_r$ is a root iff (δ_i) is a contiguous sequence, and these are all the roots. If the δ_i are noncompact, $\delta_1 + \ldots + \delta_r$ is a compact root iff r is even. Thus the simple roots of $\tilde{\ell} \cap \mathfrak{k}$ are the $\delta_1+\delta_2$, where δ_1,δ_2 are adjacent simple roots of $\tilde{\ell}$. Such a pair is clearly of the form $(g_{i,j(i)}^-, g_{i,j(i)+1}^+)$ or $(g_{i,j(i)}^+, g_{i+1,j(i+1)}^-)$. By the remarks in the proof that $\tilde{\lambda}$ is $\mathfrak{n}_0$ dominant, exactly one root of such a pair is in $\Delta(\ell)$. The corresponding compact roots are $f_{j(i),j(i)+1}$ or $e_{i,i+1}$ respectively;

we saw that μ and ν are necessarily singular with respect to such roots. Since $\Delta_\sigma^+(\mathfrak{k}) \subseteq \Delta(\tilde{\mathfrak{l}} \cap \mathfrak{k})$, it follows that $\nu = \sigma^{-1}\nu$, $\mu = \sigma^{-1}\mu$; and since by definition the roots in $\Delta(\tilde{\mathfrak{l}})$ annihilate $\tilde{\lambda}$, $\sigma^{-1}(\tilde{\lambda}) = \tilde{\lambda}$.

Now $\tilde{\mathfrak{l}}$ decomposes as a direct sum of simple subalgebras $\tilde{\mathfrak{l}}^1,\ldots\tilde{\mathfrak{l}}^s$ (plus center) corresponding to maximal contiguous sets of simple roots in $\Delta(\tilde{\mathfrak{l}})$; we call these components <u>blocks</u>. Assume the blocks are ordered in accordance with the lexicographic ordering of $i\mathfrak{t}_0'$: if $g^{\pm}_{i_1 j_1} \in \Delta(\tilde{\mathfrak{l}}^u)$, $g^{\pm}_{i_2 j_2} \in \Delta(\tilde{\mathfrak{l}}^v)$, $u < v$, then $i_1 < i_2$. The first simple root in a fixed block is either g^+_{ij} (type I) or g^-_{ij} (type II). The key observation is that, since $\Delta_\sigma^+(\mathfrak{k}) \subseteq \Delta(\tilde{\mathfrak{l}} \cap \mathfrak{k})$, and $\Delta_\tau^+(\mathfrak{g}) \subseteq \Delta(\tilde{\mathfrak{l}})$, both τ and σ respect the block decomposition. We write $\mathfrak{h}^u$ for the part of $\mathfrak{h}$ in $\tilde{\mathfrak{l}}^u$.

Recall (4.9) that $\gamma = \sigma^{-1}(\mu-\Sigma\beta_{ij}+\rho_c)-\rho_c$; thus

$$\begin{aligned}
\gamma &= \sigma^{-1}(\mu+2\rho_c+\tfrac{1}{2}\nu-\rho)-\sigma^{-1}(\rho_c+\tfrac{1}{2}\nu-\rho+\Sigma\beta_{i_j})-\rho_c \\
&= \sigma^{-1}(\tilde{\lambda})-\sigma^{-1}(\rho_c-\sigma\cdot\rho_c+\tfrac{1}{2}\nu-(\rho-\Sigma\beta_{i_j}))-2\rho_c \\
&= \tilde{\lambda}-\sigma^{-1}(\Sigma\alpha_{i_j}+\tfrac{1}{2}\nu-(\rho-\Sigma\beta_{i_j})-2\rho_c \qquad \text{(by 3.7)} \\
&= \mu+2\rho_c+\tfrac{1}{2}\nu-\rho-\tfrac{1}{2}\sigma^{-1}\nu+\sigma^{-1}(\rho-(\Sigma\alpha_{i_j}+\Sigma\beta_{i_j}))-2\rho_c
\end{aligned}$$

$\gamma = \mu + \frac{1}{2}(\nu - \sigma^{-1}\nu) - \rho + \sigma^{-1}(\tau\cdot\rho)$ (by 4.12)

(4.21) $\gamma = \mu + \frac{1}{2}(\nu - \sigma^{-1}\nu) - (\rho - (\sigma^{-1}\tau)\cdot\rho)$

(Here we used only the fact that the roots in $\Delta^{+}_{\sigma}(\mathfrak{k})$ annihilate $\tilde{\lambda}$; accordingly we may apply 4.21 in later computations.) In the present case, of course, $\nu = \sigma^{-1}\nu$, so the middle term may be ignored. We claim that for each u, $\gamma = \mu$ on $\mathfrak{h}^u$, and length $(\tau|\mathfrak{h}^u)$ is even. The claim is proved by induction on u; suppose it is known for $u' < u$. Consider first the type II case, so that the first simple root in $\Delta(\tilde{\ell}^u)$ is $g^{-}_{i_0,j(i_0)}$. From the proof that $\tilde{\lambda}$ is $\mathfrak{n}^0$ dominant, we deduce that $g^{-}_{i_0,j(i_0)} \in \Delta(\ell)$, and that the simple roots of $\tilde{\ell}^u$ are $g^{-}_{i_0,j(i_0)}$, $g^{+}_{i_0,j(i_0)+1}$, $g^{-}_{i_0+1,j(i_0)+1}$, $g^{+}_{i_0+1,j(i_0)+2}$, etc.; every second root, i.e. each $g^{-}_{i,j(i)}$, is in $\Delta(\ell)$. Since $\langle\rho,\delta\rangle_{\mathfrak{k}} = 1$ for each simple root δ, it follows that the $\tilde{\ell}^u$ part of ρ is

$$\begin{array}{cc} (i_0 \text{ place}) & j(i_0)\text{place} \\ \downarrow & \downarrow \\ {[(\ldots\ n-1,\ n-3,\ n-5\ \ldots)} & (\ldots\ n,\ n-2,\ n-4\ \ldots)] \end{array}$$

Suppose that $\mu \neq \gamma$ on the i coordinates of $\tilde{\ell}^u$. By induction, $\mu = \gamma$ on the earlier i coordinates; since

$\gamma \succ \mu$ in the lexicographic ordering, we deduce that there is an $i_1 \geq i_0$ so that $\gamma_i = \mu_i$ for $i < i_1$, and $\gamma_{i_1} > \mu_{i_1}$. Since μ is annihilated by the compact roots in $\mathfrak{l}$, the sequence (μ_i) is constant on each block; similarly for (μ_j). On the other hand, γ is dominant, so (γ_i) is decreasing; it follows that $i_1 = i_0$ is the leading i coordinate of $\tilde{\mathfrak{l}}^u$. Recall that $W(\tilde{\mathfrak{l}}^u, \mathfrak{h}^u)$ acts by permuting the coordinates. We deduce from (4.21) that the $\sigma^{-1}\tau(i_0)$ coordinate of ρ is greater than the i_0 coordinate of ρ, which is possible only if $\sigma^{-1}\tau(i_0) = j(i_0)$. Suppose that for some i in the $\mathfrak{l}^u$ block, $\tau(i) = j_0$. $g^-_{i,j(i)}$ is a simple root of $\mathfrak{l}$, and thus cannot occur in $\Delta^+_\tau(\mathfrak{g}) \subseteq \Delta(\mathfrak{n})$; but

$$\begin{aligned} \langle(\tau^{-1})\cdot g^-_{i,j(i)}, \rho\rangle_{\mathfrak{k}} &= \langle g^-_{i,j(i)}, \tau\cdot\rho\rangle \\ &= \langle -\varepsilon^1_i + \varepsilon^2_j, \tau\cdot\rho\rangle \\ &= -\rho_{\tau(i)} + \rho_{\tau(j(i))} \\ &= -n + (n-k) < 0 \end{aligned}$$

by our description of ρ. This proves $g^-_{i,j(i)} \in \Delta^+_\tau(\mathfrak{g})$, a contradiction. So $j(i_0) = \tau(j_2)$ for some $j_2 = j(i_2)$. Consider the last j coordinate j_3 of the block $\tilde{\mathfrak{l}}^u$. Clearly $\rho_{\sigma^{-1}\tau(j_3)} \geq \rho_{j_3} - 1$; since γ is dominant, (4.21) implies that $\rho_{\sigma^{-1}\tau(j)} \geq \rho_j - 1$ for all j in the block $\tilde{\ell}^u$. In particular, $\sigma^{-1}\tau(j(i_0)) = j(i_0)$ or i_0. Since $\sigma^{-1}\tau(i_0) = j(i_0)$ we must have $\sigma^{-1}\tau(j(i_0)) = i_0$, and thus (by the dominance of γ again) $\sigma^{-1}\tau(j) = i(j)$ for all j. Setting $j = j_2$, we get $i(j_2) = \sigma^{-1}\tau(j_2) = \sigma^{-1}(j_0) = j_4$ since $\sigma \in W_K$ which preserves the i and j coordinates. Since no i coordinate can equal a j coordinate, this is a contradiction; thus $\mu = \gamma$ on the i coordinates of $\mathfrak{h}^u$. By (4.21), $\rho_i = \rho_{\sigma^{-1}\tau(i)}$ for each i coordinate in $\mathfrak{h}^u$. It follows that $\sigma^{-1}\tau|\tilde{\mathfrak{h}}^u$ preserves the i coordinates, and hence also the j coordinates. Every such permutation is in $W_K(\tilde{\ell}^u)$; since $\sigma \in W_K$, $\tau|\mathfrak{h}^u \in W_K(\tilde{\ell}^u)$. The compact roots in $\Delta^+_\tau(\mathfrak{g})$ are $\{\alpha_{i_j}\} = \Delta^+_\sigma(\mathfrak{k})$; so in fact $\tau = \sigma$ on $\mathfrak{h}^u$. By (4.21) again, $\mu = \gamma$ on $\mathfrak{h}^u$. The proof that length $(\tau|\mathfrak{h}^u)$ is even proceeds just as in the spherical case. The argument for the type I case proceeds along the same lines; the assumption that $\mu \neq \gamma$ on the i coordinates in $\mathfrak{h}^u$ leads almost immediately to a contradiction. We omit the details.

Since length $(\tau|\mathfrak{h}^u)$ is even for all u, length (τ) is even. (Actually, one can see fairly easily that we have deduced that $|\{\alpha_{i_j}\}| = |\{\beta_{i_j}\}|$. This will be needed later on for the program described in the following paragraph.) But length $(\tau) = 2J-1$; a contradiction. So no such γ can exist, and μ is indeed $\mathfrak{n}$ minimal. This completes the proof of 4.6 (d).

Although it would be difficult to formulate a general lemma, we will need the arguments above repeatedly. What happens is that the more complicated algebras contain assorted subalgebras isomorphic to su(p,q). It will often be possible to deal with these separately; this will generally be done with a reference to the "su(p,q) theory." We have tried in such cases to arrange the notation so as to make the argument clear, and details will be left to the reader. The next case illustrates the technique.

Proof of 4.6 (d) Here $\mathfrak{g}_0 = sp(p,q)$, $p \leq q$. We take for T the usual Cartan subgroup of $K = SP(p) \times SP(q) \subseteq SP(p,q)$; T consists of diagonal matrices. This gives an identification of $i\mathfrak{t}_0'$ with $\mathbb{R}^p \oplus \mathbb{R}^q$ with respect to the usual inner product. Again we refer to $\mathbb{R}^p$ as the "i coordinates," with basis $\{\varepsilon_i^1\}$, and $\mathbb{R}^q$ as the "j coordinates" with basis $\{\varepsilon_j^2\}$. $\mathfrak{g}_0$ is equal rank, so $\rho^{\mathfrak{t}} = 0$. The compact roots are

$e^{\pm\pm}_{ii'} = \pm \varepsilon^1_i \pm \varepsilon^1_{i'}$ $(i < i')$, $c^{\pm}_i = \pm 2\varepsilon^1_i$, $f^{\pm\pm}_{jj'} = \pm \varepsilon^2_j \pm \varepsilon^2_{j'}$ $(j < j')$ and $d^{\pm}_j = \pm \varepsilon^2_j$. The noncompact roots are $g^{\pm}_{ij} = \pm(\varepsilon^1_i - \varepsilon^2_j)$, $h^{\pm}_{ij} = \pm(\varepsilon^1_i + \varepsilon^2_j)$.

Order $it'_0 \cong \mathbb{R}^p \oplus \mathbb{R}^q$ lexicographically. The corresponding positive compact roots are $e^{+\pm}_{ii'}$, c^+_i, $f^{+\pm}_{jj'}$, and d^+_j (recall $i < i'$, $j < j'$). One checks easily that the simple compact roots are $e^{+-}_{i,i+1}$, c^+_p, $f^{+-}_{j,j+1}$, and d^+_q, and that $2\rho_c = [(2p, 2p-2, \ldots 2), (2q, 2q-2, \ldots 2)]$. Fix a k-type $\mu = [(\mu^1_1, \ldots \mu^1_p)(\mu^2_1, \ldots \mu^2_q)]$. Since μ is dominant integral, $\mu^1_i \in \mathbb{Z}$, $\mu^1_1 \geq \mu^1_2 \geq \ldots \geq \mu^1_p \geq 0$; similarly for the j coordinates. Set $\mu + 2\rho_c = a = [(a^1_1, \ldots a^1_p)(a^2_1, \ldots a^2_q)]$. Define (formally) $a^1_0 = a^2_0 = \infty$, $a^1_{p+1} = a^2_{q+1} = 0$; and for $1 \leq i \leq p$, define $j(i)$, $0 \leq j(i) \leq q$, so that $a^2_{j(i)} > a^1_i \geq a^2_{j(i)+1}$. Call the index i <u>upper transitional</u> if $a^1_{i-1} \geq a^2_{j(i)}$, and <u>lower transitional</u> if $a^2_{j(i)+1} > a^1_{i+1}$; <u>upper critical</u> if $0 < a^2_{j(i)} - a^1_i \leq 1$, and <u>lower critical</u> if $0 \leq a^1_i - a^2_{j(i)+1} < 1$. If $\mathfrak{n}^0$ is associated to a, then $\Delta(\mathfrak{n}^o \cap \mathfrak{g}) = \{h^+_{ij}\} \cup \{g^+_{ij} \mid a^1_i - a^2_j \geq 0\}$ $\cup \{g^-_{ij} \mid a^1_i - a^2_j < 0\}$. It can be deduced that the simple roots of Δ^+ are

$\{e^{+-}_{i,i+1} \mid i$ is not lower transitional and $i+1$ is not upper transitional$\}\cup$ $\{f^{+-}_{j,j+1} \mid j \neq j(i)$ for any $i\} \cup \{c^+_p$ or d^+_q according as $a_p < a_q$ or $a_p \geq a_q\} \cup \{g^-_{i,j(i)} \mid i$ is upper transitional$\}$ $\{g^+_{i,j(i)+1} \mid i$ is lower transitional$\}$. By the integrality of a, i is upper critical iff $a^2_{j(i)} - a^1_i = 1$, and lower critical iff $a^1_i - a^2_{j(i)+1} = 0$. Just as for su(p,q), every critical index is transitional. Let $\mathfrak{b}$ be the parabolic corresponding to the simple roots $\{g^-_{i,j(i)} \mid i$ is upper critical$\} \cup \{g^+_{i,j(i)+1} \mid i$ lower critical$\}$. As for su(p,q), $\mathfrak{l}$ is a product of copies of sl(2,R), and $\mu - 2\rho(\mathfrak{n} \cap \mathfrak{g})$ is principal series minimal. It remains to check that μ is $\mathfrak{n}$-minimal.

Set $v = \sum_{\substack{i \text{ lower} \\ \text{critical}}} g^+_{i,j(i)+1}$. We claim that $\tilde{\lambda} = \mu + 2\rho_c - \rho + \frac{1}{2}v$ is $\mathfrak{n}^0$ dominant. For the various simple roots of length 2 this is proved as for su(p,q). Assume for definiteness that the simple root of length 4 is d^+_q. $(\rho_c)^2_q = 1$, so $a^2_q = (\mu + 2\rho_c)^2_q \geq 2$. No coordinate of $\frac{1}{2}v$ is less than $-\frac{1}{2}$, so

$$\begin{aligned}\langle \mu + 2\rho_c + \tfrac{1}{2}\nu,\ d_q^+\rangle &= \langle \mu + 2\rho_c + \tfrac{1}{2}\nu,\ 2\varepsilon_q^2\rangle \\ &= 2(\mu + 2\rho_c + \tfrac{1}{2}\nu)_q^2 \\ &\geq 2(2 - \tfrac{1}{2}) = 3 > 2 = \frac{\langle d_q^+, d_q^+\rangle}{2} .\end{aligned}$$

It follows that $\tilde{\lambda}$ is actually nonsingular with respect to d_q^+. Let $\tilde{\ell} \supseteq \mathfrak{h}$ be the subalgebra corresponding to the roots annihilating $\tilde{\lambda}$; since the simple root of length 4 is not in $\tilde{\ell}$, one can now apply the su(p,q) theory to deduce that μ is $\mathfrak{n}$-minimal. This completes the proof of 4.6 (d).

<u>Proof of 4.6 (e)</u> Here $\mathfrak{g}_0 = sp(n,\mathbb{R})$. We use the description of $\mathfrak{t}$ given in section 5; thus $i\mathfrak{t}_0'$ is identified with $\mathbb{R}^n$, with the usual inner product for $\langle\ ,\ \rangle_{\mathfrak{k}}$. $\mathfrak{g}_0$ is equal rank, so $\mathfrak{p}^{\mathfrak{t}} = 0$. Let $\{\varepsilon_i\}$ be the usual basis of $i\mathfrak{t}_0' \cong \mathbb{R}^n$. We order $\mathbb{R}^n$ lexicographically. Then the compact roots are $e_{ij} = \varepsilon_i - \varepsilon_j$, which is positive for $i < j$. The noncompact roots are $g_{ij}^{\pm} = \pm(\varepsilon_i + \varepsilon_j)$, and $d_i^{\pm} = \pm 2\varepsilon_i$. Fix a $\mathfrak{k}$-type $\mu = (\mu_1, \ldots \mu_n)$. Since μ is dominant integral, (μ_i) is decreasing, and $\mu_i \equiv \varepsilon \bmod \mathbb{Z}$ for some fixed ε; say $0 \leq \varepsilon < 1$. One computes easily that

$2\rho_c = (n-1, n-3, \ldots -(n-1))$. Set $a = \mu + 2\rho_c = (a_1, \ldots a_n)$. Let $\mathfrak{b}^0$ be the Borel subalgebra associated to a; then

$$\Delta(\mathfrak{n}^o \cap \mathfrak{p}) = \{g_{ij}^{\operatorname{sgn}(a_i+a_j)}\} \cup \{d_i^{\operatorname{sgn} a_i}\}$$

(as usual sgn 0 = +). We want to describe the simple roots of Δ^+. Define r so that $a_i \geq 0$ for $i \leq r$, $a_j < 0$ for $j > r$; we call the coordinates up to r the <u>i coordinates</u>, and those after r the <u>j coordinates</u>. Define $a_{n+1} = -\infty$. For $1 \leq i \leq r$, define $j(i)$ so that

$$-a_{j(i)} \leq a_i < -a_{j(i)+1}.$$

(Clearly we want $r < j(i) \leq n$. With this restriction $j(i)$ may not exist, since we may have $-a_{r+1} > a_i$. This will cause no difficulties, however, so we will not correct the notation.) The index i is called <u>upper transitional</u> if $-a_{j(i)} \leq a_{i-1}$; <u>lower transitional</u> if $a_{j(i)} > a_{i+1}$; <u>upper critical</u> if $0 < -a_i - a_{j(i)+1} \leq 1$; and <u>lower critical</u> if $0 \leq a_i + a_{j(i)} < 1$. Set $\delta = 2\varepsilon$ if $2\varepsilon < 1$, and $\delta = 2\varepsilon - 1$ if $2\varepsilon \geq 1$. Then $0 \leq \delta < 1$; by the integrality of a, one sees that i is upper critical iff $-a_i - a_{j(i)+1} = 1 - \delta$, and lower critical iff $a_i + a_{j(i)} = \delta$. Just as for su(p,q), a critical index is necessarily transitional.

Define the permutation π of $\{1,\ldots n\}$ so that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq \ldots \geq |a_{\pi(n)}|$. If $|a_i| = |a_j|$, $i \neq j$, then (since (a_i) is strictly decreasing) $a_i = -a_j \neq 0$; and we list the positive element first. In terms of π, it is easy to see that the simple roots of Δ^+ are $d_{\pi(n)}^{\operatorname{sgn} a_{\pi(n)}}$, and the various

$$\eta_i = [\operatorname{sgn} a_{\pi(i)}]\varepsilon_{\pi(i)} - [\operatorname{sgn} a_{\pi(i+1)}]\varepsilon_{\pi(i+1)} \qquad (1 \leq i \leq n-1).$$

Notice that $\pi(n) = r$ or $r+1$, according as $a_r < -a_{r+1}$, or $a_r \geq -a_{r+1}$. Thus the simple roots of Δ^+ are $\{e_{i,i+1} | 1 \leq i < r$, i is not lower transitional, $i+1$ is not upper transitional$\} \cup$
$\{e_{j,j+1} | r < j < n$, j is not $j(i)$ for any $i\} \cup$
$\{g^+_{i,j(i)} | i$ is lower transitional$\} \cup$
$\{g^-_{i,j(i)+1} | i$ is upper transitional$\} \cup \{d_\pi^{\operatorname{sgn} a_\pi(n)}\}$.

Next we must define ℓ. Choose s as small as possible so that $0 \leq a_{\pi(s)} < n-s+1$, or $0 < -a_{\pi(s)} \leq n-s+1$. If these are not satisfied for any s, s is undefined, and the corresponding set of simple roots in ℓ (to be defined) is empty. Let $\mathfrak{b} \supseteq \mathfrak{b}^o$ correspond to the simple roots

$\{\eta_i,\ s \leq i < n$, and $d_{\pi(a)}^{\operatorname{sgn} a_{\pi(n)}}$ (if s is defined)$\}$

$\cup \{g^-_{i,j(i)+1} | i$ is upper critical$\} \cup \{g^+_{i,j(i)} | i$ is lower critical$\}$.

(The first set is just the simple roots supported on the coordinates $\pi(s)\ldots\pi(n)$.) Let $g^-_{i,j(i)+1} \in \Delta^+(\ell)$ be simple, so that i is upper critical; say $g^-_{i,j(i)+1} = \eta_k$. Suppose $k+1 \geq s$; then we claim that actually $k \geq s$, so that η_k is supported on $\pi(s)\ldots\pi(n)$. If not, then $k+1 = s$, and so $0 \leq a_i < n - s + 1$; since a is integral, this forces $\varepsilon \leq a_i \leq n - s + \varepsilon$. Now i is upper critical, i.e. $a_i = -a_{j(i)+1} - 1 + \delta$; so

$$\varepsilon + (1-\delta) \leq -a_{j(i)+1} \leq n - s + \varepsilon + (1-\delta).$$

Since $0 \leq \varepsilon, \delta < 1$, $\varepsilon + 1 - \delta > 0$, and $\varepsilon - 1 - \delta < 0$; thus

$$0 < -a_{j(i)+1} \leq n - (s-1) + 1 + (\varepsilon-1-\delta)$$

$$< n - (s-1) + 1.$$

This contradicts the definition of s. A similar argument may be given for simple roots $g^+_{i,j(i)} \in \Delta^+(\ell)$. We have shown that the simple roots of ℓ may be divided into two subsets: P^1, consisting of those supported on $\pi(1)\ldots\pi(s-1)$; and P^2, consisting of those supported on $\pi(s)\ldots\pi(n)$. Clearly these two sets are strongly orthogonal to each other. Arguing as for su(p,q), one sees that the first set is pairwise strongly orthogonal; the corresponding ideal of $\mathfrak{l}$ is a product of copies of $\mathfrak{sl}(2,\mathbb{R})$, and is therefore split. The ideal corresponding to P^2 is clearly just $sp(n-s+1,\mathbb{R})$, which is of course split; the strongly orthogonal spanning set is $\{d_{\pi(i)}^{\operatorname{sgn} a_{\pi(i)}}\}_{i=s}^{n}$. Thus $\mathfrak{l}$ is split. (The slightly complicated nature of $\mathfrak{l}$ reflects the slightly complicated set of conjugacy classes of Cartan subalgebras of $sp(n,R)$.)

