Associated varieties and geometric quantization

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Geometric Quantization and Applications
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Outline

First introduction: classical limits and orbit method

Second introduction: solving differential eqns

Third introduction: Lie group representations

Howe's wavefront set and the size of representations

Associated varieties and the size of representations

Turning on your computer

Advertisement for Eva Miranda

There is a tentative plan to organize a conference in Corsica next year, but **NOT** in June, July, or August.

I am instructed to tell you all that I know of the wonders of Corsica.

What's geometric quantization about?

You have been paying attention this week, haven't you?

Seek construction QUANTIZATION. **HARD.**

$$\text{coadjt orbit } X \subset \mathfrak{g}^* \rightsquigarrow \text{unitary irr repn of } G$$

Seek guidance from **EASY** classical limit

$$\text{unitary irr repn of } G \rightsquigarrow \text{coadjt orbit}$$

This talk: define, compute classical limit(unitary rep).

What's wrong with this pic: **EASY** classical limit only computes orbit at infinity...

Classical limit of rep π should mean Howe's WF$(\pi) \subset \mathfrak{g}^*$.

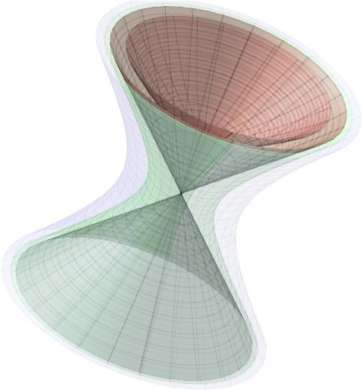
But proofs will use instead AV$(\pi_K) \subset (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k})^*_{\mathbb{C}}$

And it rained Monday, and Wednesday, and Thursday.

Tu vas pas nous sortir les violons?

Cones

Some coadjoint orbits for $SL(2, R)$.



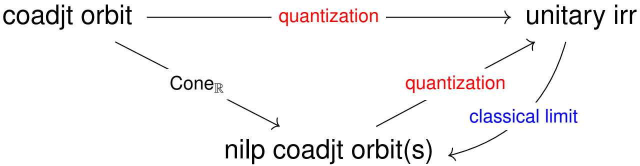
Blue, green hyperboloids are two coadjoint orbits.

Dark green cone describes both orbits at infinity.

$S \subset V$ fin diml $\rightsquigarrow$ $\text{Cone}_{\mathbb{R}}(S) = \{\lim_{i\to\infty} \epsilon_i s_i\}$ $\quad (\epsilon_i \to 0^+, s_i \in S)$.

Quantization, classical limits, and cones

Here's what classical limits tell us about quantization.



Desideratum for quantization: diagram commutes.

$\mathrm{Cone}_{\mathbb{R}}(\text{nilp coadjt orbit}) = \overline{\text{nilp coadjt orbit}} \implies$

quantization of nilpotent X must be π with

$$\text{classical limit}(\pi) = \overline{X}$$

compute classical limit$(\pi) \rightsquigarrow$
candidates for quantization(nilp orbit).

Something to do during the talk

G reductive/number field k, $\pi = \otimes_v \pi_v$ automorphic rep.

k_v local field, $G(k_v)$ reductive, $\mathfrak{g}(k_v) = \mathrm{Lie}(G(k_v))$.

Howe: $\pi_v \rightsquigarrow \mathrm{WF}(\pi_v) \subset \mathfrak{g}(k_v)^*$ nilp orbit closure[s].

Conjecture (global coherence of WF sets)

1. $\exists\ x(\pi) \in \mathfrak{g}(k)^*$, $\quad \mathrm{Cone}_{k_v}(G(k_v) \cdot x(\pi)) = \mathrm{WF}(\pi_v)$.
2. $\exists$ global version of local char expansions for π_v.

Says $G(k_v) \cdot x(\pi)$ controls asymptotics of $\pi_v|_{K_v}$.

Orbit of $x(\pi) \rightsquigarrow$ algebraic cone over $\overline{k}$

$$N(\pi) = \mathrm{Cone}_{\overline{k}}(G(\overline{k}) \cdot x(\pi)) \subset \mathcal{N}^*_{\overline{k}}$$

closure of one $G(\overline{k})$ nilpotent orbit $N(\pi)^0$.

$\mathrm{WF}(\pi_v) \subset N(\pi)_{k_v}$, but possibly $\mathrm{WF}(\pi_v) \cap N(\pi)^0_{k_v} = \emptyset$.

