Understanding restriction to K

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Wilfried Schmid Birthday Conference, May 2013

Outline

Why restrict to K?

G cplx $\supset G(\mathbb{R})$ real reductive $\supset K(\mathbb{R})$ maxl compact

Reps $(\pi, \mathcal{H}_\pi)$ of $G(\mathbb{R})$ are complicated and difficult.

Reps of $K(\mathbb{R})$ are easy, so try two things:

understand $\pi|_{K(\mathbb{R})}$; and
use understanding to answer questions about π.

$K(\mathbb{R}) \subset G(\mathbb{R}) \rightsquigarrow T \subset U$, max torus in cpt Lie.

$\mu \in \widehat{U}$ characterized by largest $\xi(\mu)$ in $\mu|_T$ (Cartan-Weyl).

Can compute $\mu|_T$ completely (Kostant).

When $\pi|_{K(\mathbb{R})}$ inf diml, can't ask for *largest* piece...

...can define $\mu(\pi)$ as a smallest piece of $\pi|_{K(\mathbb{R})}$.

(sometimes) π characterized by $\mu(\pi)$ (Schmid)

In those cases, can compute $\pi|_{K(\mathbb{R})}$ (Schmid)

Outline of talk

$G(\mathbb{R})$ real reductive $\supset K(\mathbb{R})$ max compact subgp.

Ex: $G(\mathbb{R}) = GL(n,\mathbb{C}) = \{\text{invertible linear transf of } \mathbb{C}^n\}$
$K(\mathbb{R}) = U(n) = \{\text{linear transf respecting } \langle,\rangle\}$.

Ex: $G(\mathbb{R}) = GL(n,\mathbb{R}) = \{\text{invertible linear transf of } \mathbb{R}^n\}$
$K(\mathbb{R}) = O(n) = \{\text{linear transf respecting } \langle,\rangle\}$.

Ex: $G(\mathbb{R}) = Sp(2n,\mathbb{R}) = \{\mathbb{R}\text{-lin transf of } \mathbb{C}^n \text{ resp. } \mathrm{Im}(\langle,\rangle)\}$
$K(\mathbb{R}) = U(n) = \{\mathbb{C}\text{-linear maps in } Sp(2n,\mathbb{R})\}$.

Plan for this talk:

1. Recall standard reps $I(\gamma)$ of $G(\mathbb{R})$ (Harish-Chandra)
2. Recall multiplicity calculation of $I(\gamma)|_{K(\mathbb{R})}$ (Schmid)
3. Recall (irred) = int comb (std reps) (Kazhdan-Lusztig)
4. geometric expression for irr reps restricted to $K(\mathbb{R})$
5. Open problems relating 1)–3) to 4).

Setting

G cplx conn reductive alg grp $\underbrace{\text{def over } \mathbb{R}}_{}$ (cplx conj action σ) $\rightsquigarrow G(\mathbb{R})$

σ_0 cpt form s.t. $\sigma\sigma_0 = \sigma_0\sigma \rightsquigarrow \theta =_{\text{def}} \sigma\sigma_0$ Cartan inv

$K = G^\theta$ cplx reductive alg, $K(\mathbb{R})$ max cpt in $G(\mathbb{R})$.

$\Pi_u(G(\mathbb{R}))$ = irr unitary reps/equiv: atoms of harm analysis

$\cup\!\!|$

$\Pi(G(\mathbb{R}))$ = irr quasisimple reps/infl equiv: analytic cont

$\wr\!|$

$\Pi(\mathfrak{g}, K)$ = irr HC modules: Taylor series for qsimple reps

$\Pi(K(\mathbb{R})) = \Pi(K) = \widehat{K}$ = irr reps of $K(\mathbb{R})$ = irr alg of K

$\pi \in \Pi(G(\mathbb{R})) \rightsquigarrow m_\pi \colon \widehat{K} \to \mathbb{N}$, $m_\pi(\mu)$ = mult of μ in $\pi|_K$.

Problem: compute and understand functions m_π.

Schmid's construction of discrete series

Discrete series reps $\Pi_{ds}(G(\mathbb{R})) \subset \Pi_u(G(\mathbb{R}))$

Discrete series are irr summands of $L^2(G(\mathbb{R}))$.