Next we must show that $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal for $\mathfrak{l}$. On the P^1 ideal this proceeds as for su(p,q); we omit the details. The P^2 ideal requires a little more work. Since $\Lambda^{R+S}\mathfrak{n}$ is a one dimensional $\mathfrak{l}$-module of weight $2\rho(\mathfrak{n}\cap\mathfrak{p}) + 2\rho(\mathfrak{n}\cap\mathfrak{k})$, it is equivalent to consider the weight $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p}) + 2\rho(\mathfrak{n}\cap\mathfrak{p}) + 2\rho(\mathfrak{n}\cap\mathfrak{k}) = \mu+2\rho(\mathfrak{n}\cap\mathfrak{k}) = \mu+2\rho_c - 2\rho(\mathfrak{l}\cap\mathfrak{n}^\circ\cap\mathfrak{k}) = a - 2\rho(\mathfrak{l}\cap\mathfrak{n}^\circ\cap\mathfrak{k}) = b$, say. To simplify the notation we may assume $s = 1$; thus $2\rho(\mathfrak{l}\cap\mathfrak{n}^\circ\cap\mathfrak{k}) = 2\rho_c = (n-1,\ldots-(n-1))$. Since a is decreasing, $\pi(1) = 1$ or n; assume for definiteness that $\pi(1) = n$. By the definition of s, $0 < -a_n \leq n-1+1 = n$. So $b_n = a_n + (n-1) \geq -1$. On the other hand, since $\pi(1) = n$, $-a_n > |a_i|$ for all i; in particular,

$n \geq -a_n > |a_1|$, so that $a_1 < n$. Thus $b_1 = a_1 - (n-1) < n - (n-1) = 1$. We know from the general theory that b is dominant integral for $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{k}$; so since $1 > b_1$, $b_n \geq -1$, b is of the form $(\alpha,\ldots\alpha, \alpha-1,\ldots\alpha-1)$, with $1 > \alpha \geq 0$. From Table 5.8, we see that b is principal series minimal. Thus $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal.

It remains to check that μ is $\mathfrak{n}$ minimal. It is convenient to treat several cases separately.

<u>Case I</u> <u>s undefined</u>; i.e. $a_{\pi(i)} \geq n - i + 1$ or $-a_{\pi(i)} > n - i + 1$, all i. Set

$$v = \sum_{\substack{i \text{ lower} \\ \text{critical}}} (1-\delta)g^+_{i,j(i)+1} + \sum_{\substack{i \text{ upper} \\ \text{critical}}} \delta\, g^-_{i,j(i)}, \quad \tilde{\lambda} = a - \rho + \tfrac{1}{2}v$$

By the $su(p,q)$ theory, $\tilde{\lambda}$ is dominant with respect to all the simple roots η_i. Necessarily $a_{\pi(n)} \geq 1$, or $a_{\pi(n)} < -1$. For definiteness we assume $a_{\pi(n)} \geq 1$; the other case is similar.

a) <u>$a_{\pi(n-1)} > 0$.</u> By the definition of π, and the fact that a decreases by at least 2, this forces

$$\langle\eta_{n-1}, a\rangle = a_{\pi(n-1)} - a_{\pi(n)} \geq 2.$$

Thus η_{n-1} is not critical, so the $\pi(n)$ coordinate of v is zero; and the $\pi(n-1)$ coordinate is at most 1 in absolute value. Hence

$$<\eta_{n-1},\tilde{\lambda}> = <\eta_{n-1},a-\rho+\frac{1}{2}v> \geq 2 - 1 + \frac{1}{2} = \frac{1}{2} > 0$$

(we have used $<\eta_{n-1},\rho> = 1$, since η_{n-1} has length 2.) Also $<d^{+}_{\pi(n)},\tilde{\lambda}> = <d^{+}_{\pi(n)},a-\rho+\frac{1}{2}v> \geq 2 - 2 = 0$. So $\tilde{\lambda}$ is dominant. Let $\tilde{\ell} \supset \ell$ be the subalgebra on which $\tilde{\lambda}$ is singular. Just as for $su(p,q)$, the η-minimality of μ reduces to a problem on $\tilde{\ell}$. But we have shown that $\eta_{n-1} \not\in \Delta(\tilde{\ell})$; so $\Delta(\tilde{\ell})$ decomposes into the roots supported on $(\pi(1)\ldots\pi(n-1))$, and those supported on $\pi(n)$. The first piece is handled by the $su(p,q)$ theory, and the second by the $sp(1,R) \tilde{=} su(1,1)$ theory; one sees that μ is in fact η-minimal.

b) $a_{\pi(n-1)} \leq 0$, $\eta_{n-1} = g^{-}_{\pi(n),\pi(n-1)}$ is upper critical.

Then $-a_{\pi(n-1)} > n - (n-1) + 1 = 2$, and so by integrality, $-a_{\pi(n-1)} \geq 3 - \varepsilon$. Also $-a_{\pi(n-1)} - a_{\pi(n)} = 1 - \delta$, so $a_{\pi(n)} \geq 3 - \varepsilon - (1-\delta) = 2 - \varepsilon + \delta$. In the $\pi(n-1),\pi(n)$ coordinates, $\frac{1}{2}v$ is $\frac{1}{2}\delta\, g^{-}_{\pi(n),\pi(n-1)} = (-\frac{1}{2}\delta,-\frac{1}{2}\delta)$; so the $\pi(n)$ coordinate of $\tilde{\lambda} = a - \rho + \frac{1}{2}v$ is

$$a_{\pi(n)} - \rho_{\pi(n)} + \frac{1}{2}v_{\pi(n)} = a_{\pi(n)} - 1 - \frac{1}{2}\delta$$

$$\geq 2 - \varepsilon + \delta - 1 - \frac{1}{2}\delta$$

$$= (1-\varepsilon) + \frac{1}{2}\delta > 0$$

(Here we have used $\rho_{\pi(n)} = \frac{1}{2}\langle d^+_{\pi(n)}, \rho\rangle = \dfrac{2\langle d^+_{\pi(n)}, \rho\rangle}{\langle d^+_{\pi(n)}, d^+_{\pi(n)}\rangle} = 1$.)

Hence $\tilde{\lambda}$ is dominant and non-singular with respect to $d^+_{\pi(n)}$. If we let $\tilde{\ell}$ correspond to the roots annihilated by $\tilde{\lambda}$, then the su(p,q) theory applies to $\tilde{\ell}$, and μ is π-minimal.

c) <u>$a_{\pi(n-1)} \leq 0$, $\eta_{n-1} = g^-_{\pi(n),\pi(n-1)}$ is not upper critical.</u>

Then the $\pi(n)$ coordinate of v is zero, and it is immediate that $\tilde{\lambda}$ is dominant with respect to $d^+_{\pi(n)}$. If it is non-singular, we can apply the su(p,q) theory as in case b). If it is singular, then $0 = \tilde{\lambda}_{\pi(n)} = a_{\pi(n)} - \rho_{\pi(n)} = a_{\pi(n)} - 1$; i.e. $a_{\pi(n)} = 1$. Hence $a_{\pi(n-1)}$ is a negative integer. Since s is undefined, $-a_{\pi(n-1)} > n - (n-1) + 1 = 2$, i.e. $-a_{\pi(n-1)} \geq 3$. The $\pi(n-1)$ coordinate of v is at most 1 in absolute value, so

$$\langle \eta_{n-1}, \tilde{\lambda}\rangle = \langle g^-_{\pi(n-1),\pi(n)}, \tilde{\lambda}\rangle \geq -a_{\pi(n-1)} - a_{\pi(n)} - 1 - \frac{1}{2}$$

$$\geq 3 - 1 - \frac{3}{2} = \frac{1}{2} > 0.$$

So $\tilde{\lambda}$ is nonsingular with respect to η_{n-1}, and we can argue as in case a).

Case II s is defined. Set

$$v_1 = \sum_{\substack{i \text{ lower critical} \\ g^+_{i,j(i)+1} \varepsilon P^1}} (1-\delta)\, g^+_{i,j(i)+1} + \sum_{\substack{i \text{ upper critical} \\ g^-_{i,j(i)} \varepsilon P^1}} \delta\, g^-_{i,j(i)}$$

$$v_2 = \begin{cases} 0 & \text{on coordinates } \pi(1)\ldots\pi(s-1) \\ 2(\rho - a) & \text{on } \pi(s)\ldots\pi(n). \end{cases}$$

It is not difficult to check that, since $\mu - 2\rho(\mathfrak{n}\cap\mathfrak{p})$ is principal series minimal, v_2 is of the form required by Lemma 4.20. Put $v = v_1 + v_2$, $\tilde{\lambda} = a - \rho + \frac{1}{2}v$. By the su(p,q) theory, $\tilde{\lambda}$ is dominant with respect to $\eta_1 \cdots \eta_{s-2}$; and since obviously $\tilde{\lambda} = 0$ on $\pi(s)\ldots\pi(n)$ coordinates, $\tilde{\lambda}$ is dominant with respect to $\eta_s \cdots \eta_{n-1}$, and $d_{\pi(n)}^{\operatorname{sgn} a_{\pi(n)}}$. Suppose that in fact

$$* \qquad \langle \eta_{s-1}, \tilde{\lambda} \rangle > 0.$$

Let $\tilde{\ell}$ correspond to the roots annihilated by $\tilde{\lambda}$; then $\tilde{\ell} = \tilde{\ell}^1 + \tilde{\ell}^2$, with $\tilde{\ell}^1$ corresponding to roots supported on $\pi(1)\ldots\pi(s-1)$, and $\tilde{\ell}^2$ to roots supported on $\pi(s)\ldots\pi(n)$. The su(p,q) theory applies to $\tilde{\ell}^1$; and

$\tilde{\ell}^2 \leq \ell$, so η-minimality is trivial there. It follows that μ is η-minimal. So it is enough to show (*). Assume for definiteness that $a_{\pi(s-1)} \geq 0$. By the definition of s, $a_{\pi(s-1)} \geq n-(s-1)+1$; by integrality, $a_{\pi(s-1)} \geq n-(s-1)+1+\varepsilon$. We have already seen that η_{s-1} is not a critical root. So if $v_{\pi(s-1)} \neq 0$, necessarily η_{s-2} is critical; since $a_{\pi(s-1)} \geq 0$, this forces $-a_{\pi(s-2)} - a_{\pi(s-1)} = 1-\delta$, $v_{\pi(s-1)} = -\delta$. Finally, it is easy to see that $\rho_{\pi(s-1)} = n-(s-1)+1$. Hence

$$\langle\eta_{s-1},\tilde{\lambda}\rangle = \langle\varepsilon_{\pi(s-1)} \pm \varepsilon_{\pi(s)}, a-\rho+\tfrac{1}{2}v\rangle$$

$$= a_{\pi(s-1)}-(n-(s-1)+1)+\tfrac{1}{2}v_{\pi(s-1)} \quad (\text{since } \tilde{\lambda}_{\pi(s)}=0)$$

$$\geq (n-(s-1)+1)+\varepsilon-(n-(s-1)+1)-\tfrac{1}{2}\delta = \varepsilon-\tfrac{1}{2}\delta$$

Now $\delta = 2\varepsilon$ or $2\varepsilon-1$; so $\varepsilon - \frac{1}{2}\delta \geq 0$. Hence $\langle\eta_{s-1},\tilde{\lambda}\rangle \geq 0$. Suppose equality holds. Then necessarily $a_{\pi(s-1)} = n-(s-1)+1+\varepsilon$; and if $\delta \neq 0$, $v_{\pi(s-1)} \neq 0$, so that

$$-a_{\pi(s-2)} = a_{\pi(s-1)} + 1 - \delta$$

$$= (n-(s-1)+1)+\varepsilon+1-\delta$$

$$= (n-(s-2)+1)+\varepsilon-\delta$$

Because of the equality, we also have $\varepsilon = \frac{1}{2}\delta$; so

$$= (n - (s-2) + 1) - \frac{1}{2}\delta \leq n - (s-2) + 1,$$

which contradicts the definition of s. So in fact $\delta = 0$, $\varepsilon = 0$, and $v_{\pi(s-1)} = 0$. (If we had considered instead $a_{\pi(s-1)} < 0$, we would have gotten $\langle\eta_{s-1},\tilde{\lambda}\rangle > 0$ in general.)

We must now investigate this last case, i.e. $\langle\eta_{s-1},\tilde{\lambda}\rangle = 0$. We have seen that $a_{\pi(s-1)} = n - (s-1) + 1$, and that η_{s-2} is not a critical root. It follows easily that $\langle\eta_{s-2},a\rangle \geq 2$, and hence that $\tilde{\lambda}$ is non-singular with respect to η_{s-2}. Thus we can define $= \tilde{\ell}^1 + \tilde{\ell}^2$ as usual, and treat $\tilde{\ell}^1$ by the su(p,q) theory; and we are left with $\tilde{\ell}^2$, which corresponds to roots supported on $\pi(s-1)\ldots\pi(n)$. To simplify the notation, we may assume $s = 1$. Then $a_{\pi(1)} = n$ forces $\mu = (1,1\ldots1,0\ldots0)$. It is not difficult to see that $\Delta^+(\mathfrak{n}) = \{(\varepsilon_1 \pm \varepsilon_i) | 2 \leq i \leq n\} \cup \{2\varepsilon_1\}$.

Let γ, σ, τ, $\{\beta_{i_j}\}$, $\{\alpha_{i_j}\}$ be as usual. Recall that $\Delta^+_\sigma(\mathfrak{k}) = \{\alpha_{i_j}\} \subseteq \Delta^+(\mathfrak{n} \cap \mathfrak{k}) = \{(\varepsilon_1 - \varepsilon_j)\}$. Now the compact Weyl group acts by permutation on the ε_i. It is easy to deduce that

$$\sigma(b_1 \ldots b_n) = (b_J b_1 \ldots b_{J-1} b_{J+1} \ldots b_n);$$

and in this case $\Delta_\sigma^+(\mathfrak{k}) = \{\varepsilon_1 - \varepsilon_i | 2 \leq i \leq J\}$. On the other hand, by (4.9), (recall $\sigma\rho_c - \rho_c = -\Sigma\alpha_{i_J}$)

$$\mu - \beta_{i_1} \ldots - \beta_{i_J} = \sigma \cdot \gamma + \sigma\rho_c - \rho_c$$

$$= (\gamma_J, \gamma_1, \ldots \gamma_{J-1}, \gamma_{J+1} \ldots \gamma_n) + (-(J-1), 1, \ldots 1, 0 \ldots 0)$$

$$(*) \quad \mu - \Sigma\beta_{i_j} = (\gamma_J - (J-1), \gamma_1 + 1 \ldots \gamma_{J-1} + 1, \gamma_{J+1}, \ldots \gamma_n)$$

Hence $\gamma_1 + 1 = \mu_2 - (c)$; here $c = 0$ or 1 according as $(\varepsilon_1 + \varepsilon_2)$ is or is not a β_{i_j}. (If $J = 1$, this fails; but an easy special argument shows $\gamma_1 = 0$ or -1). Hence $\gamma_1 = 0, -1,$ or -2. Since $||\gamma|| = ||\mu||$, and $\mu_1 = 1$, this implies that $\gamma < \mu$, and hence γ cannot occur. So μ is $\mathfrak{n}$-minimal in every case; which completes the proof of 4.6 (e).

<u>4.6 (f)</u> is quite similar, but much easier: the root system of $SO^*(2n)$ is just that of $SP(n, \mathbb{R})$, except that there are no long roots. One can copy the preceding argument, omitting all reference to "$\tilde{\ell}^2$".

We have now dealt with all the classical groups except the $SO(p,q)$. These are by far the most complicated. To begin with, we need a more delicate technique for proving Conjecture 4.2. Suppose $\mathfrak{b}$ is chosen so that 4.2 (ii-v) hold. Suppose furthermore that there is a θ-invariant parabolic $\tilde{\mathfrak{b}} \supseteq \mathfrak{b}$, $\tilde{\mathfrak{b}} = \tilde{\mathfrak{l}} + \tilde{\mathfrak{n}}$, with the following properties.

(4.22) a) the $\mathfrak{k}$-type μ is $\tilde{\mathfrak{n}}$-minimal.

b) the $\tilde{\mathfrak{l}} \cap \mathfrak{k}$-type $\mu - 2\rho(\tilde{\mathfrak{n}} \cap \mathfrak{p})$ is <u>uniformly minimal</u> in $\tilde{\mathfrak{l}}$; i.e. it is the minimal $\tilde{\mathfrak{l}} \cap \mathfrak{k}$ type of every irreducible Harish-Chandra module for $\tilde{\mathfrak{l}}$ in which it occurs.

c) $\mu - 2\rho(\tilde{\mathfrak{n}} \cap \mathfrak{p})$ is $\mathfrak{n} \cap \tilde{\mathfrak{l}}$-minimal in $\tilde{\mathfrak{l}}$.

Then we claim that 4.2 (i) holds, i.e. that the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on X^{μ} factors through the homomorphism ξ^{μ}. Essentially this is left to the reader, modulo the following hints. Using b), c), Theorem 3.15, and Corollary 5.5, one can deduce that $\mu - 2\rho(\tilde{\mathfrak{n}} \cap \mathfrak{p})$ occurs with multiplicity zero or one in each irreducible Harish-Chandra module for $\tilde{\mathfrak{l}}$. By results of Lepowsky [18], it follows that

$U(\tilde{\mathfrak{l}})^{\tilde{\mathfrak{l}} \cap \mathfrak{k}} / I_{\mu - 2\rho(\tilde{\mathfrak{n}} \cap \mathfrak{p})}$ is abelian. The rest of the argument is quite formal: essentially one uses Theorem 3.15 to study $H^*(\mathfrak{n} \cap \tilde{\mathfrak{l}}, H^*(\tilde{\mathfrak{n}}, X))$ instead of $H^*(\mathfrak{n}, X)$.

<u>Proof of 4.6 (g)</u> We give details only for the simplest of the various possibilities of parity, namely $\mathfrak{g}_0 = so(2p+1,2q+1)$, $p \leq q$. If $p = 0$, then G has only one conjugacy class of Cartan subgroups, and we are done; so assume $p \geq 1$. $\mathfrak{k}$ is identified with $so(2p+1) \times so(2q+1)$ in the usual way; choosing a maximal torus in each factor as usual (i.e. with $\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$ blocks on the SO(2p+1) level) we get $i\mathfrak{t}_0' \cong \mathbb{R}^p \oplus \mathbb{R}^q$. This we order lexicographically; as a basis, choose $\{\varepsilon_i^1\}_{i=1}^p$ and $\{\varepsilon_j^2\}_{j=1}^q$ as usual. Since rank $(\mathfrak{g}_0) = [\frac{2p+1 + 2q+1}{2}] = p + q + 1$, necessarily $\dim \mathfrak{p}^t = 1$. One computes (more or less easily, depending on one's point of view) that the compact roots are $\pm\varepsilon_i^1 \pm \varepsilon_{i'}^1$, $\pm\varepsilon_j^2 \pm \varepsilon_{j'}^2$, $\pm\varepsilon_i^1$, and $\pm\varepsilon_j^2$: the noncompact roots are $\pm\varepsilon_i^1 \pm \varepsilon_j^2$, $\pm\varepsilon_i^1$, and $\pm\varepsilon_j^2$. So the complex roots are just the $\pm\varepsilon_i^1$ and $\pm\varepsilon_j^2$; since all roots of $so(2p+2q+2,\mathbb{C})$ have the same length, these have <u>complex</u> length two (i.e. if $\alpha \in \Delta$, $\alpha|_{\mathfrak{t}} = +\varepsilon_i^1$, then $\langle\alpha,\alpha\rangle = \langle\alpha,\alpha\rangle_{\mathfrak{k}} + \langle\alpha,\alpha\rangle_{\mathfrak{p}} = 2$). The positive compact roots are $\varepsilon_i^1 \pm \varepsilon_{i'}^1$, ε_i^1, $\varepsilon_j^2 \pm \varepsilon_{j'}^2$, ε_j^2 $(i < i', j < j')$. Hence $2\rho_c = (2p+1, 2p-1, \ldots 1)(2q+1, 2q-1, \ldots 1)$.

Fix a $\mathfrak{k}$-type $\mu = [(\mu_1^1 \ldots \mu_p^1), (\mu_1^2 \ldots \mu_q^2)]$, and set $a = \mu+2\rho_c$. Since μ is dominant, the coordinates of μ are non-negative and decreasing on each block. Since μ is integral, all μ_i^1 are integers, or all are $\equiv \frac{1}{2} \bmod \mathbb{Z}$; similarly for the μ_j^2. The coordinates of a are thus half integers ≥ 1, decreasing by at least two on each block.

We now arrange (a) in decreasing order: define a map $\pi : \{1,2,\ldots,p+q\} \to \{\varepsilon_i^1, \varepsilon_j^2\}$ so that (with obvious notation) $\ell < m \Rightarrow a_{\pi(\ell)} \geq a_{\pi(m)}$; equality should hold only if $\pi(\ell)$ is an i coordinate and $\pi(m)$ is a j coordinate. One computes easily that $\rho_{\pi(\ell)} = p + q - \ell + 1$, and that the simple roots are $\varepsilon_{\pi(\ell)} - \varepsilon_{\pi(\ell+1)}$, $\varepsilon_{\pi(p+q)}$.

Let $\delta \in [0,1)$ be defined so that $a_i^1 - a_j^2 \equiv \delta \bmod \mathbb{Z}$, all i, j; of course $\delta = 0$ or $\frac{1}{2}$. We leave the definition of critical roots to the reader.

Let s_0 be the largest integer such that

$$(*) \qquad a_{\pi(p+q-2s_0+1)} + a_{\pi(p+q-2s_0+2)} \leq 4s_0 - 1$$

It is not hard to verify that (*) must also hold with s_0 replaced by any $\ell \leq s_0$. On the other hand, $a_{\pi(p+q)} \geq 1$; and it follows easily from the value of $2\rho_c$ that $a_{\pi(p+q-2\ell)} \geq 2\ell + 1$, all $\ell \geq 0$. If the root

$\varepsilon_{\pi(p+q-2\ell-1)} - \varepsilon_{\pi(p+q-2\ell)}$ is compact, then

$a_{\pi(p+q-2\ell+1)} \geq 2\ell + 3$, and

$$a_{\pi(p+q-2(\ell+1)+1)} + a_{\pi(p+q-2(\ell+1)+2)} \geq 4(\ell+1); \text{ so } \ell+1 > s_0.$$

So for $\ell < s_0$, $\varepsilon_{\pi(p+q-2\ell-1)} - \varepsilon_{\pi(p+q-2\ell)}$ is noncompact.

By induction, $\pi\{p+q-2\ell+1,\ p+q-2\ell+2\} = \{\varepsilon^1_{p-\ell}, \varepsilon^2_{q-\ell}\}$ $(1 \leq \ell \leq s_0)$.

Furthermore, condition (*) forces $\mu^1_{p-\ell} + \mu^2_{q-\ell} \leq 1$.