All π_v same size EXCEPT for finite arithm set of v.

Wavefront set of distribution (locally on $\mathbb{R}^n$)

If ϕ integrable function of $x \in \mathbb{R}^n$, Fourier transform is

$$\widehat{\phi}(\xi) = \int e^{2\pi i \langle x, \xi \rangle} \phi(x) dx.$$

Still makes sense if ϕ is compactly supported distribution on $\mathbb{R}^n$: apply ϕ to $x \mapsto e^{2\pi i \langle x, \xi \rangle}$

ϕ msre of cpt support $\implies$ $\widehat{\phi}$ bounded fn of ξ.

Take m derivs of $\phi \rightsquigarrow$ multiply $\widehat{\phi}$ by degree m poly.

mth derivs(ϕ) = cpt supp msres $\implies$ $\widehat{\phi}(\xi) \le C_m/(1 + |\xi|)^m$.

Cptly supp ϕ is smooth $\iff$ $\widehat{\phi}(\xi) \le C_m/(1 + |\xi|)^m$ $(m \ge 0)$.

WF(ϕ) = directions ξ where $\widehat{\phi}(t\xi)$ fails to decay.

Wavefront set (globally on manifold)

Function f on manifold M has *support*:

$$\text{supp}(f) = \text{closure of } \{m \in M \mid f(m) \neq 0\}.$$

Generalized fn ϕ is continuous linear fnl on test densities.

Can multiply ϕ by bump f_0 at m_0 to study "ϕ near m_0."

Singular support of ϕ is where it isn't smooth:

$$M - \text{sing supp}(\phi) = \{m_0 \mid \exists \text{ bump } f_0 \text{ at } m_0,\ f_0\phi \text{ smooth}\}.$$

Wavefront set of ϕ is closed cone $\text{WF}(\phi) \subset T^*(M)$: directions in $T^*(M)$ where $\text{FT}(\phi)$ fails to decay.

Refines sing supp: $\text{sing supp}(\phi) = \{m \in M \mid \text{WF}_m(\phi) \neq 0\}$.

Summary: $\text{WF}(\phi) \subset T^*(M) \rightsquigarrow$ points $m \in M$ where ϕ not smooth, directions $\xi \in T^*_m(M)$ causing non-smoothness.

Behavior of solutions of PDEs

Suppose D is order k diff op on M.

D has *symbol* $\sigma_k(D)$: fn on T^*M, hom poly on $T^*_m(M)$.

$\rightsquigarrow$ *characteristic variety* of D

$$\mathrm{Ch}(D) =_{\mathrm{def}} \{(m,\xi) \in T^*(M) \mid \sigma_k(D)(m,\xi) = 0\}$$

$$D\phi = \psi \implies \mathrm{WF}(\phi) \subset \mathrm{WF}(\psi) \cup \mathrm{Ch}(D) :$$

solving D adds singularities only in $\mathrm{Ch}(D)$.

$D_1, \ldots, D_m$ diff ops on M $\rightsquigarrow$ char var of system

$$\mathrm{Ch}(D_1, \ldots, D_m) =_{\mathrm{def}} \mathrm{Ch}(D_1) \cap \cdots \cap \mathrm{Ch}(D_m).$$

Solns of systems: if $D_j\phi = 0$, all j, then

$$D_j\phi = 0, \forall j \implies \mathrm{WF}(\phi) \subset \mathrm{Ch}(D_1, \ldots, D_m) :$$

solving system adds singularities only in $\mathrm{Ch}(D_1, \ldots, D_m)$.

Summary of the PDE story

PDE on M $\leftrightsquigarrow$ module for diff op alg $D(M)$.

Noncomm alg $D(M) \approx$ comm alg Poly$(T^*(M))$.

$=$ Smooth fns that are polys along each $T^*_m(M)$.

Solns of PDE $\approx$ (graded) modules for Poly$(T^*(M))$.

(graded) Poly$(T^*(M))$-module $\leftrightsquigarrow$ alg cone in $T^*(M)$.

Cone is common zeros of all symbols of diff eqs.

Cone controls where solutions can have WF.

Summary of the representation theory story

I know I didn't tell you the story yet, but I get excited...

Representation of G $\leftrightsquigarrow$ module for algebra $U(\mathfrak{g}_{\mathbb{C}})$.

Noncomm alg $U(\mathfrak{g}_{\mathbb{C}}) \approx$ comm alg $\text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$.