Harish-Chandra: exist iff $G(\mathbb{R}) \supset T(\mathbb{R}) \subset K(\mathbb{R})$, cpt Cartan

Harish-Chandra: $\widehat{T(\mathbb{R})}_{\mathrm{reg}}/W(\mathbb{R}) \overset{\approx}{\longleftrightarrow} \Pi_{ds}(G(\mathbb{R})), \quad \lambda \to I(\lambda)$

$\mathcal{B}$ = complete flag variety of Borel subalgs $\mathfrak{b} \subset \mathfrak{g}$.

$$\lambda \rightsquigarrow \mathfrak{b}(\lambda) \supset \mathfrak{t} \rightsquigarrow X(\lambda) = G(\mathbb{R}) \cdot \mathfrak{b}(\lambda) \simeq G(\mathbb{R})/T(\mathbb{R}) \subset \mathcal{B}$$

$$\cup|$$

$$Z(\lambda) = K(\mathbb{R}) \cdot \mathfrak{b}(\lambda) \simeq K(\mathbb{R})/T(\mathbb{R})$$

$\mathcal{L}(\lambda + \rho) \to X(\lambda)$ holomorphic line bundle induced by $\lambda + \rho$.

Kostant-Langlands: $I(\lambda) \overset{?}{\approx} H^s(X(\lambda), \mathcal{L}(\lambda + \rho))$ ($s = \dim Z(\lambda)$).

Probs: $X(\lambda)$ noncpt; cohom not Hilbert space.

Schmid: Taylor exp of cohom along cpt subvar $Z(\lambda)$.

$$I(\lambda)|_{K(\mathbb{R})} \approx H^s(Z(\lambda), \mathcal{L}(\lambda + \rho) \otimes S(\underbrace{\mathfrak{g}/(\mathfrak{k} + \mathfrak{b}(\lambda))}_{\text{conorm to } Z \text{ in } X})^*).$$

Understanding Blattner

$G(\mathbb{R}) \supset K(\mathbb{R}) \supset T(\mathbb{R})$, $\quad \mathcal{B} =$ variety of Borel subalgs.

$\lambda \in \widehat{T(\mathbb{R})}_{\text{reg}} \rightsquigarrow \mathfrak{b}^-(\lambda) \supset \mathfrak{t}$ Borel subalgebra.

$X(\lambda) = G(\mathbb{R}) \cdot \mathfrak{b}^-(\lambda) \simeq G(\mathbb{R})/T(\mathbb{R})$ open $G(\mathbb{R})$ orbit.

$\rightsquigarrow$ finitely many $G(\mathbb{R})$ orbits on $\mathcal{B} \rightsquigarrow$ reps by geom quant.

$$I(\lambda)|_{K(\mathbb{R})} \approx H^s(Z(\lambda), \mathcal{L}(\lambda+\rho) \otimes S(\mathfrak{g}/(\mathfrak{k}+\mathfrak{b}(\lambda)))^*).$$

Starts with lowest K-type $H^s(K/(B \cap K), \mathcal{L}(\lambda+\rho))$.

This is irr of highest weight $\mu(\lambda) = \lambda + \rho - \underbrace{2\rho_c}_{\text{sum cpt pos}}$

Borel-Weil-Bott-Kostant $\rightsquigarrow$ entire restriction $I(\lambda)|_{K(\mathbb{R})}$.

Thm (Schmid, Hecht-Schmid) Discrete series $I(\lambda)$ contains lowest K-type $\mu(\lambda)$ with mult 1; all others are $\mu(\lambda) +$ (sum ncpt pos). These props characterize $I(\lambda)$.

Standard representations

$G(\mathbb{R}) \supset H(\mathbb{R})$, Cartan subgp. After conj $\rightsquigarrow$ θ-stable.

$H(\mathbb{R}) = \underbrace{T(\mathbb{R})}_{\text{cpt}} \times \underbrace{A}_{\text{vec gp}}$, cplxfication $T = H^\theta = H \cap K$.

$\widehat{T(\mathbb{R})} = X^*(T) = \underbrace{X^*(H)/(1-\theta)X^*(H)}_{\text{small if } H \text{ ncpt}}, \quad \widehat{A} = \underbrace{\mathfrak{a}^*}_{\text{big if } H \text{ ncpt}}.$

Harish-Chandra: $\widehat{H(\mathbb{R})}/W(\mathbb{R}) \rightsquigarrow \Pi_{std}(G(\mathbb{R})), \quad \gamma \to I(\gamma)$

Need also Ψ = *pos imag roots* making λ *dom*.