We define $s = s_0$ <u>unless</u> $\pi(p+q-s_0+1) = \varepsilon^1_{p-s_0}$, $\mu_{\pi(p+q-s_0+1)} = 1$, and $\mu^2_{q-s_0-1} \geq 1$; in this case $s = s_0-1$. Set

P^1 = {critical roots supported on $\pi\{1,\ldots p+q-2s_0\}$}

P^2 = {roots supported on $\pi\{p+q-2s+1,\ldots p+q\}$}

We let $\mathfrak{p}$ be the parabolic corresponding to the simple roots in $P^1 \cup P^2$; then $\ell = \ell^1 + \ell^2$, according to $\Delta(\ell) = P^1 \cup P^2$. Clearly $\ell^2 \cong so(2s+1,2s+1)$, and ℓ^1 is a product of copies of $sl(2,R)$; so ℓ is split. By the $su(p,q)$ theory, $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})|_{\ell^1}$ is principal series minimal. On the ℓ^2 piece, we must check that $\mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$ satisfies Conjecture 4.2 (ii-iv). Since $\mathfrak{k} \cap [\ell^2,\ell^2] \cong so(2s+1)\times so(2s+1)$ is centerless, and $2\rho(\mathfrak{n}\cap\mathfrak{p})$ is the weight of a one-dimensional $\ell\cap\mathfrak{k}$-module, this amounts to considering

$[(\mu^1_{p-s},\ldots\mu^1_p),\ (\mu^2_{q-s},\ldots\mu^2_q)]$. We have seen that

$\mu^1_{p-\ell} + \mu^2_{q-\ell} \le 1$; so on these coordinates μ is

$[(0\ldots0)(1\ldots1,0\ldots0)]$, $[(\frac{1}{2}\ldots\frac{1}{2})(0\ldots0)]$, $[(0\ldots0)(\frac{1}{2}\ldots\frac{1}{2})]$, $[(\frac{1}{2},\ldots\frac{1}{2}),\ (\frac{1}{2}\ldots\frac{1}{2})]$, or $[(1\ldots1,\ 0\ldots0)]$. By Table 5.8, the first four are principal series minimal, and hence satisfy 4.2 (ii-iv). The last is a small $\mathfrak{l}\cap\mathfrak{k}$-type, and the only associate one is $[(0\ldots0)(1,1\ldots1,0\ldots0)]$. Since by the definition of s, $\mu^2_{p-s-1} < 1$, this is not dominant for $\mathfrak{k}$; so 4.2 (iii-iv) are satisfied.

Henceforth we assume s is defined, i.e. $s \le p+q$; the other case is rather easy and is left to the reader. It should be pointed out that, although $\varepsilon_{\pi(p+q-2s)} - \varepsilon_{\pi(p+q-2s+1)}$ may be a critical root, it is not included in $\mathfrak{l}$.

Set
$$v_1 = \sum_{\substack{i \text{ lower} \\ \text{critical} \\ \pi^{-1}(i)<p+q-2s}} (1-\delta)(\varepsilon^1_i - \varepsilon^2_{j(i)+1}) + \sum_{\substack{i \text{ upper} \\ \text{critical} \\ \pi^{-1}(i)\le p+q-2s}} \delta(\varepsilon^2_{j(i)} - \varepsilon^1_i)$$

$$v_2 = \begin{cases} 0 & \text{on } \pi(1)\ldots\pi(p+q-2s) \\ 2(\rho-a) & \text{on } \pi(p+q-2s+1)\ldots\pi(p+q). \end{cases}$$

Just as for $sp(n,R)$, one can easily check that v_2 is a sum of positive roots in P^2, with coefficients $0 \le c_\alpha \le 1$.

Set $v = v_1 + v_2$, $\tilde{\lambda} = a - \rho + \frac{1}{2}v = \lambda + \frac{1}{2}v$. Then clearly $\tilde{\lambda} = 0$ on $\pi(p+q-2s+1)\ldots\pi(p+q)$; by the $su(p,q)$ theory, $\tilde{\lambda}$ is dominant with respect to $(\varepsilon_{\pi(1)}-\varepsilon_{\pi(2)})\ldots$ $(\varepsilon_{\pi(p+q-2s-1)} - \varepsilon_{\pi(p+q-2s)})$. To show that $\tilde{\lambda}$ is dominant, we must show that $a_{\pi(p+q-2s)} + \frac{1}{2}v_{\pi(p+q-2s)}$ $- \rho_{\pi(p+q-2s)} \geq 0$ (recall $\tilde{\lambda}_{\pi(p+q-2s+1)} = 0$). Now $\rho_{\pi(p+q-2s)} = 2s + 1$, and $a_{\pi(p+q-2s)} = \mu_{\pi(p+q-2s)} + 2s + 1$; and $\frac{1}{2}v_{\pi(p+q-2s)} = 0$ unless $\varepsilon_{\pi(p+q-2s-1)} - \varepsilon_{\pi(p+q-2s)}$ is a critical root, in which case it is $\geq -\frac{1}{2}$. Recalling the definition of critical, we see that the desired inequality holds, unless $\mu_{\pi(p+q-2s)} = 0$, and $\mu_{\pi(p+q-2s-1)} = 0$ or $\frac{1}{2}$; also $\pi\{p+q-2s, p+q-2s-1\}$ must consist of one i coordinate and one j coordinate. But it is easy to see that this contradicts the definition of s. So $\tilde{\lambda}$ is dominant.

Let $\tilde{\ell}$ correspond to the simple roots annihilated by $\tilde{\lambda}$. We want to split $\tilde{\ell}$ into two pieces, roughly corresponding to P^1 and P^2; so suppose $\tilde{\lambda}$ is zero on $\pi(r)\ldots\pi(p+q)$, and $\tilde{\lambda}_{\pi(r-1)} \neq 0$, (of course $r \leq p+q-2s+1$). Then $\ell = \tilde{\ell}^1 + \tilde{\ell}^2$, corresponding to roots supported on $\pi(r)\ldots\pi(r-1)$, and $\pi(r)\ldots\pi(p+q)$ respectively. Suppose

γ, σ, τ, $\{\alpha_{i_j}\}$, and $\{\beta_{i_j}\}$ are as usual. By the su(p,q) theory, $\gamma = \mu$ and there are equally many $\{\alpha_{i_j}\}$ and $\{\beta_{i_j}\}$ in P^1; so we need only consider $\tilde{\ell}^2$. If $r = p + q - 2s + 1$, then $\tilde{\ell}^2 = \ell^2$, so that $\mathcal{m} \cap \tilde{\ell}^2 = 0$, and $J = |\{\beta_{i_j}\}| = 0$, a contradiction. So suppose $r < p + q - 2s + 1$. This means that $\tilde{\lambda}_{\pi(p+q-2s)} = 0$; so by the remarks of the preceding paragraph $\mu_{\pi(p+q-2s)} + \frac{1}{2}v_{\pi(p+q-2s)} = 0$. Since $\frac{1}{2}v_{\pi(p+q-2s)} \geq -\frac{1}{2}$, and $\mu_{\pi(p+q-2s)}$ is a non-negative half-integer, there are very few possibilities:

I) $\underline{\mu_{\pi(p+q-2s)} = \frac{1}{2},\ v_{\pi(p+q-2s)} = -1.}$ In this case, by the definition of critical root, $\mu_{\pi(p+q-2s-1)} = \frac{1}{2}$, and $\{\pi(p+q-2s-1), \pi(p+q-2s)\}$ consists of one i coordinate and one j coordinate. This contradicts the definition of s.

II) $\underline{\mu_{\pi(p+q-2s)} = 0,\ v_{\pi(p+q-2s)} = 0.}$ In this case, by the definition of v, $v_{\pi(p+q-2s-1)} \leq 0$. Since $\tilde{\lambda}_{\pi(p+q-2s-1)} \geq 0$, this forces $a_{\pi(p+q-2s-1)} \geq \rho_{\pi(p+q-2s-1)} = 2s + 2$. Suppose that in fact $\tilde{\lambda}_{\pi(p+q-2s-1)} = 0$. It follows easily that $\varepsilon_{\pi(p+q-2s-1)} - \varepsilon_{\pi(p+q-2s)}$ is noncompact; so that $(2\rho_c)_{\pi(p+q-2s-1)} = 2s + 1$, and $\mu_{\pi(p+q-2s-1)} = 1 - v_{\pi(p+q-2s-1)}$. So $\mu_{\pi(p+q-2s-1)} = 1$ or $\frac{3}{2}$. One easily rules out $\frac{3}{2}$; for in that case $\delta = \frac{1}{2}$, so that $v_{\pi(p+q-2s-1)} = 0$ or $-\frac{1}{4}$. So $\mu_{\pi(p+q-2s-1)} = 1$. This contradicts the definition of s unless $\pi(p+q-2s-1)$ is an i-coordinate, and $\pi(p+q-2s)$ is a j-coordinate. (Note that in this case $\varepsilon_{\pi(p+q-2s-1)} - \varepsilon_{\pi(p+q-2s)}$ is not a critical root.) Also we must have (again from the definition of s) $\mu^2_{q-s-2} \geq 1$; it follows easily that $\tilde{\lambda}_{\pi(p+q-2s-2)} > 0$.

To summarize, II) breaks into two cases:

a) $\underline{\tilde{\lambda}_{\pi(p+q-2s-1)} > 0}$, and

b) $\underline{\tilde{\lambda}_{\pi(p+q-2s-1)} = 0}$. In this case $\tilde{\lambda}_{\pi(p+q-2s-2)} > 0$, $\mu_{\pi(p+q-2s-1)}$ (an i coordinate) is 1, and $\mu_{\pi(p+q-2s)}$ (a j coordinate) is 0.

Consider first a). Since we are interested only in $\tilde{\ell}^2$, we may as well assume $p+q = 2s-1$; thus $\mathfrak{g}_0 = so(2(s-1)+1,2s+1)$, $\mu = [(c,\ldots c,0\ldots 0)(0\ldots 0)]$ $(c = 0, \frac{1}{2}$, or $1)$ and ℓ is the $so(2(s-1)+1,2(s-1)+1)$ which excludes the first j-coordinate. (We may have interchanged the i and j coordinates here, but the reader may convince himself that this is irrelevant.) Since $\Delta_\sigma^+(\mathfrak{k}) \subseteq \Delta(\mathfrak{n})$, it is immediate that σ is trivial on the first $so(2(s-1)+1)$ factor of $\mathfrak{k}$. Hence $\sigma \cdot \mu = \mu$; so by (4.9)

$$\gamma = \mu - \sigma^{-1}(\Sigma\beta_{i_j}) - (\rho_c - \sigma^{-1}\rho_c)$$

$$= \mu - \sigma^{-1}(\Sigma\beta_{i_j}) + \sigma^{-1}(\rho_c - \sigma\cdot\rho_c)$$

$$(**) \quad \gamma = \mu - \sigma^{-1}(\Sigma\beta_{i_j} - \Sigma\alpha_{i_j}) \quad \text{by} \quad (3.7).$$

Now $\Delta(\mathfrak{n})$ consists of those positive roots with support on the first j coordinate ε_1^2. Clearly $\pi(2s-1) = \varepsilon_1^2$, so these are ε_1^2 and $\varepsilon_1^2 \pm \varepsilon_{\pi(\ell)}$ $(1 \leq \ell < p+q)$. There are J β_{i_j} and J-1 α_{i_j}; so $\Sigma\beta_{i_j} - \Sigma\alpha_{i_j} = 1$ in the ε_1^2 coordinate. Now the Weyl group of $so(2s+1)$ acts by sign changes and permutation on the ε_j^2; it is easy to deduce that

$$\sigma(b_1^2 \ldots b_s^2) = (\pm b_r^2, b_1^2 \ldots b_{r-1}^2, b_{r+1}^2 \ldots b_s^2).$$

Since $\Sigma\alpha_{i_j} = \rho_c - \sigma\cdot\rho_c$, we have

$$\Sigma\alpha_{i_j} = (t, -1, \ldots -1, 0 \ldots 0) \quad \text{on the j coordinates.}$$

Hence $\Sigma\beta_{i_j} - \Sigma\alpha_{i_j} = (1, 1\ldots1, 0\ldots0)$ on the j coordinates,

and $\sigma^{-1}(\Sigma\beta_{i_j} - \Sigma\alpha_{i_j}) = (1,\ldots1, \pm 1, 0\ldots0)$; so by (**),

$\gamma = (-1, \ldots -1, \pm 1, 0\ldots0)$ on the j coordinates.

This forces $r = 1$, $\sigma(b_1^2\ldots b_s^2) = (-b_1^2, b_2^2\ldots b_s^2)$,

$\gamma = (1,0\ldots0)$ on the j-coordinates. Also $\Delta_\sigma^+(\mathfrak{k}) = \Delta(\mathfrak{n}\cap\mathfrak{k})$, which has $2(s-1)+1$ elements. So $J = 2(s-1)+2$; but $|\Delta(\mathfrak{n}\cap\mathfrak{p})| = 2(s-1)+1$, a contradiction. So μ is $\mathfrak{n}$-minimal in this case.

For b), we must apply (4.22). In this case $\mathfrak{g}_0 = so(2s+1,2s+1)$, $\mu = [(1,1\ldots1,0\ldots0)(0\ldots0)]$, and $\mathfrak{l}$ is the $so(2s-1,2s-1)$ omitting the first i-coordinate and the first j-coordinate. Let $\tilde{\mathfrak{l}}$ be the $so(2s-1,2s+1)$ omitting the first i-coordinate, with $\tilde{\mathfrak{l}} \supset \tilde{\mathfrak{l}}$ the obvious (maximal) parabolic. By the argument just given (used twice) μ is $\tilde{\mathfrak{n}}$-minimal, and $\mu|_{\tilde{\mathfrak{l}}}$ is $\mathfrak{n}\cap\tilde{\mathfrak{l}}$ minimal in $\tilde{\mathfrak{l}}$. One can use the subquotient theorem to show that $\mu|_{\tilde{\mathfrak{l}}}$ is uniformly minimal in $\tilde{\mathfrak{l}}$; or one can simply observe that the preceding paragraph never uses the minimality of $||\mu||$: γ cannot exist for purely algebraic reasons. In either case, this completes the argument for 4.6 (g); as promised, we will spare the reader a detailed argument for so(2p+1,2q) and so(2p,2q).

Any reader who has actually followed the computations to this point could undoubtedly treat the exceptional groups mentioned in (4.6) in his sleep: no machinery of any kind is needed, other than a knowledge of the root systems in a convenient basis. We therefore omit the details. Q.E.D.

5. The Subquotient Theorem

This section is devoted to the proof of Theorem 4.3.

Choose a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$, and an associated system of positive roots, so that $G = KAN$ is an Iwasawa decomposition of G. Let M denote the centralizer of A in K; then MAN is a closed subgroup of G, and N is normal in MAN. Let $\sigma \in \hat{M}$ be a (necessarily finite dimensional) irreducible representation of M, and $\nu \in \hat{A}$ a non-unitary character of A. Then $\delta \otimes \nu$ defines a representation of $MA \cong {}^{MAN}/N$, and hence of MAN. Define

$I_{\delta\otimes\nu} = \underset{MAN\uparrow G}{\mathrm{Ind}} (\delta \otimes \nu)$, a <u>principal series representation</u> of G. This representation is non-unitary in general; appropriate ρ's are introduced in the definition of induced representation so that Theorem 5.1 holds as stated below. Let M' denote the normalizer of A in K: ${}^{M'}/M$ is a finite group W, the <u>Weyl group of A</u> (in G). W acts on $\hat{A}$ and $\hat{M}$; e.g. if $w \in M'$, $\bar{w} \in W$, then $\bar{w}\cdot\delta$ is a representation of M on the representation space of δ defined by

$$(\bar{w}\cdot\delta)(m) = \delta(w^{-1}mw).$$

Theorem 5.1 (Bruhat, Harish-Chandra) $I_{\delta\otimes\nu}$ is an admissible representation with a finite composition series. $I_{\delta\otimes\nu}$ and $I_{\delta'\otimes\nu'}$ have equivalent composition series iff $(\delta',\nu') = (\sigma\cdot\delta,\sigma\cdot\nu)$ for some $\sigma \in W$. For each δ, $I_{\delta\otimes\nu}$ is irreducible for almost all ν.

Theorem 5.2 (Harish-Chandra's subquotient theorem) Every admissible irreducible representation of G is infinitesimally equivalent to a composition factor ("subquotient") of some $I_{\delta\otimes\nu}$.

It should be remarked that Lepowsky has given a purely algebraic proof of this result ([18]); in fact even his arguments can be substantially simplified for the cases we will need.

From the Iwasawa decomposition it is clear that $I_{\delta\otimes\nu}|_K = \operatorname{Ind}_{M\uparrow K} \delta$. By general facts about induced representations, $\operatorname{Ind}_{M\uparrow K} \delta$ is equivalent to $\operatorname{Ind}_{M\uparrow K} \sigma\cdot\delta$ for any $\sigma \in W$. By Frobenius reciprocity, the multiplicity with which any K-type μ occurs in $\operatorname{Ind}_{M\uparrow K} \delta$ is just the multiplicity of δ in $\mu|M$.

Suppose that G is split. Let $A(\delta) \subseteq \hat{K}$ be the collection of K-types μ such that δ occurs in $\mu|M$, and that $||\mu||$ is minimal with respect to this property.

Definition 5.3 The K-type μ is small if $\mu \in A(\delta)$, some $\delta \in \hat{M}$. It is principal series minimal if it is the minimal element of $A(\delta)$ with respect to $\prec$. Two K-types μ_1, μ_2 are associate if $\mu_1, \mu_2 \in A(\delta)$, some $\delta \in \hat{M}$.

Another characterization of small K-types is given by Proposition 5.11.

Theorem 5.4 Suppose μ is small, say $\mu \in A(\delta)$. Then $\mu|M$ is the sum of the M representations in the W orbit of δ in $\hat{M}$, each occurring with multiplicity one.

(It is very likely that this result is known by Bernstein, Gelfand, and Gelfand ([1]); but it was obtained independently for this thesis.)

Assume Theorem 5.4 for a moment.

Proof of Theorem 4.3. Since μ is small, $\mu \in A(\delta)$ for some δ. Set $\pi_\mu^\nu = I_{\delta\otimes\nu}$. For i), the multiplicity of μ in π_μ^ν is just the multiplicity of γ in μ, which is one. Suppose π_μ^ν contains the small K-type $\mu' \in A(\delta')$; we claim $A(\delta) = A(\delta')$. δ must occur in $\mu'|M$, since μ occurs in $\operatorname{Ind}_{M\uparrow K} \delta$. Since μ' is small, Theorem 5.4 implies that $\delta' = \sigma\cdot\delta$ for some $\sigma \in W$. By the remarks following

Theorem 5.2, $\operatorname{Ind}_{M\uparrow K} \delta = \operatorname{Ind}_{M\uparrow K} \delta'$; taking the "small" elements of each side, $A(\delta) = A(\delta')$. Statements ii) and iii) are just Theorem 5.1; the subgroup W_μ mentioned in iii) is the stabilizer of δ in W. For iv), suppose the K-type μ occurs in an admissible irreducible representation π of G. By Theorem 5.2, π is infinitesimally equivalent to a composition factor of some $I_{\delta'\otimes\nu}$; in particular, δ' occurs in $\mu|M$. By Theorem 5.3, $\delta' = \sigma\cdot\delta$ for some $\sigma \in W$. By Theorem 5.1, $\pi_\mu^{\sigma^{-1}\nu} = I_{\delta\otimes\sigma^{-1}\nu}$ and $I_{\delta'\otimes\nu}$ have equivalent composition series. Thus π is infinitesimally equivalent to some subquotient of $\pi_\mu^{\sigma^{-1}\nu}$. But the only irreducible subquotient of $\pi_\mu^{\sigma^{-1}\nu}$ containing the K-type μ is $\overline{\pi_\mu^{\sigma^{-1}\nu}}$. Q.E.D.

Recall the ideal $I_\mu \subseteq U^{k}$; set $R_\mu = U^{k}/I_\mu$. Another consequence of Theorem 5.4 is

<u>Corollary 5.5</u> <u>Suppose $\mu \in A(\delta)$ is small. Let $W_\delta \subseteq W$ be the stabilizer of δ in W (with respect to the action of W on $\hat{M}$.) Then there is an injection</u>

$$R_\mu \hookrightarrow U(\mathfrak{a})^{W_\delta} \subseteq U(\mathfrak{a}).$$

<u>The image of $\mathfrak{Z}(\mathfrak{g})$ is precisely $U(\mathfrak{a})^W$</u>.

<u>Proof</u>. This follows easily from results of Lepowsky [18].

It should be noted that $U(\mathfrak{a})^{W_\delta}$ need not be a polynomial ring. It is known (cf. Knapp and Stein [13]) that W_δ admits a factorization $W_\delta = W(\Delta')\cdot R$; here $W(\Delta')$ is a normal subgroup generated by reflections, and R is a product of Z_2's. (Knapp and Stein assume G is linear, but the non-linear cases can easily be checked from the proof of Theorem 5.4.) It seems likely that we may arrange $|R| = |A(\delta)|$; this would imply that $U(\mathfrak{a})^{W_\delta}$ is a polynomial ring when $|A(\delta)| = 1$. It can be shown that R_μ and $U(\mathfrak{a})^{W_\delta}$ have the same field of fractions, so that their integral closures are isomorphic. It would be nice if the map given by the corollary were an isomorphism; but if $W_\delta \neq W$, this is probably very hard to prove.

<u>Proof of Theorem 5.4</u>. Let $\tilde{G}$ denote the universal covering group of G. Since A is connected, the centralizer of $A_{\tilde{G}}$ in $K_{\tilde{G}}$ is just the preimage of M in $\tilde{G}$; similarly for M'. Thus $W_G = W_{\tilde{G}}$, so it is enough to prove the

theorem for $\tilde{G}$. Now $\tilde{G}$ is a direct product of simply connected groups with simple Lie algebras, and abelian groups; by standard facts about the representation theory of direct products, it suffices to prove the theorem for each factor. For the abelian factors, $M = M' = K$, and the result is trivial. It remains to investigate the universal covers of the simple split groups: up to coverings, these are $SP(n,\mathbb{R})$ $(n \geq 1)$, $SL(n,\mathbb{R})$ $(n \geq 3)$, $SO(n,n)$ $(n \geq 4)$, $SO(n+1,n)$ $(n \geq 3)$, and the split forms of G_2, F_4, E_6, E_7, and E_8. These we check one by one. Except for the universal covering group of $SP(n,\mathbb{R})$, all have finite center; so K is compact and M is finite. Accordingly we make heavy use of simple arguments from representation theory for finite groups, notably the following ones. If $\{\delta_1 \ldots \delta_\ell\} = \hat{M}$, and the representation space of δ_i has dimension d_i, then $\Sigma d_i^2 = |M|$. (This is usually used to show that some set of representations of M exhausts $\hat{M}$.) If a representation of M' (for example a K-type restricted to M') contains an M-type δ, then it contains all the M-types in the W orbit of δ in $\hat{M}$, each with the same multiplicity. If some M-type occurs with multiplicity one, and the representation is irreducible under M', then it is the sum of the M types in the W orbit of δ, each occurring with multiplicity one.

In every case the argument will run along roughly similar lines. First we compute M. Next we exhibit a collection of representations of K, and show that, after restriction to M, these exhaust the representations of M; and that the representations of M occur with multiplicity one. The proof that the K-types listed are actually the small ones is often left to the reader. (Details are given for SP(n,R), which is one of the most tedious cases.) The final step is to show that the K-types listed are irreducible under M'. Thus it is never necessary to compute M' completely, but only to exhibit enough elements to act irreducibly on the K-types in question.

Some attempt has been made to keep notation consistent with Helgason ([9]), especially Chapter IX, section 4. If $H \subseteq G$, we write $\tilde{H}$ for the preimage of H in $\tilde{G}$. Thus $\tilde{K}$ is the universal covering group of K, so that $\hat{\tilde{K}} \tilde{=} \hat{\mathfrak{k}}$; i.e. the representations of $\tilde{K}$ are indexed precisely by the dominant integral weights for $\mathfrak{k}$.

<u>Case 1</u> $G = SP(n,R) = \{g \in GL(2n,R) \mid {}^t g \cdot J_n g = J_n\}$ (Helgason ([9]) p. 342; recall that $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$).

$$A = \left\{ \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & \lambda_1^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_n^{-1} \end{pmatrix} \middle| \lambda_i > 0 \right\}.$$

K consists of the matrices $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ such that $X + iY \in U(n)$, and this defines an isomorphism of K onto $U(n)$. Since A contains elements with all diagonal entries distinct, $M = K^A$ consists of the diagonal elements in K, i.e.

$$M = \left\{ \begin{pmatrix} \varepsilon_1 & & & & & \\ & \ddots & & & & \\ & & \varepsilon_n & & & \\ & & & \varepsilon_1 & & \\ & & & & \ddots & \\ & & & & & \varepsilon_n \end{pmatrix} \middle| \varepsilon_i = \pm 1 \right\}.$$

In the $U(n)$ picture, M is therefore the diagonal matrices with all entries ± 1. Clearly $M' \subseteq \left\{ \begin{pmatrix} P & O \\ O & P \end{pmatrix} \middle| P \text{ is a permutation matrix} \right\}$; so in $U(n)$, M' includes all the permutation matrices. (The other generators of M', as elements of $U(n)$, are the diagonal matrices with all entries ± 1 or $\pm i$. It follows that the index of M in M' is $2^n \cdot n!$, which is indeed the order of the Weyl group of $sp(n)$.)