Polynomial functions on $\text{Lie}(G)_{\mathbb{C}}^*$.

Repn of $G \approx$ (graded) module for algebra $\text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$.

(graded) $\text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$-module $\leftrightsquigarrow$ alg cone in $\mathfrak{g}_{\mathbb{C}}^*$.

Cone is zeros of symbols of $U(\mathfrak{g}_{\mathbb{C}})$ elts "killing" repn.

Representation $\approx$ algebraic functions on cone.

What groups?

$G = G(\mathbb{R}, \sigma)$ real points of complex connected reductive algebraic group G, σ_c compact real form of G commuting with σ, $K = G(\mathbb{R}, \sigma) \cap G(\mathbb{R}, \sigma_c)$ maximal compact subgroup of G.

(That's for postdocs. They should sweat a little.)

$G \subset GL(n, \mathbb{R})$ closed, transpose-stable, $K = O(n) \cap G$.

(That's for the PDE people. Thank you for showing up!)

Also keep in mind $G = GL(m, \mathbb{H})$, $G = SO(p, q)$.

$G = GL(n, \mathbb{R})$, $K = O(n)$.

(That's what senior professors should think about.)

What representations?

Secs of $K(\mathbb{C})$-eqvt reg holonomic $\mathcal{D}$-mod on flag variety.

Example: Normal derivs of Borel-Weil-Bott realization of $K(\mathbb{C})$-rep on $K(\mathbb{C})/B \cap K(\mathbb{C}) \subset G(\mathbb{C})/B$.

(That's for postdocs. Sweat a medium amount.)

Finite length quasisimple Fréchet rep of moderate growth.

Example: Smooth secs of eqvt vec bdle on $\mathrm{Gr}(k, n)$.

(That's for PDE people. Although demise of language requirements means only the French will know whether the accent on "Fréchet" is correct.)

Trig polys on the circle, as module for

$$\mathrm{Span}(d/d\theta, \cos(2\theta)d/d\theta, \sin(2\theta)d/d\theta) \simeq \mathfrak{sl}(2, \mathbb{R}).$$

(Senior professors should think about that. By now they are asleep, so question is purely theoretical.)

What I'm aiming to do

system of PDE $D_j\phi = 0$ on $M \rightsquigarrow \mathrm{Ch}(D_1, \ldots, D_m) \subset T^*(M)$ controlling singularities of solns.

Want analogue of $\mathrm{Ch}(D_1, \ldots, D_m)$ for repn (π, V) of G:

$$\mathrm{WF}_{\mathrm{big}}(\pi) \subset T^*(G) \simeq G \times \mathfrak{g}^*.$$

Desideratum: $\mathrm{WF}_{\mathrm{big}}(\pi)$ closed cone, left and right G-invt.

Left invt $\implies$ $\mathrm{WF}_{\mathrm{big}}(\pi)$ determined by real closed cone

$$\mathrm{WF}(\pi) =_{\mathrm{def}} \mathrm{WF}_{\mathrm{big}}(\pi) \cap T^*_e(G) \simeq \mathfrak{g}^*$$

Right invt $\implies$ $\mathrm{WF}(\pi)$ is $\mathrm{Ad}(G)$-invt: union of orbits.

Next goal: Howe's def of $\mathrm{WF}(\pi)$.

Characters and sizes

Info about lin op A on n-diml V encoded by char poly:

$$\det(tI - A) = t^n - t^{n-1}\operatorname{tr}(A) + \cdots + (-1)^n \det(A).$$

Lower order coeffs are poly fns of $\operatorname{tr}(A),\ \operatorname{tr}(A^2), \cdots, \operatorname{tr}(A^n)$.

Info about n-diml rep (π, V) encoded by character:

$$\Theta_\pi \colon G \to \mathbb{C}, \qquad \Theta_\pi(g) = \operatorname{tr}(\pi(g)).$$

$$\text{Size of } \pi = n = \Theta_\pi(e).$$

If V inf-diml, $\pi(g)$ isn't trace class, so Θ_π isn't function.

But Θ_π is often a generalized function: if μ is test density on G, then linear operator

$$\pi(\mu) = \int_G \pi(g) d\mu(g)$$

is a smoothing of π, and often is trace class.

Can often define generalized fn $\Theta_\pi(\mu) = \operatorname{tr}(\pi(\mu))$.

Size of π $\leftrightsquigarrow$ singularity of Θ_π at e.