Choose T to be the diagonal matrices $\begin{pmatrix} e^{i\theta_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot\ e^{i\theta_n} \end{pmatrix}$ in $U(n)$. $M \subseteq T$, so $\tilde{M} \subseteq \tilde{T}$. We use the θ_i as coordinates in $\mathfrak{t}$; after normalization, the restriction of the invariant form $<\ ,\ >$ is the negative of the usual inner product. $i\mathfrak{t}_0'$ may be identified with n-tuples $(a_1 \ldots a_n)$ of real numbers in the dual coordinates to θ_i; so $<\ ,\ >$ is the usual inner product on $\mathbb{R}^n$. The compact roots are $\alpha_{ij}^{\pm} = (0,\ldots 0,\pm 1,0,\ldots 0,\mp 1,0,\ldots 0)$ for $i < j$; of course the non-zero entries are in the i^{th} and j^{th} places. By the Cartan-Weyl theory, a weight $a = (a_1 \ldots a_n)$ gives rise to a character of $\tilde{T}$ iff $\frac{2<a,\alpha>}{<\alpha\ \alpha>} \in \mathbb{Z}$ for every compact root α. This says $\frac{2\cdot(a_i - a_j)}{1+1} = a_i - a_j \in \mathbb{Z}$, i.e. $a_i \equiv a_j \bmod \mathbb{Z}$ for all i,j.

T may be identified as $\mathfrak{t}_0$ modulo the kernel Λ of the exponential map for G. Clearly Λ is just the lattice $\mathbb{Z}^n$ with respect to the coordinates θ_i. $\Lambda' = (2\mathbb{Z})^n$ is the preimage of M, so $M \cong \tilde{\Lambda}'/\Lambda$. Λ contains a smaller lattice Λ_0 (the dual of the integral weights in $i t_0'$) such that $\tilde{T} \cong t_0/\Lambda_0$. Clearly $\tilde{M} \cong \Lambda'/\Lambda_0$. Using these identifications, it is essentially obvious that two weights a and a' have the same restriction to $\tilde{M}$ iff $a_i \equiv a_i'$ mod 2 for all i. The permutation matrices in M' normalize T, and in fact act on $i t_0'$ by permutation. Thus if (a_i') is a permutation of (a_i), the restrictions of (a_i) and (a_i') to $\tilde{M}$ lie in the same W orbit in $\hat{\tilde{M}}$. (For $SP(n,\mathbb{R})$ the groups W_K and W are closely related; but they should not be confused.)

Order $i t_0'$ lexicographically, i.e. $(a_1 \ldots a_n) \prec (a_1' \ldots a_n')$ iff $a_1 = a_1' \ldots a_i = a_i'$, $a_{i+1} < a_{i+1}'$. The corresponding set of positive (compact) roots is $\{\alpha_{ij}^+\}$. A weight $(a_1 \ldots a_n)$ is dominant if its inner product with every positive root is non-negative. This says $a_i - a_j \geq 0$ when $i < j$, i.e. $(a_1 \ldots a_n)$ is <u>decreasing</u>. The sum of the positive roots is easily computed to be $2\rho_c = (n-1, n-3, \ldots -(n-1))$; so $(2\rho_c)_i = n - 2i + 1$. Let δ be a fixed element of $\hat{\tilde{M}}$, and let μ be a small $\tilde{K}$-type containing δ. μ has highest weight $(\mu_1,\ldots\mu_n)$; since μ is dominant integral, $\mu_i-\mu_j$ is a non-negative integer for $i < j$. Some weight $b = (b_1\ldots b_n)$ of μ must transform according to δ under $\tilde{M}$. We know that $A(\delta)$ depends only on the W orbit of δ; so, replacing δ by a W-translate $\sigma\cdot\delta$, we may permute the b_i and assume b is dominant integral (every weight of a representation is integral.) Since the permutation matrices in $\tilde{M}'$ normalize all of $\tilde{T}$, b is still a weight of the $\tilde{K}$-type μ. It is well known that whenever a is a weight of μ, $\langle\mu+\rho_c,\mu+\rho_c\rangle \geq \langle a+\rho_c,a+\rho_c\rangle$, with equality iff $a = \mu$; the same argument shows $||\mu|| \geq ||a||$, with equality iff $\mu = a$. Suppose $\mu \neq b$. By the preceding observation, the $\tilde{K}$-type b (i.e. the $\tilde{K}$-type with highest weight b) is strictly smaller than μ in the sense of section 4; but the K-type $b|\tilde{M}$ still contains δ, since the weight $b|\tilde{M}$ is $\sigma\cdot\delta$. This contradicts the minimality of μ. Thus $\mu = b$; i.e. the highest weight of μ transforms under $\tilde{M}$ according to an element of the W orbit of δ. So among all dominant weights whose restriction to M is $\sigma\cdot\delta$, μ has the smallest possible norm $||\mu||$.

Obviously some element of the W orbit of δ is the restriction of a $\tilde{T}$-weight of the form $(\beta,\ldots\beta,\beta-1,\beta-1\ldots\beta-1)$; here the first r' terms are β, and we may assume $0 \leq \beta < 2$.

If $\beta \geq 1$, replace this by $(\beta-1,\ldots\beta-1,\beta-2\ldots\beta-2)$; here the first $n-r'$ terms are $\beta-1$. The restriction of this weight to $\tilde{M}$ is in the same W-orbit as γ. So in any case, there is a dominant weight $(\varepsilon,\ldots\varepsilon,\varepsilon-1,\ldots\varepsilon-1)$, with the first r terms $= \varepsilon$, $0 \leq \varepsilon < 1$, which restricts to an element of $\hat{\tilde{M}}$ in the W orbit of γ. If $\varepsilon = 0$, there is also $(1,\ldots1,0,\ldots0)$, with the first $n-r$ terms $= 1$.

We know that if $(b_1,\ldots b_n)$ is such that r of the b_i are $\equiv \varepsilon \bmod 2\mathbb{Z}$, and the rest are $\equiv \varepsilon-1 \bmod 2\mathbb{Z}$, then the restriction of b to $\tilde{M}$ is in the W orbit of δ. Clearly this produces $\binom{n}{r}$ (the binomial coefficient) elements of this orbit. On the other hand, the representation of $\tilde{K}$ with highest weight $(\varepsilon,\ldots\varepsilon,\varepsilon-1,\ldots\varepsilon-1)$ (or $(1,\ldots1,0\ldots0)$) has dimension $\binom{n}{r}$; this is well known, or may be checked using Weyl's dimension formula. So the W orbit of δ has at most $\binom{n}{r}$ elements, so we have them all. Furthermore the restriction of this $\tilde{K}$-type to $\tilde{M}$ must be the sum of one copy of each element of the orbit. To prove Theorem 5.4 in this case, it remains only to check that $\mu = (\varepsilon,\ldots\varepsilon,\varepsilon-1,\ldots\varepsilon-1) \in A(\delta)$ (and $(1,\ldots1,0\ldots0) \in A(\delta)$ if $\varepsilon = 0$). This we do in several steps. In each step, it is shown that if μ did not have some property, there would be a $\tilde{K}$-type μ' with

$\mu'|\tilde{M}$ in the same W orbit as $\mu|\tilde{M}$, but $||\mu'|| < ||\mu||$. By the minimality of μ, it follows that μ must have the stated property.

i) <u>$\mu_i = \mu_{i+1}$ or $\mu_i = \mu_{i+1} + 1$, all i.</u>

<u>Proof</u>. Suppose not. Since μ_j is decreasing, it follows that $\mu_i \geq \mu_{i+1} + 2$ for some i. If $\mu_i + (n-2i+1) > 1$, set $\mu_i' = \mu_i - 2$, $\mu_j' = \mu_j$ for $i \neq j$. Then μ' is dominant, and $\mu'|M = \mu|\tilde{M}$, but

$$
\begin{aligned}
||\mu|| - ||\mu'|| &= \langle\mu+2\rho_c, \mu+2\rho_c\rangle - \langle\mu'+2\rho_c, \mu'+2\rho_c\rangle \\
&= (\mu_i + (n-2i+1))^2 - ((\mu_i - 2) + (n-2i+1))^2 \\
&= 4(\mu_i + (n-2i+1) - 1) > 0.
\end{aligned}
$$

So $\mu' < \mu$, a contradiction. If $\mu_i + (n-2i+1) \leq 1$, then $\mu_{i+1} - (n-2(i+1)+1) \leq \mu_i + (n-2i+1) - 4 \leq -3 < -1$, and one can use the same argument with $\mu'_{i+1} = \mu_{i+1} + 2$. //

ii) <u>The sequence (μ_i) contains no subsequence $(\ldots x+1, x \ldots x, x-1 \ldots)$.</u>

<u>Proof</u>. Suppose that such a subsequence occurs, with the last x+1 in the i^{th} place and the first x-1 in the j^{th} place. Setting $\mu' = (\ldots \underbrace{x}_{\text{ith place}}, x \ldots x, \underbrace{x-1}_{\text{jth place}}, x-1, \ldots)$

we see that $\mu|\tilde{M}$ and $\mu'|\tilde{M}$ are in the same W orbit (μ' is obtained from μ by transposing the ith and (j-1)th places, and subtracting 2 from the (j-1)th place.)
Also

$$\begin{aligned} ||\mu||-||\mu'|| &= \langle\mu+2\rho_c,\mu+2\rho_c\rangle - \langle\mu'+2\rho_c,\mu'+2\rho_c\rangle \\ &= (x+1+(n-2i+1))^2-(x+(n-2i+1))^2 \\ &+ (x+(n-2(j-1)+1))^2 - (x-1+(n-2(j-1)+1))^2 \\ &= 4((n-i-j+2)+x) \end{aligned}$$

which is positive if $x > -(n-i-j+2)$. Similarly,

$$\mu'' = (\ldots \underbrace{x+1}_{\text{ith place}}, x+1, x, \ldots x, \underbrace{x}_{\text{jth place}}, x-1 \ldots)$$ is smaller than μ if $x < -(n-i-j)$; so in any case the minimality of μ is contradicted. //

It follows from i) and ii) that μ is of the form $(\beta,\beta\ldots\beta,\beta-1,\ldots\beta-1)$; it remains to show that $0 \leq \beta \leq 1$. Say the first r' terms are β.

iii) $0 \leq \beta < 2$

Proof. Suppose not, e.g. $\beta \geq 2$. Set $\mu' = (\beta-2,\ldots\beta-2,\beta-3,\ldots\beta-3)$, with the first r' terms $\beta-2$. Then $\mu|\tilde{M} = \mu'|\tilde{M}$, but

$$||\mu||-||\mu'|| = \sum_{i=1}^{r'} [(\beta+(n-2i+1))^2-(\beta-2+(n-2i+1))^2]$$

$$+ \sum_{i=r'+1}^{n} [(\beta-1+(n-2i+1))^2-(\beta-3+(n-2i+1))^2]$$

$$= \sum_{i=1}^{r'} 4(\beta-1+(n-2i+1)) + \sum_{i=r'+1}^{n} 4(\beta-2+(n-2i+1))$$

$$= 4 \sum_{i=1}^{n} (\beta-1+(n-2i+1)) + 4 \sum_{i=r'+1}^{n} (-1)$$

$$= 4n(\beta-1) - 4(n-r') \quad \text{since} \quad \sum_{i=1}^{n} (n-2i+1) = 0$$

$$\geq 4n - 4(n-r) \geq 0.$$

We get equality only if $\beta = 2$ and $r' = 0$. In that case we may rewrite $\mu = (\beta',\ldots\beta')$, with $\beta' = 1$.

iv) <u>$0 \leq \beta \leq 1$.</u>

<u>Proof</u>. Suppose not, i.e. $1 < \beta < 2$. Set $\mu' = (\beta-1,\ldots\beta-1,\beta-2,\ldots\beta-2)$; here the first $n-r'$ terms are $\beta-1$. Then $\mu'|\tilde{M}$ is in the same W orbit as $\mu|\tilde{M}$, and

$$||\mu||-||\mu'|| = \sum_{i=1}^{r'} (\beta+n-2i+1)^2 + \sum_{i=r'+1}^{n} (\beta-1+n-2i+1)^2$$

$$- \sum_{i=1}^{n-r'} (\beta-1+n-2i+1)^2 - \sum_{i=n-r'+1}^{n} (\beta-2+n-2i+1)^2$$

$$= \sum_{i=1}^{n} ((\beta-1)+n-2i+1)^2 + \sum_{i=1}^{r'} [2(\beta-1+(n-2i+1))+1]$$

$$- \sum_{i=1}^{n} (\beta-1+(n-2i+1))^2 + \sum_{i=n-r'+1}^{n} [2(\beta-1+(n-2i+1))-1].$$

Changing the last index,

$$= \sum_{i=1}^{r'} \{[2(\beta-1+(n-2i+1))+1] + [2(\beta-1-(n-2i+1))-1]\}$$

$$= 4r'(\beta-1) \geq 0.$$

Equality holds iff $r' = 0$. In that case, $\mu = (\beta-1,\ldots\beta-1)$; defining $r = n$, $\varepsilon = \beta - 1$, μ is in the desired form. This completes Case 1.

Case 2 $G = SL(2n,\mathbb{R})$, $n \geq 2$. Here of course,

$$A = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \middle| \lambda_i > 0, \Pi\lambda_i = 1 \right\}, \text{ and } K \text{ is } SO(2n).$$

One knows (cf. Chevalley [5]) that the fundamental group of K has order 2; so $\tilde{G}$ is a two-sheeted cover of G, and $\tilde{K}$ = Spin (2n). Clearly

$$M = \left\{ \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} \middle| \varepsilon_i = \pm 1, \Pi\varepsilon_i = 1 \right\}, \text{ and } M' \text{ consists}$$

of the permutation matrices of determinant 1. So $|M| = 2^{2n-1}$, and $|\tilde{M}| = 2^{2n}$. Since M is abelian, $\tilde{M}$ has at least 2^{2n-1} one dimensional representations. $\tilde{K}$ = Spin (2n) has two representations of dimension 2^{n-1}, which we call $spin^+$ and $spin^-$. We will eventually show that these are irreducible and inequivalent as representations of $\tilde{M}$. Since $|\tilde{M}| = 2^{2n} = 2^{2n-1}\cdot(1^2) + (2^{n-1})^2 + (2^{n-1})^2$, these exhaust the representations of $\tilde{M}$. Leaving to the reader the easy fact that $spin^+$ and $spin^-$ are actually the small $\tilde{K}$-types of the corresponding representations of $\tilde{M}$, this will prove Theorem 5.3 for the $\tilde{M}$-types $spin^{\pm}|\tilde{M}$. So we consider the one dimensional representations of M first.

Embed M in the full group D of diagonal matrices in $O(n)$. D has order 2^{2n}. If $S \subseteq \{1,2...2n\}$, define

$$\delta_S \in D \text{ by } \delta_S\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_{2n}\end{pmatrix} = \prod_{i\in S}\varepsilon_i. \text{ This exhausts } \hat{D},$$

so the $\delta_S|M$ exhaust $\hat{M}$. Since $\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n\end{pmatrix} \in M$ iff

$\Pi\varepsilon_i = 1$, $\delta_S|M = \delta_{S'}|M$ iff $S = S'$ or $S = (S')^c$.

If $\sigma \in W$, and $\tilde{\sigma}$ is the corresponding permutation (recall that M' consists of permutation matrices) $\sigma\cdot\delta_S = \delta_{\tilde{\sigma}\cdot S}$. It follows that the orbits of W in $\hat{M}$ each contain exactly one representation $\delta_r = \delta_{\{1,2...r\}}$ for $0 \leq r \leq n$. Consider first the case $r < n$. Then $W\cdot\delta_r$ is clearly in

1-1 correspondence with the r element subsets of $\{1,\ldots,2n\}$; so $|W\cdot\delta\ | = \binom{2n}{r}$. Let μ_r be the representation of $SO(2n)$ on $[\Lambda^r(\mathbb{R}^{2n})]_{\mathbb{C}}$. If $e_1\ldots e_{2n}$ is the usual basis of $\mathbb{R}^{2n}$, then $e_1\wedge\ldots\wedge e_r$ clearly transforms according to δ_r under M. Thus δ_r occurs in $\mu_r|M$. On the other hand, $\dim\mu_r = \binom{2n}{r} = |W\cdot\delta_r|$, so $\mu_r|M$ must contain every element of $W\cdot\delta_r$ exactly once. (In particular μ_r is irreducible, which is of course well known.) An argument along the lines of that given for $SP(n,\mathbb{R})$ shows that μ_r is in fact the unique small K-type containing δ_r, which proves Theorem 5.3 for the δ_r, $r < n$.

To consider δ_n and the spin representations, we need a little more structure. Let

$$T = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & 0 & \\ & & \ddots & & \\ & 0 & & \cos\theta_n & \sin\theta_n \\ & & & -\sin\theta_n & \cos\theta_n \end{pmatrix} \middle| \theta_i \in \mathbb{R} \right\}$$

be the usual torus in K. We use the θ_i as coordinates in $\mathfrak{t}$, again identifying $i\mathfrak{t}_0'$ with $\mathbb{R}^n$ (with $\langle\ ,\ \rangle$ the usual inner product). The Weyl group of $\mathfrak{t}$ in K, W_K, acts by permuting the coordinates and changing an <u>even</u> number of the signs of the coordinates. We order $i\mathfrak{t}_0'$ lexicographically in these coordinates. The positive (compact) roots are then $(0,\ldots 1,0\ldots,\pm 1,\ldots 0)$; so $2\rho_c = (2n,2n-2,\ldots 2,0)$. A weight $(a_1\ldots a_n)$ is dominant if it is decreasing and $a_{n-1} + a_n \geq 0$; it is integral if every a_i is an integer or if every $a_i - \frac{1}{2}$ is an integer. All this may be checked just as we checked the corresponding statements for U(n) above.

Consider now δ_n. The pairs $\{S,S^c\}$ of n-element subsets of $\{1,2\ldots 2n\}$ are in 1-1 correspondence with the W orbit of δ_n in $\hat{M}$. It follows that this orbit has $\frac{1}{2}\cdot\binom{2n}{n}$ elements. The weights of the $(\mathbb{R}^{2n})_{\mathbb{C}}$ representation of K are $(0,\ldots 0,\pm 1,0\ldots 0)$, a ± 1 in the ith place corresponding to the weight vector $e_{2i-1} \mp ie_{2i}$ in the usual basis. These weights occur with multiplicity one. Hence the weights of the $(\Lambda^n\mathbb{R}^{2n})_C$ representation are $(\varepsilon_1\ldots\varepsilon_n)$; here r of the ε_i are 1, s are -1, and the rest are zero; furthermore $n-(r+s)$ is necessarily even, since +1 weights and -1 weights can cancel only in pairs. In particular the dominant weights

$\mu_n^+ = (1,\ldots 1)$ and $\mu_n^- = (1,\ldots 1,-1)$ occur, and $\mu_n^{\pm} + \alpha$ does not occur for any compact positive root α. It follows that $(\Lambda^n \mathbb{R}^{2n})_{\mathbb{C}}$ contains the two irreducible K-types $\mu_n^{\pm}$. By Weyl's dimension formula, $\dim \mu_n^{\pm} = \frac{1}{2} \cdot \binom{2n}{n} = \frac{1}{2} \dim (\Lambda^n \mathbb{R}^{2n})_{\mathbb{C}}$, so $(\Lambda^n \mathbb{R}^{2n})_{\mathbb{C}} = \mu_n^+ \oplus \mu_n^-$. Using the usual basis of $\mathbb{R}^{2n}$, it is clear that $\Lambda^n \mathbb{R}^{2n}$ contains the M-type δ_n twice (the corresponding vectors are $e_1 \wedge \ldots \wedge e_n$ and $e_{2n+1} \wedge \ldots \wedge e_{2n}$). Hence $\Lambda^n \mathbb{R}^{2n}$ contains each element of $W \cdot \delta_n$ at least twice; since $|W \cdot \delta_n| = \frac{1}{2}\binom{2n}{n}$, each element occurs exactly twice. It follows that the conclusion of Theorem 5.4 holds for either μ_n^+ or μ_n^-. One can compute that $||\mu_n^+|| = ||\mu_n^-||$, and it is easily verified that $A(\delta_n) = \mu_n^{\pm}$.

It remains to show that spin^+ and spin^- are irreducible and inequivalent under $\tilde{M}$. By Schur's lemma, it is enough to show that $\pi = (\text{spin}^+ \oplus \text{spin}^-) \otimes (\text{spin}^+ \oplus \text{spin}^-)^*$ contains only two copies of the trivial $\tilde{M}$ type. But it is well known that $\pi \cong (\Lambda^* \mathbb{R}^{2n})_{\mathbb{C}}$, the exterior algebra of $\mathbb{R}^{2n}$. Since $\Lambda^i \mathbb{R}^{2n} \cong \Lambda^{2n-i} \mathbb{R}^{2n}$ as a K-module, our previous results show that the only M invariants in π are $\Lambda^0 \mathbb{R}^{2n} \oplus \Lambda^{2n} \mathbb{R}^{2n}$. This completes Case 2. The case of

$G = SL(2n+1,\mathbb{R})$ is considerable simpler for more or less obvious reasons, so we leave it to the reader.

Case 3 $G = SO(2p+1,2p)$, $p \geq 2$. G is the subgroup of $SL(4p+1)$ consisting of matrices preserving the quadratic form $-x_1^2 - \dots - x_{2p+1}^2 + x_{2p+2}^2 + \dots + x_{4p+1}^2$ (cf. Helgason ([9]) p. 340). Let $\{e_{ij}\}$ be the usual basis of the $4p+1 \times 4p+1$ matrices. As a basis for the Lie algebra $\mathfrak{a}_0$ of A, we may use $\{e_{i,2p+1+i} + e_{2p+1+i,i} \mid 1 \leq i \leq 2p\}$.

K is $\left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \middle| \; X \in SO(2p+1),\ Y \in SO(2p) \right\}$. A matrix $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ commutes with $\mathfrak{a}$ iff $X_{ij} = Y_{ij} = 0$ for $i \neq j$, $i,j \leq p$, and $X_{ii} = Y_{ii}$ for $i \leq p$. For $(X,Y) \in SO(2p+1) \times SO(2p)$, this implies

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \left(\begin{array}{c|c} \begin{matrix} \varepsilon_1 & & & 0 \\ & \ddots & & \\ & & \varepsilon_{2p} & \\ 0 & & & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{2p} \end{matrix} \end{array}\right), \quad \varepsilon_i = \pm 1,\ \Pi \varepsilon_i = 1. \quad \text{If}$$

P is a $2p \times 2p$ permutation matrix, $\det P = 1$, then

$\left(\begin{array}{cc|c} P & 0 & \\ 0 & 1 & 0 \\ \hline & 0 & P \end{array}\right) \in M'$; and if D_i is diagonal, $\det D_i = 1$, then

$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \varepsilon M'$. These assertions may be checked by direct calculation.

Clearly M has order 2^{2p-1}. On the other hand, $\tilde{K} = \text{Spin}(2p+1) \times \text{Spin}(2p)$ is a 4 sheeted cover of K. so $|\tilde{M}| = 2^{2p+1}$. G has a 2 sheeted linear cover Spin (2p+1,2p). For a linear group, M is abelian (cf. Warner [25], theorem 1.4.1.5, or the proof of Lemma 5.5 below). So $\tilde{M}$ has at least 2^{2p} one dimensional representations.

Let π_2 denote the projection of $\tilde{K}$ onto Spin (2p). Then $\pi_2(\tilde{M})$ is "$\tilde{M}$ for $\widetilde{SL}(2p)$," and $\pi_2(\tilde{M}')$ contains "$\tilde{M}'$ for $\widetilde{SL}(2p)$;" this follows from the corresponding remarks on the SO(2p+1,2p) level. If $\tilde{H}$ is a subgroup of $\tilde{K}$, and R_2 is a representation of Spin (2p), then the representation $1 \otimes R_2$ (outer tensor product) of $\tilde{K}$, restricted to $\tilde{H}$, is trivial on ker π_2, and corresponds to the representation R_2 restricted to $\pi_2(\tilde{H})$. It follows immediately that the various $1 \otimes \mu_r$, $0 \leq r < n$, and $1 \otimes \mu_n^-$, are irreducible under M', and correspond to 2^{2p-1} distinct one dimensional representations of M. Also the two representations $1 \otimes \text{spin}^{\pm}$ are irreducible under $\tilde{M}$ and inequivalent; each has dimension 2^{p-1}.

Next, notice that $\tilde{M}'$ contains "$\tilde{M}$ for $\widetilde{SL}(2p+1)$" $\times$ "$\tilde{M}$ for $\widetilde{SL}(2p)$;" again this follows from the corresponding statement for M' on the $SO(2p+1,2p)$ level. It follows that $\tilde{K}$-type spin $\otimes$ $spin^-$ is irreducible under $\tilde{M}'$; it has dimension $2^p.2^{p-1} = 2^{2p-1}$. (Here "spin" is the 2^p dimensional spin representation of $SO(2p+1)$. The proof of Theorem 5.4 for $G = SL(2p+1,R)$ shows that spin is irreducible under "$\tilde{M}$ for $SL(2p+1,\mathbb{R})$;" this proceeds exactly as for $SL(2p,\mathbb{R})$.) It is not difficult to verify that spin $\otimes$ $spin^-$ actually lives on the two-sheeted cover of K in the linear cover of G, but not on K. $\tilde{M}$ at this level is abelian, so spin $\otimes$ $spin^-$ provides 2^{2p-1} more one dimensional elements of $\hat{\tilde{M}}$ - we need only show that they are distinct. This can be done in the following way. Let H denote the standard $SO(2p) \subseteq SO(2p+1)$, $h \hookrightarrow \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$.

Let $M_1 \subseteq H$ denote "M for $SL(2p)$", and let M_2 be the corresponding subgroup of $\pi_2(K) \cong SO(2p)$. Then $M_1 \cong M_2$, so it makes sense to speak of the diagonal subgroup $\Delta \subseteq M_1 \times M_2$; Δ is just M. Thus $\tilde{M} \supseteq \Delta(\tilde{M}_1 \times \tilde{M}_2) = \Delta' \cong \tilde{M}_1$. We will show that spin $\otimes$ $spin^-|\Delta'$ already consists of 2^{2p-1} distinct components. First, one knows that $spin|_{\tilde{H}} = spin^+ \oplus spin^-$. So as a representation of

$\tilde{\Delta}' \cong \tilde{M}_1$, spin ⊗ spin^- is simply (spin^+ ⊕ spin^-) $\dot{\otimes}$ spin^- (here $\dot{\otimes}$ means inner tensor product.) As a representation of $\Delta(\tilde{H} \times$ [illegible] = Spin (2p), this is known to be $[(\sum_{r=0}^{p-1} \mu_r) \oplus \mu_n^-]$, which we know (from Case 2) cons[illegible] of 2^{2p-1} distinct $\tilde{M}_1 \cong \tilde{\Delta}'$-types.

Finally, consider the $\tilde{K}$-type spin ⊗ 1. Let π_1 denote the projection of $\tilde{K}$ on Spin (2p+1). Arguing as for 1 ⊗ $\text{spin}^{\pm}$, we see that spin ⊗ 1 is equivalent as an $\tilde{M}$ representation to the representation of $\pi_1(\tilde{M})$ on spin | $\pi_1(\tilde{M})$. Now $\pi_1(\tilde{M}) \cong \tilde{M}_1 \subseteq \tilde{H}$ (since this holds on the SO(2p+1,2p) level) and spin$|_{\tilde{H}}$ = $\text{spin}^+ \oplus \text{spin}^-$. So spin ⊗ 1 splits into two irreducible $\tilde{M}$ components of dimension 2^{p-1}; they are inequivalent. We need to show that spin ⊗ 1 is $\tilde{M}'$ irreducible; but $\pi_1(\tilde{M}') \supseteq$ "$\tilde{M}$ for SL(2p+1)," which acts irreducibly on spin.

In this way (i.e. by hook and by crook) we get $2^{2p-1} + 2^{2p-1}$ one dimensional representations of $\tilde{M}$, and four 2^{p-1} dimensional representations. Since $|\tilde{M}| = 2^{2p+1} = 2^{2p-1} + 2^{2p-1} + 4 \cdot (2^{p-1})^2$, this exhausts $\hat{\tilde{M}}$. Leaving to the reader the proof that the $\tilde{K}$ types exhibited are the small ones, we have proved Theorem 5.4 in Case 3.

The various oth... ...plit orthogonal groups are similar and easier.

For the exceptional groups, the following lemmas are helpful.

<u>Lemma 5.6.</u> <u>Let $\pi : K \to SO(\mathfrak{p}_0)$ be the $\mathfrak{p}$ representation of K. Choose $\alpha_1 \ldots \alpha_n$ a strongly orthogonal set of noncompact imaginary roots, and let $X^{\alpha_1} \ldots X^{\alpha_n}$ be non-zero elements of the corrresponding weight spaces of $\mathfrak{p} = (\mathfrak{p}_0)_{\mathbb{C}}$. Suppose that for each i there is an element σ_i in the normalizer of T in K, such that $\sigma_i \cdot \alpha_i = -\alpha_i$, and $\pi(\sigma_i^2) = 1$. Then the space $\mathfrak{a}$ spanned by $\{X^{\alpha_i} + \pi(\sigma_i) \cdot X^{\alpha_i}\}$ and $\mathfrak{p}^t$ is an abelian subalgebra of $\mathfrak{p}$. Also $\bar{\mathfrak{a}} = \mathfrak{a}$, so that $\mathfrak{a} = (\mathfrak{a}_0)_{\mathbb{C}}$ for some abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$.</u>

<u>Proof.</u> That $\mathfrak{a}$ is abelian follows immediately from the strong orthogonality of the α_i; notice that $\mathfrak{p}^t$ commutes with X^{α_i} since α_i is an imaginary root. Since $\overline{\mathfrak{p}^t} = \mathfrak{p}^t$, it suffices to see that

$\overline{(X^{\alpha_i} + \pi(\sigma_i)X^{\alpha_i})} = C_{\alpha_i}(X^{\alpha_i} + \pi(\sigma_i)X^{\alpha_i})$ for each i; here $C_{\alpha_i} \varepsilon \mathbb{C}$. For simplicity of notation, we drop the subscript i.

$\pi(\sigma)X^{\alpha}$ clearly has weight $-\alpha$. By (2.6), $\overline{X^{\alpha}}$ has weight $-\theta\alpha = -\alpha$ since α is imaginary. Since the weight spaces are one dimensional, $\overline{X^{\alpha}} = C_{\alpha}\pi(\sigma)X^{\alpha}$. Apply $\pi(\sigma)$ to both sides; since π is a real representation, $\pi(\sigma)$ commutes with conjugation. Thus $\overline{\pi(\sigma)X^{\alpha}} = C_{\alpha}\pi(\sigma)\pi(\sigma)X^{\alpha} = C_{\alpha}\pi(\sigma^2)X^{\alpha} = C_{\alpha}X^{\alpha}$; so $(X^{\alpha}+\pi(\sigma)X^{\alpha}) = \overline{X^{\alpha}} + C_{\alpha}X^{\alpha} = C_{\alpha}(\pi(\sigma)X^{\alpha}+X^{\alpha})$. QED.

Lemma 5.7. Suppose that G is a split semisimple matrix group. Then M is abelian, and $|M| \leq 2^{\text{rank } G}$.

Proof. By hypothesis, $G \subseteq GL(n,R)$, so that $\mathfrak{g}_0 \subseteq \mathfrak{gl}(n,R)$, and $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$. Let $G_{\mathbb{C}}$ denote the connected subgroup of $GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$. If bar denotes the natural conjugation in $GL(n,\mathbb{C})$, then $G \subseteq G_{\mathbb{C}}$ is just the identity component of $\{g \in G_{\mathbb{C}} | \overline{g} = g\}$.

The center of a connected Lie group consists precisely of the elements which act trivially in the adjoint representation. Since $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$, it follows that the center of G is contained in the center of $G_{\mathbb{C}}$. A complex semisimple Lie group has finite center, so the center of G is finite, and therefore K is compact. Since $M \subseteq K$ is closed and discrete, M is finite.

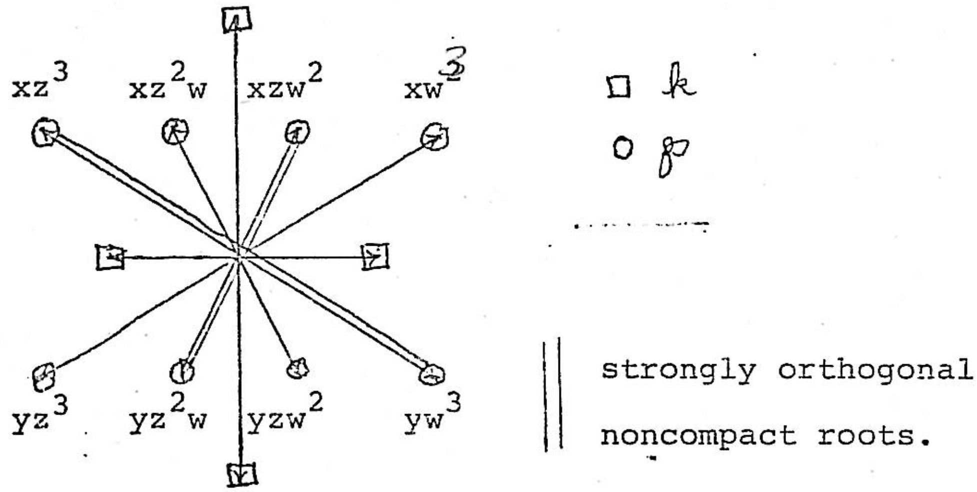
Let $A_{\mathbb{C}} = \exp \mathfrak{a}$. One knows (or it follows from the results of section 2) that $A_{\mathbb{C}} = G^{\mathfrak{a}}$; hence $M \subseteq A_{\mathbb{C}}$. The kernel of $\exp|_{\mathfrak{a}}$ is a lattice $\Lambda \subseteq i\mathfrak{a}_0$, and $\exp i\mathfrak{a}_0$ is isomorphic to a product of $\dim_{\mathbb{C}} \mathfrak{a}$ = rank G circles. Suppose $m \in M$; choose $x,y \in \mathfrak{a}_0$ so that $\exp(x+iy) = m$. Since M is finite, $m^N = 1$ for some positive integer N; this implies $Nx + Niy \in \Lambda$. Since $\Lambda \subseteq i\mathfrak{a}_0$, it follows that $x = 0$, so $m \in \exp i\mathfrak{a}_0$. On the other hand, since $m \in G$, $m = \overline{m} = \overline{\exp(iy)} = \exp(-iy) = m^{-1}$, so $m^2 = 1$. A product of r circles has exactly 2^r elements of order 2, so $|M| \leq 2^{\text{rank } G}$. QED.

One can give a much shorter proof using known results - for instance the fact that $M \subseteq \exp(i\mathfrak{a}_0)$ is a special case of a theorem of Osborne and Rader (cf. Warner [25], proposition 1.4.1.3). But is seems likely that anyone familiar with these results knows the preceding proposition.

<u>Case 4</u>. G = simply connected split form of G_2. Here

$$K = SU(2) \times SU(2) = \left\{\left(\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix}\right) \middle| |\alpha|^2+|\beta|^2=|\gamma|^2+|\delta|^2=1\right\}.$$

We let the first factor here correspond to the long compact root in the diagram below. Each SU(2) acts on polynomials in two variables (say (x,y) and (z,w) respectively) so that K acts on polynomials in x, y, z, and w. The irreducible representations of K are the actions on polynomials of degree n in (x,y) and m in (z,w); we call this representation $(n+1) \otimes (m+1)$, these numbers being dimensions. Weight vectors with respect to the usual Cartan subgroup T of K (consisting of pairs of diagonal matrices) are the monomials. The $\mathfrak{p}$ representation is isomorphic to $2 \otimes 4$. Such an isomorphism is indicated in the following root diagram for G with respect to T.



Let $\sigma \in N(T)$ be the element

$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. $\sigma^2 = (-I,-I)$; since $-I$ acts by -1 in every even dimensional irreducible representation of $SU(2)$, σ^2 acts by $(-1)\cdot(-1) = 1$ in $\mathfrak{p}$. The root vectors xzw^2 and xz^3 correspond to strongly orthogonal roots, so we may apply Lemma 5.6. Since $\sigma\cdot(x,y,z,w) = (-y,x,-w,z)$, $\sigma\cdot(xzw^2) = (-y)(-w)z^2 = yz^2w$, and $\sigma\cdot(xz^3) = (-y)(-w)^3 = yw^3$; so we may take $xzw^2 + yz^2w$ and $xz^3 + yw^3$ as a basis for $\mathfrak{a}$ (of course $\mathfrak{g}_0$ is equal rank, so $\mathfrak{p}^t = 0$). It is easy to check that the eight elements

$$\left\{\left(\begin{pmatrix} e^{-3i\theta} & 0 \\ 0 & e^{3i\theta} \end{pmatrix}, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) \middle| \; e^{4i\theta} = 1 \right\} \cup$$

$$\left\{\left(\begin{pmatrix} 0 & e^{-3i\theta} \\ -e^{3i\theta} & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix}\right) \middle| \; e^{4i\theta} = 1 \right\} \text{ fix } \mathfrak{a} .$$

We claim that G is a two-sheeted cover of its adjoint group. The center of G is just the centralizer of $\mathfrak{p}$ in the center of K. The center of K is $(\pm I, \pm I)$; $(I,-I)$ and $(-I,I)$ act by -1 on $\mathfrak{p}$, so the center of G is $\{(I,I), (-I,-I)\}$. This has order two, which proves the claim. The adjoint representation is faithful for the adjoint group, so Lemma 5.7 applies to it:

"M for the adjoint group" has at most $2^2 = 4$ elements. Thus $|M| \leq 8$, so the 8 elements listed above exhaust M. It also follows that M has at least 4 one dimensional representations. It is easy to check that $1 \otimes 3|_M$ consists of 3 inequivalent non-trivial M-types, spanned by z^2+w^2, z^2-w^2, and zw. $1 \otimes 1$ is of course the trivial M-type. We claim that $1 \otimes 2$ is irreducible under M: recalling that $\dot{\otimes}$ denotes internal tensor product, we compute

$$(1 \otimes 2) \dot{\otimes} (1 \otimes 2)^* = 1 \otimes (2 \dot{\otimes} 2) = (1 \otimes 1) \oplus (1 \otimes 3)$$

which contains only one copy of the trivial M-type by the preceding remarks. By Schur's lemma, $1 \otimes 2$ is M-irreducible. Since $3 + 1 + 2^2 = 8$, this shows that $1 \otimes 1$, $1 \otimes 2$, and $1 \otimes 3$ exhaust $\hat{M}$, each M type occurring with multiplicity one. Leaving to the reader the verification that these are small, it remains to check that they are M'-irreducible. Of course $1 \otimes 1$ and $1 \otimes 2$ are already M-irreducible. Choose $\xi \in \mathbb{C}$ so that $\xi^2 = i$; then one checks easily

that $P_1 = \left(\begin{pmatrix} \xi^{-3} & 0 \\ 0 & \xi^3 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}\right)$ and

$P_2 = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right)$ are in M'. The

three weight spaces z^2, zw, and w^2 of $1 \otimes 3$ transform according to distinct scalars under P_1; and

$P_2 \cdot (z^2) = (\frac{1}{\sqrt{2}}(z-w))^2$, which has a non-zero component in each weight space. It follows that P_1 and P_2 act irreducibly on $1 \otimes 3$. This completes Case 4.

Case 5 G = simply connected split real form of F_4. Here $K = SU(2) \times SP(3)$. A torus for K consists of the pairs of diagonal matrices

$$\left[\begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0}\end{pmatrix}, \begin{pmatrix} e^{i\theta_1} & & & & & \\ & e^{i\theta_2} & & & 0 & \\ & & e^{i\theta_3} & & & \\ & & & e^{-i\theta_1} & & \\ & 0 & & & e^{-i\theta_2} & \\ & & & & & e^{-i\theta_3}\end{pmatrix}\right] = [e^{i\theta_0},\dots,e^{i\theta_3}].$$

We use the θ_i as coordinates in t_0; then it_0' is identified with $\mathbb{R}^4$ in the dual coordinates, with the dual basis $\{e_0,e_1,e_2,e_3\}$; after normalization, the invariant inner product is the usual one. The compact roots are $\pm 2e_0$ (the roots of SU(2)) and $\pm 2e_i$, $\pm e_i \pm e_j$ for $1 \leq i, j \leq 3$, $i \neq j$ (the roots of SP(3)). The noncompact roots are $\pm e_0 \pm e_i$, $1 \leq i \leq 3$, and $\pm e_0 \pm e_1 \pm e_2 \pm e_3$. (This is the dual of the usual presentation of the roots of F_4; cf. Humphreys ([11]) p. 43.)

Let $\sigma = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\right] \varepsilon$ K. Then $\sigma^2 = (-I,-I) = [-1,-1,-1,-1] = (e^{i\pi},e^{i\pi},e^{i\pi},e^{i\pi})$, which is central in K. We claim that $\bar{\sigma}^2$ acts trivially on $\mathfrak{p}$; since $\mathfrak{p}$ is K-irreducible, and σ^2 is central in K, it is enough to check this on one weight vector of $\mathfrak{p}$, say

(1,1,1,1). Now $[e^{i\theta_0},\ldots e^{i\theta_3}]$ acts on a weight (x_0,x_1,x_2,x_3) by the scalar $e^{i\Sigma\theta_i x_i}$; so σ^2 acts on (1,1,1,1) by $e^{4\pi i} = 1$.

As a strongly orthogonal system of noncompact roots, one can choose $(1, \pm 1, \pm 1, \pm 1)$, with an odd number of minus signs; clearly there are four such roots. Apply Lemma 5.6 with $\sigma_i = \sigma$ for all i. Since $\mathfrak{p}^{\mathfrak{t}} = 0$, we get a maximal abelian subalgebra $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$, with $\sigma \in M$. We want to compute $M \cap T$. $[e^{i\theta_0}\ldots e^{i\theta_3}]$ is in $M \cap T$ iff it acts trivially on each of the strongly orthogonal weights (and their negatives.) This amounts to $\theta_0-\theta_1+\theta_2+\theta_3$, $\theta_0+\theta_1-\theta_2+\theta_3$, $\theta_0+\theta_1+\theta_2-\theta_3$, and $\theta_0-\theta_1-\theta_2-\theta_3$ being elements of $2\pi\mathbb{Z}$. Using this condition, one checks easily that $[\pm 1, \pm 1, \pm 1, \pm 1]$ (even number of minus signs) and $[\pm i, \pm i, \pm i, \pm i]$ (odd number of minus signs) are in $M \cap T$, for 16 elements in all. Also $\sigma \in M$; clearly σ and $M \cap T$ generate a group of order 32. On the other hand, a computation like that given for G_2 shows that G is a two-sheeted cover of its adjoint group, so that $|M| \leq 2.2^4 = 32$. So this exhausts M.

M has a direct product decomposition

$M = \langle[-i,i,i,i], [-1,-1,-1,-1], \sigma\rangle \times \langle 1,1,-1,-1]\rangle \times \langle[1,-1,-1,1]\rangle$

$= P \times Q \times R$ (here < > denotes "group generated by".) This is quite easy to check, using the fact that if $t \in T$, $\sigma t \sigma^{-1} = t^{-1}$. P is the non-abelialn group of order 8, which was studied as "M for G_2." Q and R are of course isomorphic to $\mathbb{Z}_2$. Thus the representations of M come in sets of four, each set corresponding to an element of $\hat{P}$ and parametrized by the action of Q and R. We continue to denote representations of $SU(2)$ by their dimensions; also write "1" and "6" for the trivial and six dimensional representations of $SP(3)$ respectively. Then the small K-types are $1 \otimes 1$, $2 \otimes 1$, $3 \otimes 1$, $1 \otimes 6$, and $2 \otimes 6$. $1 \otimes 1$ is obviously the trivial M type. $2 \otimes 1$ and $3 \otimes 1$ are trivial on $Q \times R$, since this group projects to I in the first factor (i.e. if $(T_1,T_2) \in Q \times R$, then $T_1 = I \in SU(2)$). The projection of P onto the first factor is "M for G_2", so $2 \otimes 1$ is the 2-dimensional representation of P, and $3 \otimes 1$ is the 3 non-trivial one dimensional representations of P. $[-1,-1,-1,-1] \in P$ acts by -1 on $1 \otimes 6$, so this must split into three copies of the two-dimensional representation of P: if x_1,x_2,x_3,y_1,y_2,y_3 is the usual basis of $\mathbb{C}^6$ as an $SP(3)$ module, the corresponding components are $\langle x_i,y_i\rangle$, as is easy to check. $Q \times R$ acts on each of these pieces by a different character, the three non-trivial characters of $Q \times R$. Finally, the 2 dimensional representation of P tensored with itself is the sum of the 4 one dimensional representations; from our description of $2 \otimes 1$ and $1 \otimes 6$, this implies that $2 \otimes 6$ consists of the 12 one dimensional representations of M which are non-trivial on $Q \times R$. Thus all M types occur in the given list, each with multiplicity one. Leaving the verification that these are small to the reader, it remains to show M' irreducibility. This is of course obvious for $1 \otimes 1$ and $2 \otimes 1$.

To compute some elements of M', observe first that $\mathfrak{a} = \mathfrak{p}^M$. For it is easy to see that the only noncompact roots on which $M \cap T$ acts trivially are the given strongly orthogonal set and their negatives; and $\mathfrak{a}$ is precisely the space of σ-invariants in the span of those 8 root vectors. If $g \in K$, it follows that $g \cdot \mathfrak{a} = \mathfrak{p}^{gMg^{-1}}$, so that M' is the normalizer of M in K. Suppose $t \in T$. Then $t\sigma t^{-1} = t(\sigma t^{-1}\sigma^{-1})\sigma = t \cdot t \cdot \sigma = t^2\sigma$, which is in M iff $t^2 \in M$. Since T normalizes $M \cap T$, $\underline{t \in M' \text{ iff } t^2 \in M}$. Using this, it is easy to check that each of the weight spaces in any one of the representations $3 \otimes 1$, $1 \otimes 6$, and $2 \otimes 6$ transforms according to a different character of $M' \cap T$. To prove irreducibility, it is therefore enough to show that any of these weight spaces is cyclic under M'. For definiteness, we restrict attention to $2 \otimes 6$. Let (x_0, y_0) be the usual basis of $\mathbb{C}^2$ as an $SU(2)$ module. Considering the action of σ, it is enough to show that one of the subspaces $V_i = \langle x_0 \otimes x_i, y_0 \otimes y_i \rangle$, $W_i = \langle y_0 \otimes x_i, x_0 \otimes y_i \rangle$, $i = 1, 2, 3$ has components in all the others under the action of M'. Let ψ be a 3×3 permutation matrix such that $\psi^2 = I$; then $\psi = \psi^t = \psi^{-1}$, so $\gamma_\psi = \left(I, \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix}\right)$ commutes with σ, and it follows easily that $\gamma_\psi \in M'$. Clearly $(\gamma_\psi) \cdot V_i = V_{\psi(i)}$, $(\gamma_\psi) \cdot W_i = W_{\psi(i)}$. So it is enough to show that $M' \cdot (\Sigma V_i)$ has a component in ΣW_i. For this we use the element $g = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix}\right) \in K$. It can be checked that g commutes with Q and R, $g^{-1}\sigma g = [i,-i,-i,-i]$, and $g^{-1}[i,-i,-i,-i]g = -\sigma$. So $g \in M'$; and $g \cdot (x_0 \otimes x_1) = \frac{1}{2}(x_0 + iy_0) \otimes (x_1 - iy_1)$, which has a component in ΣW_i. So $2 \otimes 6$ is M' irreducible, which completes Case 5.