Howe's wavefront set

(π, V) nice repn of nice Lie group G.

$$\mathrm{WF}(\pi) =_{\mathrm{def}} \mathrm{WF}_e(\Theta_\pi) \subset \mathfrak{g}^* = T_e^*(G).$$

How do you control that?

$U(\mathfrak{g}) =_{\mathrm{def}}$ left-invt diff ops on G; V is $U(\mathfrak{g})$-module.

Rt transl preserves $U(\mathfrak{g})$, $\rightsquigarrow$ alg auts $\mathrm{Ad}(g)$.

Symbols $=$ left-invt polys on T^*G, or polys on $\mathfrak{g}^*$.

$\mathfrak{Z}(\mathfrak{g}) =_{\mathrm{def}} U(\mathfrak{g})^{\mathrm{Ad}(G)} =$ left and right invt diff ops.

Symbols of $\mathfrak{Z}(\mathfrak{g}) = \mathrm{Ad}(G)$-invt polys on $\mathfrak{g}^*$.

Schur's lemma: $\mathfrak{Z}(\mathfrak{g})$ acts by scalars on V.

$\implies$ diff eqs for Θ_π: $z \cdot \Theta_\pi = \lambda(z)\Theta_\pi$ $(z \in \mathfrak{Z}(\mathfrak{g}))$.

$\implies$ $\mathrm{WF}(\pi) \subset$ zeros of symbol of z.

Real nilpotent cone

$$\mathcal{N}^*_{\mathbb{R}} =_{\text{def}} \text{zeros of } \mathrm{Ad}(G)\text{-invt homog polys} \subset \mathfrak{g}^*.$$

Proved: $\mathrm{WF}(\pi) \subset \mathcal{N}^*_{\mathbb{R}}$, $\mathrm{Ad}(G)$-invt.

Howe's wavefront set defines

$$(\text{irr of } GL(n,\mathbb{R})) \overset{\mathrm{WF}}{\rightsquigarrow} (\text{conj class of nilp mats}).$$

$$(\text{irr of } G) \overset{\mathrm{WF}}{\rightsquigarrow} (G \text{ orbit on } \mathcal{N}^*_{\mathbb{R}}).$$

Size of π = one half real dimension of orbit.

Howe's $\mathrm{WF}(\pi)$ is the perfect classical limit:

$$\text{group representation} \overset{\mathrm{WF}}{\rightsquigarrow} \text{symplectic manifold}$$

in a simple, natural, and meaningful way.

But after forty years, it's still a royal pain to compute.
Next: (computable) algebraic analogue of $\mathrm{WF}(\pi)$.

Analytic roots of algebraic rep theory

Typical $GL(n, \mathbb{R})$ rep is $C^\infty(\mathrm{Gr}(p, n))$, smooth fns on Grassmann variety of p-diml planes in $\mathbb{R}^n$.

Compact subgroup $O(n)$ acts transitively on $\mathrm{Gr}(p, n)$: smooth functions have nice Fourier expansions.

(Remember that I asked the senior professors to think about trigonometric polynomials on the circle?)

Harish-Chandra understood that this works for all reps of all reductive G, with K = max cpt subgp.

(π, V) any smooth rep of $G \rightsquigarrow$

$$V_K =_{\text{def}} \{v \in V \mid \dim\langle\pi(K)v\rangle < \infty\} \quad K\text{-finite vecs}$$
$$\approx \text{spherical harmonics.}$$

Action of $U(\mathfrak{g}_\mathbb{C})$ preserves V_K.

Fourier_K, easy diff eqns $\rightsquigarrow$ recover G action on V.

$(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-mod: making rep theory algebraic

Last slide suggested $V_K = K$-finite vectors in V as algebraic substitute for smooth G rep V.

Definition A $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module is cplx vec space with $U(\mathfrak{g}_{\mathbb{C}})$ action, and alg rep of $K_{\mathbb{C}}$, so that

1. deriv of $K_{\mathbb{C}}$ action equal to $\mathfrak{k}$ action (from $U(\mathfrak{g}_{\mathbb{C}})$); and
2. Actions compatible: $k \cdot (u \cdot v) = \mathrm{Ad}(k)(u) \cdot (k \cdot v)$.

Thm (Harish-Chandra) (π, V) irr smooth quasisimple rep of $G \implies V_K$ irr $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-mod.
Conversely, every irr $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-mod is a V_K.