<u>Case 6</u> G = simply connected split form of E_6. Here $K = SP(4)$. We use the coordinates θ_i in the Lie algebra of the torus

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & & & & & & 0 \\ & \ddots & & & & & & \\ & & e^{i\theta_4} & & & & & \\ & & & e^{-i\theta_1} & & & & \\ & & & & \ddots & & & \\ 0 & & & & & & & e^{-i\theta_4} \end{pmatrix} = [e^{i\theta_1}, \ldots e^{i\theta_4}] \right\}.$$

With respect to the usual basis $e_1 \ldots e_4$ of $i\mathfrak{t}_0' \cong \mathbb{R}^4$, the compact roots are $\pm 2e_i$, $\pm e \pm e_j$ $(i \neq j)$. $\mathfrak{p}$ is a 42 dimensional subrepresentation of $\Lambda^4 \mathbb{C}^8$, the fourth exterior power of the standard $\mathbb{C}^8$ representation of $SP(4)$; its weights are $\pm e_1 \pm e_2 \pm e_3 \pm e_4$, $\pm e_i \pm e_j$ $(i \neq j)$, and two zero weights. By the general results of section 2, the imaginary noncompact roots are those which are not also compact, i.e. $\pm e_1 \pm e_2 \pm e_3 \pm e_4$. Notice in particular that $\mathfrak{p}$ contains any element of $\Lambda^4 \mathbb{C}^8$ of weight $\pm e_1 \pm e_2 \pm e_3 \pm e_4$, since these occur with multiplicity one in $\Lambda^4 \mathbb{C}^8$. Also the center of K, which has order 2, acts trivially in $\mathfrak{p}$; so the center of G has order 2, and

$|M| \leq 2 \cdot 2^6 = 128$.

Let $\sigma \in SP(4)$ denote the element $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Consider the group $D = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, r, s, rs\}$ $(r^2 = s^2 = 1)$. Interpret $\mathbb{C}^4$ as $L^2(D)$, and take $x_1, \ldots, x_4, y_1 \ldots y_4$ as a basis for $\mathbb{C}^8 = \mathbb{C}^4 \oplus \mathbb{C}^4$. For $g \in D$, let P_g denote the 4×4 matrix defining the action of g in the regular representation: e.g. $P_r = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$.

Clearly $P_{g_1} P_{g_2} = P_{g_1 g_2}$; in particular $P_g^2 = 1$, so $P_g^t = P_g^{-1} = P_g$. Define $\sigma_g \in SP(4)$ by $\sigma_g = \begin{pmatrix} P_g & 0 \\ 0 & P_g \end{pmatrix}$; then the σ_g commute with σ. Now consider $x_1 \ldots x_4$, $y_1 \ldots y_4$ as the usual basis of the SP(4) module $\mathbb{C}^8$; notice that $\sigma \cdot x_i = -y_i$, $\sigma \cdot y_i = x_i$. If χ is a character of D, then $\chi(g) = +1$ or -1 for $g \in D$; $\frac{1-\chi(g)}{2} = 0$ or 1 accordingly. Set

$$b_\chi = \sigma^{\frac{1-\chi(1)}{2}} \cdot x_1 \wedge \ldots \wedge \sigma^{\frac{1-\chi(rs)}{2}} \cdot x_4 \in \Lambda^4 \mathbb{C}^8.$$

Since x_i has weight e_i, and y_i has weight $-e_i$, b_χ has weight

$\beta_\chi = (\chi(1),\chi(r),\chi(s),\chi(rs))$; so $b_\chi \in \mathfrak{p}$. It is easy to check that $\{\beta_\chi\}_{\chi \in \hat{D}}$ is a strongly orthogonal set of real noncompact roots, and of course $\sigma \cdot \beta_\chi = -\beta_\chi$. By Lemma 5.6, since $\sigma^2 = -I$ is central in G, we may take $a_\chi = b_\chi + \sigma \cdot b_\chi$, together with $\mathfrak{p}^{\mathfrak{t}}$, as a basis of $\mathfrak{a}$.

We claim that σ and all σ_g are in M. Obviously σ fixes the a_χ; and

$$\sigma_g \cdot a_\chi = \sigma_g \cdot ([\sigma^{\frac{1-\chi(1)}{2}} \cdot x_1 \wedge \ldots \wedge \sigma^{\frac{1-\chi(rs)}{2}} \cdot x_4] + \sigma[\quad])$$

$$= ([\sigma^{\frac{1-\chi(1)}{2}} \cdot (\sigma_g \cdot x_1) \wedge \ldots \wedge \sigma^{\frac{1-\chi(rs)}{2}} \cdot (\sigma_g \cdot x_4)] + \sigma[\quad])$$

(since σ and σ_g commute)

$$= ([\sigma^{\frac{1-\chi(1)}{2}} \cdot x_{g\cdot 1} \wedge \ldots \wedge \sigma^{\frac{1-\chi(rs)}{2}} \cdot x_{g\cdot 4}] + \sigma[\quad]).$$

Since $\sigma^2 = 1$, it is easy to see that

$$\sigma^{\frac{1-\chi(g)}{2}} \cdot \sigma^{\frac{1-\chi(g_1)}{2}} = \sigma^{\frac{1-\chi(gg_1)}{2}}.$$

Therefore (recalling $g^2 = 1$)

$$= \sigma^{\frac{1-\chi(g)}{2}} \cdot ([\sigma^{\frac{1-\chi(g\cdot 1)}{2}} x_{g\cdot 1} \wedge \ldots \wedge \sigma^{\frac{1-\chi(g\cdot rs)}{2}} \cdot x_{g\cdot 4}] + \sigma[\quad])$$

But the permutation of (1, 2, 3, 4) defined by the regular action of g is necessarily even; so

$$= \sigma^{\frac{1-\chi(g)}{2}}([\sigma^{\frac{1-\chi(1)}{2}} x_1 \wedge \ldots \wedge \sigma^{\frac{1-\chi(rs)}{2}} x_4] + \sigma[\quad])$$

$$= \sigma^{\frac{1-\chi(g)}{2}} \cdot a_\chi = a_\chi$$

since a_χ is fixed by σ. Thus a_χ is fixed by σ_g. That σ and σ_g fix $\mathfrak{p}^t$ is just an explicit computation, which we only sketch. Write $z_i = x_i \wedge y_i$; then $z_i \wedge z_i = 0$, $z_i \wedge z_j = z_j \wedge z_i$, and $\mathfrak{p}^t$ is spanned by those combinations of $z_i \wedge z_j$ which lie in $\mathfrak{p} \subseteq \Lambda^4 \mathbb{C}^8$. We have seen that $x_1 \wedge \ldots \wedge x_4 \in \mathfrak{p}$; applying appropriate elements of the Lie algebra of $SP(4)$ to this, one gets $(z_1 - z_2) \wedge (z_3 - z_4)$ and $(z_1 - z_3) \wedge (z_2 - z_4)$ in $\mathfrak{p}$; by dimension, these span $\mathfrak{p}^t$. Now $\sigma \cdot z_i = \sigma \cdot x_i \wedge \sigma \cdot y_i = -y_i \wedge x_i = x_i \wedge y_i = z_i$; so σ fixes $\mathfrak{p}^t$. For reasons of symmetry, it is enough to consider any non-identity σ_g, say σ_r. One checks that $\sigma_r(z_1) = z_2$, $\sigma_r(z_2) = z_1$, $\sigma_r(z_3) = z_4$, and $\sigma_r(z_4) = z_3$; hence $\sigma_r((z_1 - z_2) \wedge (z_3 - z_4)) = (z_2 - z_1) \wedge (z_4 - z_3) = (z_1 - z_2) \wedge (z_3 - z_4)$, and $\sigma_r((z_1 - z_3) \wedge (z_2 - z_4)) = (z_2 - z_4) \wedge (z_1 - z_3) = (z_1 - z_3) \wedge (z_2 - z_4)$. This completes the proof that σ and the σ_g are in M.

Let M_W be the 16 element abelian group generated by σ and the σ_g. Notice that M_W normalizes T. The 16 element (abelian) subgroup of T consisting of $[\pm 1, \pm 1, \pm 1, \pm 1]$ (even number of minus signs) and $[\pm i, \pm i, \pm i, \pm i]$ (even number of minus signs) is also contained in M; this is checked exactly as for F_4. If we call this subgroup M_T, then M_W normalizes M_T; $M_W \cap M_T = \{\pm I\}$; so $M_T M_W$ has $\frac{16 \cdot 16}{2} = 128$ elements, and is therefore all of M. (Recall that we saw $|M| \leq 128$.)

Next, we claim that $\mathfrak{a} = \mathfrak{p}^M$; in fact $\mathfrak{a} = \mathfrak{p}^{\langle M_T, \sigma \rangle}$. For one checks that the only weights of $\mathfrak{p}$ invariant under M_T are $\pm \beta_\chi$ and 0, and the two dimensional space corresponding to the weights $\pm \beta_\chi$ has only one σ invariant. It follows as before that M' is precisely the normalizer of M in K.

Since $M/\pm 1$ is abelian, there are at least 64 one dimensional representations in $\hat{M}$; one may check that each occurs with multiplicity one in the representation $\mathbb{C}^8 \otimes \mathbb{C}^8$ of K. It follows by Schur's lemma that $\mathbb{C}^8$ is irreducible under M; since $128 = 8^2 + 64$, this exhausts $\hat{M}$. Now $\mathbb{C}^8 \otimes \mathbb{C}^8$ decomposes as a K representation into $S^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8) = S^2 \oplus \pi_2 \oplus 1$; here S^2 is the symmetric product, 1 is the exterior form defining SP(4) (a trivial

K-type) and π_2 is a 27 dimensional subrepresentation of $\Lambda^2(\mathbb{C}^8)$; these are irreducible. We claim that 1, $\mathbb{C}^8$, π_2, and S^2 are small; the proof is as usual left to the reader. It remains to check that π_2 and S^2 are M' irreducible.

Let Σ_4 denote the permutation group on the 4 element set D; recalling $\mathbb{C}^4 \cong L^2(D)$, an element of Σ_4 corresponds to a 4×4 matrix P, which in turn gives an element $\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \in SP(4)$. D is a normal subgroup of Σ_4, and the conditions defining M_T are invariant under permutation of coordinates; so it is not hard to check that $\Sigma_4 \subseteq M'$. Also the elements $[-1,1,1,1]$, $[1,-1,1,-]$,... $[1,1,1,-1]$ of T are in M'. Using these, together with M_T, one sees that every weight space of π_2 or S^2 transforms according to a distinct character of $M' \cap T$. Hence every M'-irreducible component of those representations is a sum of weight spaces. For definiteness we restrict attention to S^2. Considering the action of Σ_4 and σ, one sees that every M'-irreducible component of S^2 is a sum of certain of the four spaces $V_1 = \langle x_i^2, y_i^2 \rangle$, $V_2 = \langle x_i x_j, y_i y_j \mid i \neq j \rangle$,

$$V_3 = \langle x_i y_j \mid i \neq j \rangle,\ V_4 = \langle x_i y_i \rangle.$$

To prove irreducibility, we exhibit two more elements of M' which mix these subspaces. Put $g_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ -iI & I \end{pmatrix}$

and $g_2 = \frac{1}{2}\begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix}$, where $\chi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ is

the matrix of characters of D. One may thing of g_1 and g_2 as diagonalizing the commutative families $\{1,\sigma\}$ and $\{\sigma_g\} \cong D$ respectively. From this perspective, or by explicit computation, one can check that g_1 and g_2 normalize M, and hence belong to M'. Clearly $g_1 \cdot V_1$ meets V_4, $g_1 \cdot V_2$ meets V_3, and $g_2 \cdot V_1$ meets V_2. Thus S^2 is M' irreducible, which completes case 6.

<u>Case 7</u> G = simply connected split forms of E_7 and E_8. These are entirely analogous to E_6; since E_8 involves some rather messy computations in a Clifford algebra, we give details in that case. So K = Spin (16). Using the usual Cartan subgroup on the SO(16) level to give an identification of $i\mathfrak{t}_0'$ with $\mathbb{R}^8$, the highest weight of $\mathfrak{p}$ is $(\frac{1}{2},\dots \frac{1}{2})$; $\mathfrak{p}$ is the positive half spin representation. One knows that two of the four elements in the center of K act trivially on $\mathfrak{p}$ (this will be

seen explicitly below) so the center of G has order 2. Thus $|M| \leq 2 \cdot 2^8 = 512$.

Define $D = Z_2 \times Z_2 \times Z_2$, with generators r, s, t. Writing D = {1,r,s,rs,t,rt,st,rst}, we identify $i\mathfrak{t}_0' \cong \mathbb{R}^8$ with $L^2(D)$. A strongly orthogonal set of weights for $\mathfrak{p}$ is

$$\{\tfrac{1}{2}\chi \mid \chi \in \hat{D}\} = \{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),\ (\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}),\ \text{etc.}\}.$$

For each χ we choose $u_\chi \in \mathfrak{p}$ of weight $\chi/2$. There is a representative σ of the -1 element of W_K so that $\sigma^2 = 1$ (as will be seen below.) By Lemma 5.6, $\{u_\chi + \sigma \circ u_\chi\}$ is a basis for $\mathfrak{a}$. M should now be generated by $M \cap T = M_T$, σ, and a certain copy of D in the normalizer of T in K. Unfortunately I know of no easy way to guarantee the existence of this copy of D other than direct computation. So...

Consider the complexified Clifford algebra C generated by $\mathbb{R}^{16}$ with the usual basis $e_1 \ldots e_{16}$ (cf. Chevalley [5]); then of course $e_i^2 = -1$, $e_i e_j = -e_j e_i$ for $i \neq j$. There is an exponential map $\exp : C \to C^*$, the regular elements of C. The <u>real</u> subspace $C_2 = \langle e_i e_j \mid i \neq j \rangle$ is a Lie algebra, and

$K = \exp C_2$ is Spin (16). For $j = 1, \dots 8$, define

$$x_j = e_{2j-1} + ie_{2j} \quad y_j = e_{2j-1} - ie_{2j} \quad z_j = \frac{1}{2}(e_{2j-1}e_{2j}).$$

Then
$$\begin{aligned} z_j x_j &= \frac{1}{2}(e_{2j-1}e_{2j}e_{2j-1} + ie_{2j-1}e_{2j}^2) \\ &= \frac{1}{2}(-e_{2j-1}^2 e_{2j} + ie_{2j-1}e_{2j}^2) \\ &= \frac{1}{2}(e_{2j} - ie_{2j-1}) = -\frac{i}{2}(e_{2j-1} + ie_{2j}) \\ &= -\frac{i}{2}x; \end{aligned}$$

The other identities below are similar.

$$z_j x_j = -\frac{i}{2}x_j \qquad z_j y_j = \frac{i}{2}y_j$$

$$(*) \quad \left.\begin{array}{lll} x_i x_j = -x_j x_i & x_i y_j = -y_j x_i & y_i y_j = -y_j y_i \\ z_i z_j = z_j z_i & z_i x_j = x_j z_i & z_i y_j = y_j z_i \end{array}\right\} \; i \neq j.$$

We may take $\langle z_i \rangle$ as the Lie algebra of the Cartan subgroup $T \subseteq K$. Since $z_i^2 = -\frac{1}{4}$, one easily computes $\exp(\lambda z_i) = \cos\frac{\lambda}{2} + 2z_i \sin\frac{\lambda}{2}$; in particular, $\exp \pi z_i = 2z_i$, and $\exp(2\pi z_i) = 1$.

One knows (as was pointed out to me by Kostant) that $Y = y_1 \cdots y_8$ generates a maximal left ideal in C; so the unique irreducible C-module is CY. This has a basis indexed by subsets S of $\{1,\ldots 8\}$; $X_S = (\prod_{i\varepsilon S} x_i)Y$. Set $\sigma = e_1e_3e_5 \ldots e_{13}e_{15}$. Then $\sigma^2 = 1$, and $\sigma z_i\sigma^{-1} = -z_i$. (It is easy to see that $\sigma = \prod_{j=1}^{4} \exp(\frac{\pi}{2}e_{4j-3}e_{4j-1}) \varepsilon K$.) By (*), X_S has weight $(\pm\frac{i}{2}, \pm\frac{i}{2}, \ldots \pm\frac{i}{2})$ with respect to $\mathfrak{t}$; here the minus signs correspond precisely to elements of S. So we may take as a model for the $\mathfrak{p}$ representation the span of the vectors X_S, where S has even order. If $\chi \varepsilon \hat{D}$, let S_χ denote the set where $\chi = -1$; of course $|S_\chi| = 4$ or 0. Then $\mathfrak{a}$ has basis $\{X_{S_\chi} + \sigma\cdot X_{S_\chi}\}$. Exactly as for the other exceptional groups, one easily checks that $M \cap T$ contains

$$\prod_{i\notin S_\chi} 2z_i, \quad \prod_{i\varepsilon S_\chi} 2z_i, \quad \left[\prod_{i\varepsilon S_\chi} \frac{1}{\sqrt{2}}(1\pm 2z_i)\right]\left[\prod_{i\notin S_\chi} \frac{1}{\sqrt{2}}(1\mp 2z_i)\right]$$

(one choice of sign throughout.) These four elements are distinct for each χ, giving 32 in all. For $\chi = 1$, the element $\prod_{i\notin S_\chi} 2z_i = 2^8\cdot z_1\cdot z_2\ldots z_8\cdot = e_1e_2e_3e_4\ldots e_{15}e_{16}$ is the promised central element of K which acts trivially on $\mathfrak{p}$. Clearly σ is also in M, and normalizes T. The resulting 64 element group already picks out $\mathfrak{a}$: in fact, set $M_0 = \langle\sigma, 2^4z_1z_2z_3z_4, 2^4z_1z_3z_5z_7, 2^4z_1z_2z_5z_6\rangle \subseteq M$.

This is a 16 element abelian group, which is in some sense dual to the group generated by D and σ (once we show $D \subseteq M$.)

We claim that $\mathfrak{A} = \mathfrak{P}^{M_0}$. Arguing as for E_6, it is enough to show that the various $\pm \frac{X}{2}$ are the only weights annihilated by $M_0 \cap T$. Notice that, for example, $2^4 z_1 z_2 z_3 z_4$ acts on the weight $(\frac{\varepsilon_1}{2}, \ldots \frac{\varepsilon_8}{2})$ $(\varepsilon_i = \pm 1)$ by $(\varepsilon_1 i)(\varepsilon_2 i)(\varepsilon_e i)(\varepsilon_4 i) = \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4$. Thus an even number of $\varepsilon_1 \ldots \varepsilon_4$ are ± 1; if $(\frac{\varepsilon_i}{2})$ is a weight of $\mathfrak{P}$, it follows also that an even number of $\varepsilon_5 \ldots \varepsilon_8$ are ± 1. Proceeding along these lines, it is easy to see that $(\varepsilon_1 \ldots \varepsilon_8)$ or $-(\varepsilon_1 \ldots \varepsilon_8)$ is a character of D; details are left to the reader. It follows that any element of K which conjugates the four generators of M_0 into M, is necessarily in M'.

We want to exhibit D inside M. The element of W_K transposing z_i and z_j has a representative $\sigma_{ij} = \frac{1}{2}(1 + e_{2i-1}e_{2j-1})(1 + e_{2i}e_{2j})$. Every non-identity element of D, acting in the regular representation on $L^2(D) \cong i\mathfrak{t}_0' \cong \mathbb{R}^8$, may be represented as a product of four such transpositions, e.g. $\sigma_r = \sigma_{12}\sigma_{34}\sigma_{56}\sigma_{78}$. We claim that there is a certain choice of this representative σ_g such that $\sigma_g \in M$. Clearly each σ_{ij} commutes with σ, so it is enough to show that if $g \in D$, $\sigma_g \cdot X_{S_\chi} = X_{S_\chi}$ or $\sigma \cdot X_{S_\chi}$ according as $\chi(g) = 1$ or -1. For purely formal reasons, these equations hold up to sign; verifying that they hold exactly is the only reason for considering the Clifford algebra. Set $w_{ij} = e_{2i-1}e_{2j-1}$. Suppose $\{(i_1,j_1),\ldots(i_4,j_4)\} = \{1,\ldots 8\}$; let ε be the sign of the permutation $((i_\ell,j_\ell)) \to (1,\ldots 8)$. Then by (*) $\sigma = (\varepsilon) \prod_{\ell=1}^{4} w_{i_\ell j_\ell}$. By direct computation,

$$\sigma_{ij}\cdot(y_iy_j) = y_iy_j \qquad \sigma_{ij}(x_ix_jy_iy_j) = x_ix_jy_iy_j$$

$$(**) \qquad \sigma_{ij}(x_iy_iy_j) = x_jy_iy_j \qquad w_{ij}(x_iy_iy_j) = x_jy_iy_j$$

$$\sigma_{ij}(x_jy_iy_j) = -x_iy_iy_j \qquad w_{ij}(x_jy_iy_j) = -x_iy_iy_j$$

σ_{ij}, w_{ij} commute with x_k, y_k for $k \neq i,j$.

σ_{ij} and w_{ij} commute with $\sigma_{k\ell}$ and $w_{k\ell}$ if $\{i,j\} \neq \{k,\ell\}$.

Fix $g \neq 1$, and a character $\chi \in \hat{D}$. Suppose first that $\chi(g) = 1$. Write $\{1,\ldots 8\} = \{(i_1,j_1)\ldots(i_4,j_4)\}$ with $g\cdot i_\ell = j_\ell$, and set $\sigma_g = \sigma_{i_1j_1}\cdots\sigma_{i_4j_4}$. (Later we will put some conditions on the choice of the (i_ℓ, j_ℓ).) Since $\chi(g) = 1$, $\chi(i_\ell) = \chi(j_\ell)$; set $L_\chi = \{\ell \mid \chi(i_\ell) = \chi(j_\ell) = -1\}$. Then for some fixed $\varepsilon_1 = \pm 1$, we have

$$X_{S_\chi} = \varepsilon_1 (\prod_{\ell \varepsilon L_\chi} x_{i_\ell} x_{j_\ell} y_{i_\ell} y_{j_\ell})(\prod_{\ell \not\varepsilon L_\chi} y_{i_\ell} y_{j_\ell}).$$

Thus $\sigma_g \cdot X_{S_\chi} = \varepsilon_1 [\prod_{\ell \varepsilon L_\chi} \sigma_{i_\ell j_\ell} \circ (x_{i_\ell} x_{j_\ell} y_{i_\ell} y_{j_\ell})][\prod_{\ell \not\varepsilon L_\chi} \sigma_{i_\ell j_\ell} (y_{i_\ell} y_{j_\ell})]$

$$= \varepsilon_1 \prod_{\ell \varepsilon L_\chi} (x_{i_\ell} x_{j_\ell} y_{i_\ell} y_{j_\ell}) \prod_{\ell \not\varepsilon L_\chi} (y_{i_\ell} y_{j_\ell})$$

by (**); $= X_{S_\chi}$.

So the desired equation holds for any representative if $\chi(g) = 1$. If $\chi(g) = -1$, then $\chi(i_\ell) = -\chi(j_\ell)$. Set $L_\chi = \{\ell \mid \chi(i_\ell) = -1\}$. Since the order of the various pairs (i_ℓ, j_ℓ) is at our disposal, we may assume that $(1, \ldots 8) \to ((i_1, j_1) \ldots (i_4, j_4))$ is an <u>even</u> permutation; then $\sigma = \prod_{\ell=1}^{4} w_{i_\ell j_\ell}$. On the other hand, for some $\varepsilon_2 = \pm 1$, we have $X_{S_\chi} = \varepsilon_2 (\prod_{\ell \varepsilon L_\chi} x_{i_\ell} y_{i_\ell} y_{j_\ell})(\prod_{\ell \not\varepsilon L} x_{j_\ell} y_{i_\ell} y_{j_\ell})$

Using the middle equations of (**) just as in the previous case, one sees that $\sigma_g \cdot X_{S_\chi} = \sigma \cdot X_{S_\chi}$. So $\sigma_g \varepsilon M$.