Quasisimple = Schur's lemma true for π: avoid pathology.

Irreducible for $(\pi, V) \leftrightsquigarrow$ closed subspaces.

Irreducible for $V_K \leftrightsquigarrow$ pure algebra.

Making the algebra commutative

Recall idea of WFs and PDE:

$$D = \text{diff ops on } M \approx \text{Poly}(T^*(M))$$

$$\text{system of PDEs} = D\text{-module} \approx \text{Poly}(T^*(M))\text{-module}$$

$$\text{Poly}(T^*(M))\text{-module} \leftrightsquigarrow \text{cone in } T^*(M)$$

$$U(\mathfrak{g}_{\mathbb{C}}) = \text{left-invt cplx diff ops on } G \approx \text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$$

Precisely: $U(\mathfrak{g}_{\mathbb{C}})$ filtered by deg, gr $U(\mathfrak{g}_{\mathbb{C}}) \simeq \text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$.

fin length $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-mod X has $K_{\mathbb{C}}$-stable good filt,

$$\begin{aligned} \text{gr}\, X &= \text{fin. gen. graded } (\text{Poly}(\mathfrak{g}_{\mathbb{C}}^*/\mathfrak{k}_{\mathbb{C}}^*), K_{\mathbb{C}})\text{-module} \\ &= K_{\mathbb{C}}\text{-eqvt coherent sheaf on } \mathfrak{g}_{\mathbb{C}}^*/\mathfrak{k}_{\mathbb{C}}^* \end{aligned}$$

$$\text{AV}(X) =_{\text{def}} \text{supp}\, \text{gr}\, X,$$

a $K_{\mathbb{C}}$-stable algebraic cone in $\mathfrak{g}_{\mathbb{C}}^*/\mathfrak{k}_{\mathbb{C}}^*$.

What sort of invariant is that?

$$\mathcal{N}^*_\theta =_{\text{def}} \text{zeros of Ad}(G)\text{-invt homog polys} \subset \mathfrak{g}^*_{\mathbb{C}}/\mathfrak{k}^*_{\mathbb{C}}.$$

WF(π) proof $\rightsquigarrow$ AV(π_K) $\subset \mathcal{N}^*_\theta$, Ad($K_{\mathbb{C}}$)-invt.

Associated variety defines

$$(\text{irr } (\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})\text{-module } X) \overset{\text{AV}}{\rightsquigarrow} K_{\mathbb{C}}\text{-orbits on } \mathcal{N}^*_\theta).$$

Size of X = complex dim of orbit.

AV(X) is the perfect algebraic classical limit:

$$(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})\text{-module} \overset{\text{AV}}{\rightsquigarrow} \text{algebraic cone}$$

in a simple, natural, and meaningful way.

One way to understand the meaning:

$$X|_{K_{\mathbb{C}}} \simeq (\text{gr } X)|_{K_{\mathbb{C}}}$$
$$= (\text{coherent sheaf on AV}(X))|_{K_{\mathbb{C}}}.$$

How to calculate AV

X finite length $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module...
$\rightsquigarrow$ gr X $K_{\mathbb{C}}$-eqvt coherent sheaf on $\mathcal{N}_\theta^*$...
$\rightsquigarrow$ AV(X) algebraic cone in $\mathcal{N}_\theta^*$.

Key property: $X|_{K_{\mathbb{C}}} \simeq$ (coherent sheaf on AV(X))$|_{K_{\mathbb{C}}}$.

KNOW how to calculate $X|_{K_{\mathbb{C}}}$. So...

FIND eqvt sheaf M on $\mathcal{N}_\theta^*$ such that $X|_{K_{\mathbb{C}}} = M|_{K_{\mathbb{C}}}$.

KNOW how to do that as well. Pet computers are awesome.

CONCLUDE AV(X) = supp(M).

Restate: $AV(X)$ = what can carry the K-types of X.

Such thms $\rightsquigarrow$ Kashiwara & Vergne (Luminy 1978).

Connect 1978$\leftrightsquigarrow$2018 needs $(\mathcal{N}_{\mathbb{R}}^*)/G \leftrightsquigarrow (\mathcal{N}_\theta^*)/K_{\mathbb{C}}$.

Such relation $\rightsquigarrow$ Vergne(1995).

How that looks for $SL(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$, $K = SO(2)$, $\widehat{K} = \mathbb{Z}$.

Standard representations are

1. holomorphic (lims of) disc series $I^+(m)$ ($m \geq 0$),
   $I^+(m)|_K = \{m+1, m+3, m+5, \dots\}$
2. antihol (lims of) disc series $I^-(m)$ ($m \geq 0$),
   $I^-(m)|_K = \{-m-1, -m-3, -m-5, \dots\}$
3. spher princ series $I^{\text{even}}(\nu)$,
   $I^{\text{odd}}(\nu)|_K = \{0, \pm 2, \pm 4 \dots\}$
4. nonspher princ series $I^{\text{odd}}(\nu)$, $\nu \neq 0$,
   $I^{\text{odd}}(\nu)|_K = \{\pm 1, \pm 3, \pm 5 \dots\}$

   N.B. $I^{\text{odd}}(0) = I^+(0) + I^-(0)$.

Three nilp $SO(2,\mathbb{C})$ orbits on $\mathcal{N}_\theta^*$: $\mathcal{O}^+$, $\mathcal{O}^-$, $\{0\}$.

$\underbrace{\text{Coherent sheaves}}_{\text{restriction to } K}$ on $\overline{\mathcal{O}^+}$: $\underbrace{[I^{\text{even}}(0)] - [I^-(1)]}_{\{0,2,4,\dots\}}$, $\underbrace{[I^+(0)]}_{\{1,3,5,\dots\}}$.

Coherent sheaves on $\overline{\mathcal{O}^-}$: $\underbrace{[I^{\text{even}}(0)] - [I^+(1)]}_{\{0,-2,-4,\dots\}}$, $\underbrace{[I^-(0)]}_{\{-1,-3,\dots\}}$.

Coh on $\{0\}$: $\underbrace{[I^{\text{even}}(0) - I^+(1) - I^-(1)]}_{\{0\}}$, $\underbrace{[I^+(m) - I^+(m+2)]}_{\{m+1\} \quad (m\geq 0)}$, $\underbrace{[I^-(m) - I^-(m+2)]}_{\{-m-1\} \quad (m\geq 0)}$.

Algorithm for AV(X)

0. Make formulas (Achar theory, `atlas` practice)

$$\mathcal{S}_j = \text{coh shf on } \mathcal{O}_j) = \sum_i s_j^k [I_k] \qquad (I_k \text{ standard rep})$$

1. Write (KL theory, `atlas` practice) char formula

$$X = \sum_i m_i I_i \qquad (I_i \text{ standard rep}).$$

2. Restrict to K: set cont parameters equal to zero.
3. Write (linear algebra)

$$\sum_i m_i I_i|_K = \sum n_j \mathcal{S}_j$$

4. Biggest $\mathcal{O}_j$ needed give AV(X).

Here's how that looks for $SL(2,\mathbb{R})$ (reprise)

Library of coherent sheaves on orbit closures:

$\underbrace{\text{Coherent sheaves}}_{\text{restriction to } K}$ on $\overline{\mathcal{O}^+}$: $\underbrace{[I^{\text{even}}(0)] - [I^-(1)]}_{\{0,2,4,\dots\}}$, $\underbrace{[I^+(0)]}_{\{1,3,5,\dots\}}$.

Coherent sheaves on $\overline{\mathcal{O}^-}$: $\underbrace{[I^{\text{even}}(0)] - [I^+(1)]}_{\{0,-2,-4,\dots\}}$, $\underbrace{[I^-(0)]}_{\{-1,-3,\dots\}}$.

Coh on $\{0\}$: $\underbrace{[I^{\text{even}}(0) - I^+(1) - I^-(1)]}_{\{0\}}$, $\underbrace{[I^+(m) - I^+(m+2)]}_{\{m+1\} \quad (m\geq 0)}$, $\underbrace{[I^-(m) - I^-(m+2)]}_{\{-m-1\} \quad (m\geq 0)}$.

Here $[I^+(0)]$ means class in Groth grp of gr $I^+(0)$.

Try X = three-diml adjoint rep, character formula

$$X = I^{\text{even}}(3) - I^+(3) - I^-(3)$$

$$\begin{aligned} X|_K &= I^{\text{even}}(0) - I^+(3) - I^-(3) \\ &= (I^{\text{even}}(0) - I^+(1) - I^-(1)) \\ &\qquad + (I^+(1) - I^+(3)) + (I^-(1) - I^-(3)). \end{aligned}$$

Three terms from orbit $\{0\}$, so $AV(X) = \overline{\{0\}}$.