It is fairly clear that the 32 elements of M_T, σ, and the 8 elements of D generate a group of order $32 \cdot 2 \cdot 8 = 512$; so they exhaust M. The subgroup M_T is normal; σ acts on it by the inverse map, and the various σ_g by permutation. With this structural information, and the various explicit formulae given, it is not hard to work out the representation theory of M: there are 256 one dimensional representations, each occurring with multiplicity one in the $(\mathbb{R}^{16} \otimes \mathbb{R}^{16})_{\mathbb{C}}$ representation of K. By Schur's lemma, $(\mathbb{R}^{16})_{\mathbb{C}}$ is M-irreducible; since $16^2 + 256 = 512$, this accounts for all of $\hat{M}$. Now $(\mathbb{R}^{16} \otimes \mathbb{R}^{16})_{\mathbb{C}} = \Lambda^2 \oplus S^2 = \Lambda^2 \oplus (\pi_2 \oplus 1)$; components of dimension 120, 135, and 1 (the invariant quadratic form defining SO(16)). These components are irreducible under K. To complete the proof, we must show that Λ^2 and π_2 are M'-irreducible. Recall that $g \in M'$ iff $gM_0g^{-1} \subseteq M$, where M_0 is generated by σ and 3 elements of M_T. As for F_4, it follows immediately that $x \in T$, $x^2 \in M \Rightarrow x \in M'$.

We now consider $\{x_i, y_i\}$ as a basis for the $(\mathbb{R}^{16})_{\mathbb{C}}$ representation of K, which we denote by $\cdot$. One knows that if $g \in K$, $g \cdot x_i = g x_i g^{-1}$. Then $\sigma \cdot x_i = y_i$, $\sigma_g \cdot x_i = x_{g \cdot i}$, $\sigma_g \cdot y_i = y_{g \cdot i}$, etc. For the differentiated representation,

$z_i \cdot x_i = -ix_i$, $z_i \cdot y_i = iy_i$, $z_i \cdot x_j = z_i \cdot y_j = 0$ for $i \neq j$.

Set $u_{ij} = \exp(\pi z_i + \pi z_j)$. Since $u_{ij} \varepsilon T$, and $u_{ij}^2 = 1 \varepsilon M$, it follows that $u_{ij} \varepsilon M'$. Consider for definiteness Λ^2. Using the u_{ij}, together with various square roots of the elements $\Pi \frac{1}{\sqrt{2}}(1 \pm 2z_i) \varepsilon M$, it is easy to see that the $M' \cap T$-primary decomposition of Λ^2 is just the weight space decomposition.

Now every σ_{ij} commutes with σ, so any product of σ_{ij} normalizing M_T is in M'. In particular, whenever the permutation π of $\{1,\ldots 8\}$ defines an automorphism of D, then π expressed as a product of σ_{ij} is in M'. D together with Aut D (D acting by the regular action) is doubly transitive on $\{1,\ldots 8\}$. It follows that the M'-irreducible components of Λ^2 are sums of the following subspaces:

$$V_1 = \langle x_i \wedge x_j,\ y_i \wedge y_j \mid i \neq j \rangle$$

$$V_2 = \langle x_i \wedge y_j,\ y_i \wedge x_j \mid i \neq j \rangle$$

$$V_3 = \langle x_1 \wedge y_1 + \ldots + x_8 \wedge y_8 \rangle$$

$$V_4 = \langle x_i \wedge y_i - x_j \wedge y_j \rangle.$$

(The zero weight space $V_3 \oplus V_4$ splits only as indicated because Aut D acts transitively on the non-trivial characters of D; one proves the splitting on the "Fourier transform side" of D.)

Consider $P = \frac{1}{\sqrt{2}}(1+e_2e_3)\frac{1}{\sqrt{2}}(1+e_6e_7)\frac{1}{\sqrt{2}}(1+e_{10}e_{11})\frac{1}{\sqrt{2}}(1+e_{14}e_{15})$.

We claim $P \in M'$; one checks that the inner automorphism of C induced by P fixes all e_i, except $e_2 \to e_3$, $e_3 \to -e_2$, $\ldots e_{14} \to e_{15}$, $e_{15} \to -e_{14}$. Then a direct computation shows that $PM_0P^{-1} \subseteq M$. It also follows that

$P\cdot x_1 = P\cdot(e_1+ie_2) = e_1+ie_3 = \frac{1}{2}(x_1+y_1)+\frac{i}{2}(x_2+y_2)$: similarly for the other x_i, y_i. Thus

a) $P\cdot(x_1\wedge x_2) = \frac{i}{2}x_1\wedge y_1 - \frac{i}{2}x_2\wedge y_2 - \frac{1}{2}x_1\wedge y_2 - \frac{1}{2}x_2\wedge y_1$

b) $P\cdot(x_1\wedge y_1) = \frac{i}{2}(x_1+y_1)\wedge(x_2+y_2)$

Because of a) $P\cdot V_1$ has non-zero components in V_2 and V_4; by b), and similar equations for $x_i\wedge y_i$ in general, V_3 is not M'-invariant. Thus Λ^2 is M'-irreducible; π_2 is similar. Q.E.D.

TABLE 5.8

Small K-types (non-spherical) for simple split groups

G	K	Principal series minimal K-types	Associate Small K-types
SP(n,R)	U(n)	$(\varepsilon,\ldots\varepsilon,\varepsilon-1,\ldots\varepsilon-1)$	$(1\ldots1.0\ldots0)$ $(\varepsilon=0)$
		$(0 \leq \varepsilon < 1)$	(#1's = #-1's)
SL(2n,R)	Spin (2n)	$\Lambda^r(\mathbb{C}^{2n})$ $1 \leq r < n$	
		$(\Lambda^n)^-(\mathbb{C}^{2n})$	$(\Lambda^n)^+(\mathbb{C}^{2n})$
		$\text{spin}^{\pm}$	
SL(2n+1,R)	Spin (2n+1)	$\Lambda^r(\mathbb{C}^{2n+1})$ $1 \leq r \leq n$	
		spin	
SO(2n,2n)	Spin (2n) × Spin (2n)	$1 \otimes \Lambda^r(\mathbb{C}^{2n})$ $1 \leq r < n$	$\Lambda^r(\mathbb{C}^{2n}) \otimes 1$
		$1 \otimes (\Lambda^n)^-(\mathbb{C}^{2n})$	$1 \otimes (\Lambda^n)^+(\mathbb{C}^{2n})$,
			$(\Lambda^n)^{\pm}(\mathbb{C}^{2n}) \otimes 1$
		$\text{spin}^- \otimes \text{spin}^{\pm}$	$\text{spin}^+ \otimes \text{spin}^{\mp}$
		$1 \otimes \text{spin}^{\pm}$	
		$\text{spin}^{\pm} \otimes 1$	

G	K	non-spherical principal series minimal K-types	Associate small K-types
$SO(2n+1,2n+1)$	$Spin(2n+1) \times Spin(2n+1)$	$1 \otimes \Lambda^r(\mathbb{C}^{2n+1})$ $1 \leq r \leq n$	$\Lambda^r(\mathbb{C}^{2n+1}) \otimes 1$
		$spin \otimes spin$	
		$spin \otimes 1$	
		$1 \otimes spin$	
$SO(2n,2n-1)$	$Spin(2n) \times Spin(2n-1)$	$1 \otimes \Lambda^r(\mathbb{C}^{2n-1})$ $1 \leq r \leq n-1$	
		$spin^- \otimes spin$	$spin^+ \otimes spin$
		$1 \otimes spin$	
		$spin^- \otimes 1$	$spin^+ \otimes 1$
$SO(2n+1,2n)$	$Spin(2n+1) \times Spin(2n)$	$1 \otimes \Lambda^r(\mathbb{C}^{2n})$ $1 \leq r < n$	
		$1 \otimes (\Lambda^n)^-(\mathbb{C}^{2n})$	$1 \otimes (\Lambda^n)^+(\mathbb{C}^{2n})$
		$spin \otimes spin^-$	$spin \otimes spin^+$
		$1 \otimes spin^{\pm}$	
		$spin \otimes 1$	

G	K	non-spherical principal series minimal K-types	Associate small K-types
G_2	$SU(2) \times SU(2)$	$1 \otimes 2,\ 1 \otimes 3$	
F_4	$SU(2) \times SP(3)$	$2 \otimes 1,\ 3 \otimes 1,\ 1 \otimes 6,\ 2 \otimes 6$	
E_6	$SP(4)$	$\mathbb{C}^8,\ S^2(\mathbb{C}^8),$	
		$\pi_2 \subseteq \Lambda^2(\mathbb{C}^8)$	
E_7	$SU(8)$	$\Lambda^2(\mathbb{C}^8)$ 28	$\Lambda^2(\mathbb{C}^8)^*$
		$S^2(\mathbb{C}^8)$ 36	$S^2(\mathbb{C}^8)^*$
		adjoint 63	
		$\mathbb{C}^8$	
		$(\mathbb{C}^8)^*$	
E_8	Spin (16)	$\Lambda^2(\mathbb{C}^{16})$	
		$\pi_2 \subset S^2(\mathbb{C}^{16})$	
		$\mathbb{C}^{16}$	

<u>Remark</u> Suppose $[\mathfrak{g}_0, \mathfrak{g}_0] \cong s\ell(2,\mathbb{R})$. Let α be the noncompact imaginary root which is $\succ 0$ in the ordering of $i\mathfrak{t}_0'$. Then μ is principal series minimal iff

$$-1 \leq \frac{2\langle\mu,\alpha\rangle_{\mathfrak{k}}}{\langle\alpha,\alpha\rangle_{\mathfrak{k}}} < 1; \quad \text{it is small iff} \quad -1 \leq \frac{2\langle\mu,\alpha\rangle_{\mathfrak{k}}}{\langle\alpha,\alpha\rangle_{\mathfrak{k}}} \leq 1.$$

(This is just the case $SP(1,\mathbb{R})$ above; we mention it separately only for convenience.)

It is worth noting another technical property of small K-types. Recall the Iwasawa decomposition G = KAN. Corresponding to every root $\alpha \in \Delta(\mathfrak{n}_0, \mathfrak{a}_0)$, there is a map $s\ell(2,\mathbb{R}) \hookrightarrow \mathfrak{g}_0$. Let $X_\alpha \in \mathfrak{n}_0$ denote the image of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is $Z_\alpha = X_\alpha + \Theta X_\alpha \in \mathfrak{k}_0$; and $\{Z_\alpha\}_{\alpha\in\Delta(\mathfrak{n}_0,\mathfrak{a}_0)}$ is an orthogonal basis of $\mathfrak{k}_0$.

<u>Definition 5.9</u> (Bernstein, Gelfand, and Gelfand [1]) The representation γ of K is <u>fine</u> if the eigenvalues of Z_α in γ all lie in $[-1,1]$. (If G is linear, they are necessarily integers.)

<u>Theorem 5.10</u> (Bernstein, Gelfand, and Gelfand [1]).

<u>Let π be a finite dimensional irreducible representation of G on a vector space V, and γ a fine K-type. Let δ be the representation of M on the (one dimensional) space V^N. Then the multiplicity of γ in $\pi|K$ is</u>

<u>precisely the multiplicity of δ in $\gamma|M$</u>.

Amazingly enough, we have

<u>Proposition 5.11</u> <u>γ is fine <=> γ is small</u>

<u>Proof</u>. It is enough to assume G is simple. We then proceed on a case-by-case basis, using the following criteria for fineness and Table 5.7. Let $\beta_1 \ldots \beta_q$ be a strongly orthogonal spanning set for $\mathfrak{t}^*_{root}$ consisting of noncompact imaginary roots. We may assume that A is obtained from $\mathfrak{h}_0^{\mathfrak{t}}$ and a Cayley transform on the β_i. Then it is easy to see that the $s\ell(2,\mathbb{R})$ through β_i is one of the "root $s\ell(2,\mathbb{R})$'s" for $\Delta(\mathfrak{n}_0, \mathfrak{a}_0)$. The corresponding Z_α is in the β_i direction in $\mathfrak{t}_0$. There are several possibilities.

i) <u>All roots of $\mathfrak{g}_{\mathbb{C}}$ have the same length</u> (SL(n,$\mathbb{R}$), SO(n,n), E_6, E_7, E_8.) In this case all Z_α are conjugate under W, so it is enough to verify the fineness condition for one of them; say in the β_i direction in $\mathfrak{t}_0$. Computing for $s\ell(2,\mathbb{R})$, one sees that the fineness condition is

$-1 \leq \dfrac{2\langle\mu,\beta_i\rangle}{\langle\beta_i,\beta_i\rangle} \leq 1$ whenever μ is a weight of γ. Since all weights lie in the convex hull of the extremal ones, this is equivalent to

$-1 \leq \frac{2\langle\gamma,\beta\rangle}{\langle\beta,\beta\rangle} \leq 1$ for each noncompact imaginary root β.

ii) There are two root lengths, both occurring among the β_i ($SO(2n,2n-1)$, G_2). Similar reasoning leads to precisely the same condition.

iii) There are two root lengths, only one occurring among the β_i ($SO(2n+1,2n)$, $SP(n,\mathbb{R})$, F_4.) The β_i are necessarily long roots (by inspection; or the reader can provide the "general" proof.) One can find two β_i, say β_1 and β_2, such that $\frac{1}{2}(\beta_1 + \beta_2)$ is a short compact imaginary root. It is easy to see that the Lie algebra corresponding to these roots and their negatives is isomorphic to $sp(2,\mathbb{R})$; and it also corresponds to certain roots of $\mathfrak{a}_0$ in $\mathfrak{n}_0$. In this way one can get explicit control on Z_α when α is a short root. Computing for $sp(2,\mathbb{R})$, and reasoning as before, one gets the criterion for fineness: $-1 \leq \frac{2\langle\gamma,\beta\rangle}{\langle\beta,\beta\rangle} \leq 1$ for every long noncompact imaginary root; and $-1 \leq \frac{\langle\gamma,\alpha\rangle}{\langle\alpha,\alpha\rangle} \leq 1$ for every short compact imaginary root.

We leave to the reader the straightforward verification that the small K-types listed in Table 5.8 are the only dominant integral weights satisfying the conditions given above. Q.E.D.

Corollary 5.12 A finite dimensional irreducible representation of a split group contains exactly one associate class of small K-types, each with multiplicity one.

Proof Combine 5.4, 5.10, and 5.11.

It follows that the $A(\delta)$ form a "strong system of minimal types" for a split group, in the sense of Lepowsky ([19]). The considerations of section 6 give analogous results for arbitrary semisimple groups; this will be considered in detail in a later paper.

6. Existence of the Representations.

It will be convenient in this section to assume that K is compact. This makes available the theory of the discrete series in its most standard form. Essentially the same results hold without this assumption, however.

Fix a K-type μ. We assume that Conjecture 4.2 holds for μ; thus we are given a θ-invariant parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ with the following properties:

i) $\mathfrak{l}$ is split; i.e. if $\mathfrak{t}^+ = \mathfrak{t} \cap$ (center of $\mathfrak{l}$), and $\mathfrak{a}_0$ is a maximal abelian subalgebra of $\mathfrak{l} \cap \mathfrak{p}_0$, then $\mathfrak{t}^+ + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{l}$ (and $\mathfrak{g}$).

Write $\mu = (\mu_1, \mu_2)$ according to the decomposition $\mathfrak{t} = \mathfrak{t}^+ + \mathfrak{t}^-$; recall that $\mathfrak{t}^- = \mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}]$.

(6.1) ii) $\mu_2 - 2\rho(\mathfrak{n} \cap \mathfrak{p})|_{\mathfrak{t}^-}$ is a small $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{k}$ type; and if $\mu_2' - 2\rho(\mathfrak{n} \cap \mathfrak{p})|_{\mathfrak{t}^-}$ is an associate small $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{k}$ type, and (μ_1, μ_2') is dominant for $\mathfrak{k}$, then $(\mu_1, \mu_2) \leq (\mu_1, \mu_2')$.

iii) $\lambda = \mu + 2\rho_c - \rho$ is dominant with respect to the imaginary roots in $(\mathfrak{t}^+)^*$.

(6.1) iv) <u>If the irreducible Harish-Chandra module X has minimal K-type μ, then the action of $U(\mathfrak{g})^K$ on X^μ factors through ξ^μ.</u>

Let P = MAN be a Langlands decomposition of a cuspidal parabolic subgroup corresponding to $\mathfrak{a}_0 = \mathfrak{a}_0(\mu) \subseteq \ell \cap \mathfrak{p}_0$. Then $L \cap MA = T^+A$ is a Cartan subgroup of G; $T^+ \subseteq K$, and the identity component T_0^+ of T^+ has Lie algebra $\mathfrak{t}_0^+$. T^+ is a compact Cartan subgroup of M.

<u>Theorem 6.2.</u> <u>Suppose Conjecture 4.2 holds for the K-type μ; choose P = MAN as above. Then there is a certain tempered unitary representation $\delta_\mu \in \hat{M}$, in the "limit of the discrete series," such that</u>

i) <u>If $\nu \in \hat{A}$, $\operatorname{Ind}_{P\uparrow G} \delta_\mu \otimes \nu = \pi_\mu^\nu$ has minimal K-type μ; and μ occurs exactly once.</u>

ii) <u>If $\overline{\pi_\mu^\nu}$ is the unique subquotient of π_μ^ν containing μ, then $\{\overline{\pi}_\mu^\nu\}$ is precisely the set of irreducible quasisimple representations of G with minimal K-type μ, up to infinitesimal equivalence.</u>

<u>Proof.</u> Suppose we can prove i). Then ii) follows for formal reasons, which we only sketch. Recall the map

$$\xi^{\mu} : U^{\mathfrak{k}} \to U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}} / I_{\mu-2\rho(\mathfrak{a}\cap\mathfrak{p})}(\mathfrak{l}).$$

The latter ring may be embedded in $U(\mathfrak{a})$ in a natural way (Corollary 5.5) so we get a map $\xi_1 : U^{\mathfrak{k}} \to U(\mathfrak{a})$. One can also construct a map $\xi_2 : U^{\mathfrak{k}} \to U(\mathfrak{a})$, so that the $U^{\mathfrak{k}}$ character of π_{μ}^{ν} on the $\mathfrak{k}$-type μ is just $\xi_2 \circ$(evaluation at ν); and ξ_1 and ξ_2 agree on $\mathfrak{z}(\mathfrak{g})$. We therefore have maps (Spec R means the set of ring homomorphisms of R into $\mathbb{C}$)

$$\begin{array}{ccccc} & \xi_1 \nearrow & \text{Spec } U^{\mathfrak{k}} & \searrow & \\ \text{Spec } U(\mathfrak{a}) & & & & \text{Spec } \mathfrak{z}(\mathfrak{g}) \\ & \xi_2 \searrow & \text{Spec } U^{\mathfrak{k}} & \nearrow & \end{array}$$

so that the diagram commutes. By 6.1 (iv), Im $\xi_2 \subseteq$ Im ξ_1. By simple properties of the map Spec $U(\mathfrak{a}) \to$ Spec $\mathfrak{z}(\mathfrak{g})$ (which is essentially a Harish-Chandra homomorphism) and a little algebraic geometry, it follows that Im ξ_1 = Im ξ_2, which is essentially statement (ii) of the theorem.

So it is enough to prove i). Let M_0 denote the identity component of M. Let G_1 be some linear form of G, such that there is a covering map $G \overset{\pi}{\to} G_1$. (One knows that G has a finite dimensional representation which is faithful on $\mathfrak{g}$.) Set $F = \pi^{-1}(\pi(M) \cap \exp(i\mathfrak{a}_0))$, a finite group; then $F \subseteq M \cap L \cap K$, and F centralizes M_0. Put $M^\flat = M_0 F$. We will produce a representation δ_μ^0 in the limit of the discrete series of M_0, and a finite dimensional representation δ_μ^1 of F, so that δ_μ^0 and δ_μ^1 act by the same scalars on $M_0 \cap F$. Thus $\delta^0 \otimes \delta^1$ will be a well-defined representation of $M^\flat$; and we will set $\delta_\mu = \operatorname{Ind}_{M^\flat \uparrow M} \delta_\mu^0 \otimes \delta_\mu^1$.

Choose a strongly orthogonal set $\beta_1 \ldots \beta_q$ of noncompact positive imaginary roots which span $\mathfrak{t}^-$. We want to borrow some arguments from [22]. (It is assumed there that G is equal rank; but the results we need can easily be generalized from that case.) Via some Cayley transform, one can define a system Ψ of positive roots of T_0^+ in M, so that Ψ corresponds to $\Delta^+ \cap (\mathfrak{t}^+)^*$, the positive imaginary roots supported on $\mathfrak{t}^+$ (Schmid [22], p. 68). Then $2\rho_c(\Psi) - \rho(\Psi) = (2\rho_c - \rho)|_{\mathfrak{t}^+}$ ([22], p. 128), so that $\lambda|_{\mathfrak{t}^+} - 2\rho_c(\Psi) + \rho(\Psi) = \mu|_{\mathfrak{t}^+} = \mu_1$. We claim that μ_1 is dominant with respect to every compact root $\alpha \in \Psi$. If α is compact as an element of $\Delta(\mathfrak{g}, \mathfrak{h})$, this is immediate from the dominance of μ. Suppose α is noncompact in $\Delta(\mathfrak{g}, \mathfrak{h})$. Then ([22], p. 68) there is a root $\beta_i \in \{\beta_1 \ldots \beta_q\}$ which is <u>not</u> strongly orthogonal to α. Since $\alpha \perp \beta_i$, $\alpha \pm \beta_i$ are roots in $\Delta(\mathfrak{g},\mathfrak{h})$; and since α and β_i are noncompact, $\alpha \pm \beta_i$ are compact. Since $\alpha \in \Delta(\mathfrak{n})$, and $\beta_i \in \Delta(\mathfrak{l})$, $\alpha \pm \beta_i \in \Delta(\mathfrak{n})$, so $\alpha \pm \beta_i$ are positive roots. Thus

$$<\mu_1,\alpha> = <\mu,\alpha> = \frac{1}{2}(<\mu,\alpha+\beta_i> + <\mu,\alpha-\beta_i>) \geq 0$$

since μ is dominant with respect to $\mathfrak{k}$.

Schmid has defined certain invariant eigendistributions $\Theta(\Psi,\lambda)$ ([22], [8]) which for λ strictly dominant with respect to Ψ are discrete series characters. We will apply the following lemma to M_0. (All of the statements are more or less known by now, although it is difficult to give a specific reference.)

Lemma 6.3 Suppose rk G = rk K, Ψ is a system of positive roots in $\Delta(\mathfrak{g},\mathfrak{t})$, $\lambda \in i\mathfrak{t}_0'$ is dominant with respect to Ψ, $\lambda+\rho$ lifts to character of T, and $\mu = \lambda-2\rho_c+\rho$ is dominant with respect to $\Psi \cap \Delta(\mathfrak{k})$. Then the invariant eigendistribution $\Theta(\psi,\lambda)$ is the character of a tempered unitary representation $\delta(\lambda)$. In the restriction of $\delta(\lambda)$ to K, μ occurs with multiplicity one; and every K-type which occurs is of the form $\mu+Q$, where Q is a sum of positive roots.

Proof. For the proof that $\Theta(\psi,\lambda)$ is tempered, see [8]. Langlands ([17]) has shown that a tempered representation is unitarizable. (Apparently $\Theta(\psi,\lambda)$ is is irreducible or zero whenever λ is dominant, but I have been unable to find a reference for this. For linear groups it is asserted in [15]. If λ is non-singular, the irreducibility is of course part of Harish-Chandra's description of the discrete series.) The other assertions are known (see for example Schmid [23]) if λ is non-singular. We pass to the general case using a technique of Zuckerman.

Let $\gamma_{-\rho}$ be the finite dimensional irreducible representation of lowest weight $-\rho$, with character $\Theta_{-\rho}$. Then the character of $\delta(\lambda+\rho) \otimes \gamma_{-\rho}$ is

$$\Theta(\Psi,\lambda+\rho)\cdot\Theta_{-\rho} = \sum_{\substack{\nu_i \text{ a weight} \\ \text{of } \gamma_{-\rho}}} m_i\, \Theta(\psi,\lambda+\rho+\nu_i) \;;$$

here m_i is the multiplicity of ν_i in $\gamma_{-\rho}$ (cf. [8]). Each ν_i is $-\rho+\Pi_i$, where Π_i is a sum of positive roots; Thus

$$\Theta(\psi,\lambda+\rho)\cdot\Theta_{-\rho} = \Sigma\, m_i\, \Theta(\psi,\lambda+\Pi_i).$$

Since λ is Ψ-dominant, it is easy to see that the only term with the same central character as $\Theta(\psi,\lambda)$ is $\Theta(\psi,\lambda)$, which occurs exactly once. So $\delta(\lambda)$ may be identified with the subspace of $\delta(\lambda+\rho) \otimes \gamma_{-\rho}$ with a certain central character. $\lambda+\rho$ is non-singular; so we know that every K-type of $\delta(\lambda+\rho)$ has highest weight $\mu+\rho+Q'$. Every weight of $\gamma_{-\rho}$ is $-\rho+Q''$; of course Q' and Q'' are sums of positive roots. So every K-type of $\delta(\lambda) \subseteq \delta(\lambda+\rho) \otimes \gamma_{-\rho}$ has highest weight $\mu+\rho+Q'-\rho+Q'' = \mu+Q$; and by being slightly more careful, one sees that μ has multiplicity at most one. It remains to show that μ in fact occurs in the piece of $\delta_{\lambda+\rho} \otimes \gamma_{-\rho}$ with the right central character. We do this by producing some cohomology. (Easier and less interesting arguments exist, but it never hurts to justify one's thesis title.)

Let X be the Harish-Chandra module of $\delta_{\lambda+\rho}$, and Y the representation space of $\gamma_{-\rho}$. Let $\mathfrak{b} \supseteq \mathfrak{t}$ be the Borel subalgebra corresponding to Ψ. Now $(\mu+\rho)+2\rho_c = \lambda+2\rho$ is dominant with respect to Ψ; since every other K-type which occurs is $\mu+\rho+Q$, with Q a sum of roots in Ψ, it follows easily that $\mu+\rho$ is the minimal K-type of X. Also $\lambda(\mu+\rho) = (\mu+\rho)+2\rho_c-\rho = \lambda+\rho$ is dominant and non-singular with respect to Ψ; so by Proposition 4.15, $\mu+\rho$ is $\mathfrak{n}$-minimal in X. By Theorem 3.12,

$$\pi_0^{\mu+\rho} : H^0(\mathfrak{n}\cap\mathfrak{k}, X)^{\mu+\rho} \otimes (\Lambda^R \mathfrak{n}\cap\mathfrak{p})^* \to H^R(\mathfrak{n}, X)^{\mu+\rho-2\rho(\mathfrak{n}\cap\mathfrak{p})}$$

is injective.

On the other hand, Y contains the K-type of lowest weight $-\rho$. By a theorem of Rao ([21]), $X \otimes Y$ contains the K-type of highest weight $(\mu+\rho)-\rho = \mu$; i.e. $H^0(\mathfrak{n}\cap\mathfrak{k}, X \otimes Y)^\mu$ is non-zero. Let Y_1 be the $\mathfrak{b}$-submodule of Y generated by all the weight vectors except the lowest one; thus Y/Y_1 is a one dimensional space of weight $-\rho$. By the long exact sequence,

$$0 \to H^0(\mathfrak{n}\cap\mathfrak{k}, X\otimes Y_1)^\mu \to H^0(\mathfrak{n}\cap\mathfrak{k}, X\otimes Y)^\mu \overset{\alpha}{\to} H^0(\mathfrak{n}\cap\mathfrak{k}, X)^{\mu+\rho} \otimes (Y/Y_1) \to \ldots$$

Using a filtration of Y_1, it is easy to see that the first group is zero, so that α is an isomorphism. (Actually it is not hard to prove directly that $H^1(\mathfrak{n}\cap\mathfrak{k}, X \otimes Y_1)^\mu = 0$; in this way one can eliminate the use of Rao's theorem.) So there is a commutative diagram (for convenience we put $Q = \Lambda^R(\mathfrak{n}\cap\mathfrak{p})^*$, and $\gamma = \mu-2\rho(\mathfrak{n}\cap\mathfrak{p})$)

$$\begin{array}{ccccccc}
H^R(\mathfrak{n}, X \otimes Y_1)^\gamma & \to & H^R(\mathfrak{n}, X \otimes Y)^\gamma & \to & H^R(\mathfrak{n}, X)^{\gamma+\rho} \otimes (Y/Y_1) \\
\uparrow & & \uparrow \pi_0^\mu & & \uparrow \pi_0^{\mu+\rho} \otimes 1 \\
0 & \to & H^0(\mathfrak{n}\cap\mathfrak{k}, X \otimes Y)^\mu \otimes Q & \xrightarrow{\alpha\otimes 1} & H^0(\mathfrak{n}\cap\mathfrak{k}, X)^{\mu+\rho} \otimes (Y/Y_1) \otimes Q
\end{array}$$

Now α is an isomorphism and $\pi_0^{\mu+\rho}$ is injective; so necessarily π_0^μ is injective. Thus there is a cocycle $\omega \in \mathrm{Hom}(\Lambda^R \mathfrak{n}, X \otimes Y)$, with values in $(X \otimes Y)^\mu$, which does not vanish on the cohomology level; and ω has weight $\mu - 2\rho(\mathfrak{n} \cap \mathfrak{p}) = (\mu + 2\rho_c - \rho) - \rho = \lambda - \rho$. It follows from Theorem 3.5 that $\mathfrak{Z}(\mathfrak{g})$ acts on $(X \otimes Y)^\mu$ according to the central character of $\Theta(\psi,\lambda)$. Q.E.D.

It should be pointed out that if λ is Q-dominant, but $\lambda - 2\rho_c + \rho$ is <u>not</u> $\mathfrak{k}$-dominant, then $\Theta(\psi,\lambda) = 0$. For in this case there is a simple compact root α (simple for $\psi \cap \Delta(k)$) such that

$$\frac{2\langle\alpha, \lambda - 2\rho_c + \rho\rangle}{\langle\alpha,\alpha\rangle} \leq -1.$$

Now $\langle\alpha,\lambda\rangle \geq 0$, $\dfrac{2\langle\alpha,\rho_c\rangle}{\langle\alpha,\alpha\rangle} = 1$ since α is simple for $\psi \cap \Delta(\mathfrak{k})$, and $\dfrac{2\langle\alpha,\rho\rangle}{\langle\alpha\ \alpha\rangle} \geq 1$. So necessarily $\langle\alpha,\lambda\rangle = 0$, and $\dfrac{2\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} = 1$. Then α is simple for ψ; and Hecht and Schmid ([8]) have proved that if λ is singular with respect to a compact root which is simple for ψ, then $\Theta(\psi,\lambda) = 0$.

Actually, one can get an existence theorem like 6.2 using only the existence of a representation with the K-decomposition specified in the lemma. This can be proved very algebraically, without invoking the theory of the discrete series (see for example Wallach [24]). But we will use the full strength of Lemma 6.3 to relate our results to Langlands' (Corollary 6.7).

Consider now the representation of M_0 with character $\Theta(\psi,\lambda|_{\mathfrak{t}^+})$. Lemma 6.3 applies; so we let δ_μ^0 be the irreducible component containing the $M_0 \cap K$-type μ_1.

Producing the representation δ_μ^1 of F requires a little more work. Recall that $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{n} \cap \mathfrak{k}$ is a parabolic in $\mathfrak{k}$. With obvious notation, therefore,

$$\mathfrak{k} = (\overline{\mathfrak{n} \cap \mathfrak{k}}) + (\mathfrak{l} \cap \mathfrak{k}) + (\mathfrak{n} \cap \mathfrak{k}), \text{ and}$$

(*) $$U(\mathfrak{k}) = U(\overline{\mathfrak{n} \cap \mathfrak{k}}) \otimes U(\mathfrak{l} \cap \mathfrak{k}) \otimes U(\mathfrak{n} \cap \mathfrak{k}).$$

Let V denote the irreducible $\mathfrak{k}$-module of highest weight μ; put $V_1 = V^{\mathfrak{n} \cap \mathfrak{k}}$. By Kostant's theorem 3.8, V_1 is the irreducible $\mathfrak{l} \cap \mathfrak{k}$ module of highest weight μ. Since the roots of $\mathfrak{t}$ in $\mathfrak{l} \cap \mathfrak{k}$ are supported on $\mathfrak{t}^-$, every weight of V_1 is of the form $(\mu_1,*)$. Using (*), one sees that every weight of V not in V_1 is of the form $(\mu_1,*) + \overline{Q}$, where $\overline{Q}$ is a (non-empty) sum of roots in $\Delta(\overline{\mathfrak{n} \cap \mathfrak{k}})$. On the other hand, the proof of Lemma 2.7 provides an element $x \in \mathfrak{t}^+$, such that $\alpha(x)$ is positive, zero, or negative according as $\alpha \in \Delta(\mathfrak{n})$, $\Delta(\mathfrak{l})$, or $\Delta(\overline{\mathfrak{n}})$. Hence a weight ξ of V is in V_1 iff $\xi(x) = \mu_1(x)$; so V_1 consists precisely of the weights $(\mu_1,*)$ occurring in V. If α is a compact positive root of Ψ, then $\alpha(x) > 0$; so V_1 is precisely the $M_0 \cap K$-highest weight space of weight μ_1 in V.

Recall that $\mu - 2\rho(\mathfrak{n} \cap \mathfrak{p})$ was a small $L \cap K$ type. Now $2\rho(\mathfrak{n} \cap \mathfrak{p})$ is the weight of a one dimensional $L \cap K$ module (namely $\Lambda^R(\mathfrak{n} \cap \mathfrak{p})$), so Theorem 5.4 applies to μ: the action of $(L \cap K)^A$ on V_1 decomposes into a Weyl group (of A in L) orbit in $((L \cap K)^A)^\wedge$, each representation occurring with multiplicity one. Let δ_μ^1 be one of the $(L \cap K)^A$ types occurring. Now one knows that $(L \cap K)^A = T_0^+ F$, and T_0^+ is central in $L = G^{T_0^+}$. Hence $\delta_\mu^1 | F$ is still irreducible.

We must check that δ_μ^1 and δ_μ^0 act by the same scalars on $F \cap M_0$. Since this group is central in F and M_0, it does act by scalars in δ_μ^1 and δ_μ^0. Since $F \cap M_0 \subseteq K \cap M_0$, necessarily $F \cap M_0 \subseteq T_0^+$ (for T_0^+ is maximal abelian in $K \cap M_0$.) But now it is obvious from the definitions that $F \cap M_0$ acts by $\mu|_{F \cap M_0}$ in both δ_μ^0 and δ_μ^1. So $\delta_\mu^0 \otimes \delta_\mu^1$ is a well defined irreducible representation of $M^\natural$.

We claim that $\delta_\mu = \operatorname{Ind}_{M^\natural \uparrow M} \delta_\mu^0 \otimes \delta_\mu^1$ is irreducible.

Now F is central in M if G is linear; and it follows easily that $M^{\natural}$ is normal in M in general. So we must show that if $1 \neq \overline{\sigma} \in M/M^{\natural}$, then $\overline{\sigma}\cdot(\delta_\mu^0 \otimes \delta_\mu^1)$ is not equivalent to $\delta_\mu^0 \otimes \delta_\mu^1$. In fact a little more is true: $\overline{\sigma}\cdot\delta_\mu^0 \not\simeq \delta_\mu^0$. For we may of course choose a representative $\sigma \in M \cap K$ of $\overline{\sigma}$, which normalizes T_0^+, and preserves the positive Weyl chamber for $M_0 \cap K$. Then (cf. Schmid [22]) σ determines a non-identity element of the complex Weyl group of T_0^+ in M_0; since $\sigma \in M \cap K$, σ preserves the set of compact roots. So $\sigma\cdot\rho_c(\psi) = \rho_c(\psi)$, and (since $\lambda|_{\mathfrak{t}^+}$ is dominant for ψ) $\sigma\cdot(\lambda|_{\mathfrak{t}^+} + \rho(\psi)) \neq \lambda + \rho(\psi)$. We have seen that δ_μ^0 has minimal $M_0 \cap K$-type μ_1; so $\overline{\sigma}\cdot\delta_\mu^0$ has minimal $M_0 \cap K$-type

$$\begin{aligned}\sigma\cdot\mu_1 &= \sigma(\lambda|_{\mathfrak{t}^+} + \rho(\psi) - 2\rho_c(\psi)) \\ &= \sigma(\lambda|_{\mathfrak{t}^+} + \rho(\psi)) - 2\rho_c(\psi) \\ &\neq \lambda|_{\mathfrak{t}^+} \rho(\psi) - 2\rho_c(\psi) = \mu_1.\end{aligned}$$

So $\overline{\sigma}\cdot\delta_\mu^0 \not\simeq \delta_\mu^0$, and δ_μ is irreducible.

Fix a character $\nu \in \hat{A}$. It remains to show that $\pi_\mu^\nu = \operatorname{Ind}_{P\uparrow G} \delta_\mu \otimes \nu$ has minimal K-type μ. Set $P^{\natural} = M^{\natural}AN$.

Since $G = KP$, and $K \cap P^{\natural} = K \cap M^{\natural}$, we have

$$\pi_\mu^\nu|_K = \operatorname*{Ind}_{K\cap M^{\natural}\uparrow K} [(\delta_\mu^0|_{K\cap M_0}) \otimes \delta_\mu^1]$$

Suppose $\gamma \in \hat{K}$. By Frobenius reciprocity, the multiplicity of γ in $\pi_\mu^\nu|_K$ is the dimension of $\mathrm{Hom}_{K\cap M^\natural}((\delta_\mu^0|_{K\cap M_0}) \otimes \delta_\mu^1, \gamma)$. It follows easily from the discussion preceding the definition of δ_μ^1 that μ occurs exactly once in $\pi_\mu^\nu|_K$. Suppose $\gamma = (\gamma_1, \gamma_2)$ occurs. Then γ contains a weight (ξ_1, ξ_2) which restricts to the highest weight of some $M^\natural \cap K$-type occurring in $\delta_\mu^0 \otimes \delta_\mu^1$. By Lemma 6.3, $\xi_1 = \mu_1 + Q$, where Q is a sum of positive roots supported on $\mathfrak{t}^+$. Consider the subspace of γ of weights $(\xi_1, *)$. This must contain elements transforming according to δ_μ^1 under F, and thus a weight (ξ_1, ξ_2'), with ξ_2' the highest weight of some $[\ell,\ell] \cap \mathfrak{k}$ type containing δ_μ^1. Recall that $\mu_2 - 2\rho(\mathfrak{n}\cap\mathfrak{p})|_{\mathfrak{t}^-}$ is a small $[\ell,\ell]\cap\mathfrak{k}$ type. Since $2\rho(\mathfrak{n}\cap\mathfrak{p}) + 2\rho(\mathfrak{n}\cap\mathfrak{k})$ is the weight of a one dimensional ℓ module, $\mu_2 + 2\rho(\mathfrak{n}\cap\mathfrak{k})|_{\mathfrak{t}^-}$ is also small. Definition 5.3 now implies (writing $|v|^2 = \langle v,v\rangle$)

$$(**) \quad |\xi_2' + 2\rho(\mathfrak{n}\cap\mathfrak{k})|_{\mathfrak{t}^-} + 2\rho(\Delta^+(\ell\cap\mathfrak{k}))|^2 \geq |\mu_2 + 2\rho(\mathfrak{n}\cap\mathfrak{k})|_{\mathfrak{t}^-} + 2\rho(\Delta^+(\ell\cap\mathfrak{k})|^2;$$

equality holds iff $\xi_2' - 2\rho(\mathfrak{n}\cap\mathfrak{p})$ is a small $[\ell,\ell]\cap\mathfrak{k}$-type associated to $\mu_2 - 2\rho(\mathfrak{n}\cap\mathfrak{p})$. Of course $2\rho(\mathfrak{n}\cap\mathfrak{k})|_{\mathfrak{t}^-} + 2\rho(\Delta^+(\ell\cap\mathfrak{k}) = 2\rho_c|_{\mathfrak{t}^-}$, so we have

$$||\gamma|| = |\gamma+2\rho_c|^2 \geq |\xi_1+2\rho_c|_{\mathfrak{t}^+}|^2 + |\xi_2'+2\rho_c|_{\mathfrak{t}^-}|^2$$

$$\geq |\mu_1+Q+2\rho_c|_{\mathfrak{t}^+}|^2 + |\mu_2+2\rho_c|_{\mathfrak{t}^-}|^2$$

$$\geq |\mu_1+2\rho_c|_{\mathfrak{t}^+}|^2 + |\mu_2+2\rho_c|_{\mathfrak{t}^-}|^2$$

$$= ||\mu||.$$

The first inequality is obvious; the second is (**); and the third is just the fact that $\langle Q,\mu+2\rho_c\rangle \geq 0$ (recall the definition of Δ^+.) Suppose equality holds. Then $\gamma = (\xi_1,\xi_2')$; $Q = 0$; and $\xi_2-2\rho(\mathfrak{n}\cap\mathfrak{p})$ is a small $[\mathfrak{l},\mathfrak{l}]\cap\mathfrak{k}$ type associated to $\mu_2-2\rho(\mathfrak{n}\cap\mathfrak{p})$. By 6.1 (ii), $\gamma \succeq \mu$; so μ is in fact the minimal K-type. This completes the proof of Theorem 6.2.

We note for future reference a corollary to the proof. If X is an arbitrary Harish-Chandra module, the $\mathfrak{k}$-type γ of X is called <u>small in X</u> if $||\gamma||$ is minimal among the $\mathfrak{k}$-types of X. Then the proof shows that the small $\mathfrak{k}$-types of π_μ^ν are the various (μ_1,γ_2), where (μ_1,γ_2) is dominant, and $\gamma_2-2\rho(\mathfrak{n}\cap\mathfrak{p})|_{\mathfrak{t}^-}$ is a small $[\mathfrak{l},\mathfrak{l}]\cap\mathfrak{k}$-type associated to $\mu_2-2\rho(\mathfrak{n}\cap\mathfrak{p})|_{\mathfrak{t}^-}$; one need only check that these actually occur (in fact with multiplicity one), which is left to the reader. Modulo Conjecture 4.2, it follows that the small $\mathfrak{k}$-types of an arbitrary irreducible X occur with multiplicity one. When G is linear, at least, and π_μ^ν is tempered, it seems to be possible to show (using Corollary 6.7 below) that every constituent of π_μ^ν contains a small K-type; in particular that every constituent occurs with multiplicity one (which was proved by Knapp [12].) I hope to pursue this matter in a later paper.

Our next goal is to relate the classification of Theorem 6.2 to that of Langlands ([17]), which we now describe. Langlands assumes that G is linear, but this is almost certainly unnecessary for the results we will quote.

Let $P = MAN$ be a Langlands decomposition of an arbitrary parabolic subgroup of G. Let $\delta \in \hat{M}$ be an irreducible tempered representation, and let $\nu \in \hat{A}$ be arbitrary. Put $I_{\delta\otimes\nu}^P = \operatorname{Ind}_{P\uparrow G} \delta \otimes \nu$. By computing characters, one finds that if P' is any other parabolic with $M'A' = MA$, then $I_{\delta\otimes\nu}^P$ and $I_{\delta\otimes\nu}^{P'}$ have equivalent composition series.

<u>Definition 6.4</u> $\nu \in \hat{A}$ is dominant if for every root α of A in N, $\operatorname{Re} \langle\nu,\alpha\rangle \geq 0$; it is <u>strictly dominant</u> if $\operatorname{Re} \langle\nu,\alpha\rangle > 0$.

<u>Theorem 6.5</u> (Langlands [17]). <u>Suppose G is linear,</u>

P is as above, and $\nu \in \hat{A}$ is strictly dominant. Then $I_{\delta\otimes\nu}$ is a cyclic module. Let $J^P_{\delta\otimes\nu}$ denote the unique irreducible quotient. Then every irreducible quasisimple representation of G is infinitesimally equivalent to such a $J_{\delta\otimes\nu}$; the conjugacy class of (MA, $\delta\otimes\nu$) is unique.

Langlands does not state explicitly that $J_{\delta\otimes\nu}$ is cyclic; the observation that this follows immediately from the proof of Lemma 3.13 of [17] is due to Milicic.

Suppose ν is dominant. The case Re $\nu = 0$ amounts to ν a unitary character. In general, it is easy to see that there is a parabolic $P' \supseteq P$, so that the induction from P' to P is unitary (say $\mathrm{Ind}_{P\uparrow P'}(\delta\otimes\nu) = \delta'\otimes\nu'$) and ν' is strictly dominant. Unitary induction preserves temperedness, so δ' is tempered. As such it splits into finitely many irreducible constituents $\delta'_1 \ldots \delta'_r$, which are tempered. Using this construction, one easily deduces the following corollary of Theorem 6.2.

Corollary 6.6. Suppose Conjecture 4.2 holds for G. Let π be an irreducible representation with minimal K-type μ. Then there is a parabolic P = MAN, a tempered irreducible representation $\delta \in \hat{M}$, and a strictly dominant character $\nu \in \hat{A}$, so that $I_{\delta\otimes\nu}$ has minimal K-type μ, which occurs with multiplicity one; and if we let $\tilde{J}^P_{\delta\otimes\nu}$ denote the

<u>unique subquotient containing μ, then π is infinitesimally equivalent to $\tilde{J}^P_{\delta\otimes\nu}$</u>.

Combining this with Theorem 6.5, we obtain

<u>Corollary 6.7. Suppose G is linear, and that Conjecture 4.2 holds for G. Let $P = MAN$, δ, and ν be as above, with ν strictly dominant. Then $J^P_{\delta\otimes\nu} = \tilde{J}^P_{\delta\otimes\nu}$; i.e. the minimal K-type of $\operatorname{Ind}_{P\uparrow G} \delta \otimes \nu$ is cyclic.</u>

<u>Proof</u>. There are only finitely many (non-conjugate) P', δ', ν' so that $I_{\delta'\otimes\nu'}$ has the same central character as $I_{\delta\otimes\nu}$. We claim that for each of these, $J^{P'}_{\delta'\otimes\nu'} = \tilde{J}^{P'}_{\delta'\otimes\nu'}$. This is proved by downward induction (with respect to the ordering of section 4) on the minimal K-type μ' of $I^{P'}_{\delta'\otimes\nu'}$. If μ' is maximal, every irreducible subquotient of $I^{P'}_{\delta'\otimes\nu'}$ must (by Corollary 6.6) contain μ'. By Corollary 6.6 again, μ' occurs with multiplicity one, so $I_{\delta'\otimes\nu}$ is irreducible; so $J_{\delta'\otimes\nu'} = I_{\delta'\otimes\nu'} = \tilde{J}_{\delta'\otimes\nu'}$. In general, suppose that $J_{\delta'\otimes\nu'} \neq \tilde{J}_{\delta'\otimes\nu'}$. Then the minimal K-type γ' of $J_{\delta'\otimes\nu'}$ is greater than μ'. Applying Corollary 6.6 to $\pi = J_{\delta'\otimes\nu'}$, we see that $J_{\delta'\otimes\nu'}$ is infinitesimally

equivalent to $\tilde{J}_{\delta''\otimes\nu''}$, which is just $J_{\delta''\otimes\nu''}$ by inductive hypothesis. But this contradicts the uniqueness statement of Theorem 6.5, and completes the induction. QED.

Corollary 6.7 is of course full of possibilities. For example, using the fact that every discrete series representation is a quotient of some principal series (see [14]), one can get Casselman's theorem that <u>any</u> representation is a quotient of some principal series ([3]). Of course this proof uses Conjecture 4.2, is confined to linear groups, and is much harder than Casselman's argument; but one gets more specific information about which principal series to choose, as well as generalizations. All of this will be pursued in a later paper.

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