$\gamma = (\lambda, \nu) \in \widehat{T(\mathbb{R})} \times \widehat{A}$

1. $I(\gamma, \Psi)$ *tempered* $\iff$ γ unitary $\iff$ $\nu \in i\mathfrak{a}_0^*$
2. $I(\gamma, \Psi)|_K$ depends only on $\lambda \in X^*(T)$

$I(\gamma, \Psi)|_K =_{\text{def}} I(\lambda, \Psi)$ known: hard case disc series.

Basis for virtual reps

$H(\mathbb{R}) = T(\mathbb{R}) \times A$, $\gamma = (\lambda, \nu) \in \widehat{H(\mathbb{R})}$, Ψ pos imag.

All $I(\gamma, \Psi)$ lin ind unless $\langle \gamma, \alpha^\vee \rangle = 0$ ($\alpha \in \Psi$ simple)...

...Hecht-Schmid character identities

1. α noncompact:
$$\underbrace{I(\gamma_\alpha, \Psi_\alpha)}_{\text{split } H_\alpha \subset SL(2)_\alpha} = \begin{cases} I(\gamma, \Psi) + I(\gamma, s_\alpha \Psi) & s_\alpha \notin W(\mathbb{R}) \\ I(\gamma, \Psi) & s_\alpha \in W(\mathbb{R}). \end{cases}$$
2. α compact: $\quad I(\gamma, \Psi) = 0.$

(γ, Ψ) *final* if not on left of a Hecht-Schmid identity.

$\iff$ $\nexists \alpha$ real, $\langle \nu, \alpha^\vee \rangle = 0$, $\lambda(m_\alpha) = -1$, and

$\nexists \alpha$ compact simple in Ψ, $\langle \lambda, \alpha^\vee \rangle = 0$.

Thm (Langlands, Knapp-Zuckerman, Hecht-Schmid)

1. $\{I(\gamma, \Psi) \mid (\gamma, \Psi) \text{ final}\}$ basis for virtual reps of $G(\mathbb{R})$.
2. $\{I(\gamma, \Psi) \mid (\gamma, \Psi) \text{ final, unitary}\} = \Pi_{temp}(G(\mathbb{R}))$.
3. $I(\gamma, \Psi)$ has quotient rep $J(\gamma, \Psi)$; irr if (γ, Ψ) final.

Irreducibles and standards

$\mathcal{P}(G(\mathbb{R})) = \{(\gamma, \Psi) \text{ final } \mid \gamma \in \widehat{H(\mathbb{R})}, \ \Psi \text{ pos imag}\}$
Langlands parameters; write $x \in \mathcal{P}$ for (γ, Ψ).

$I(x)$ standard rep $\twoheadrightarrow J(x)$ irr Langlands quotient.

$\mathcal{P}(G(\mathbb{R}))$ basis$/\mathbb{Z}[q, q^{-1}]$ for Hecke alg module.

KL analysis $\rightsquigarrow$ Two kinds of KL polys in $\mathbb{N}[q]$...

1. $Q_{z,y}(1) =$ mult of $J(z)$ in $I(y)$

$$I(y) = \sum_{z \le y} Q_{z,y}(1) J(z).$$

2. $P_{y,x}(1) = (-1)^{\ell(x)-\ell(y)} \cdot$ (coeff of $I(y)$ in char of $J(x)$)

$$J(x) = \sum_{y \le x} (-1)^{\ell(x)-\ell(y)} P_{y,x}(1) I(y).$$

(x, y, z in $\mathcal{P}(G(\mathbb{R}))$).

Consequence: computable branching law

$$J(x)|_K = \sum_{y \le x} (-1)^{\ell(x)-\ell(y)} P_{y,x}(1) I(y)|_K.$$

Basis for restrictions to K

$\{I((\lambda,\nu),\Psi)\}$ are all the standard reps.

Restriction to K independent of continuous param ν.

Thm (How standard reps restrict to K.)

1. $\{I(\lambda,\Psi) \mid \lambda \in X^*(T) \text{ final}\}$ basis for (virtual reps)$|_K$.
2. $I(\lambda,\Psi)$ has unique lowest K-type $\mu(\lambda,\Psi)$.
3. $\{\text{temp, real infl char}\} \leftrightarrow \{I(\lambda,\Psi) \mid \lambda \in X^*(T) \text{ final}\} \leftrightarrow \widehat{K}$

Ex: $G = SL(2) \times SL(2)$, $K = SL(2)_\Delta$, torus $= H \times H$

$T = H_\Delta \rightsquigarrow$ final params (temp reps $SL(2,\mathbb{C})$, real infl)...

$\{n \mid n \geq 0\}$, $\quad I(n)|_{SL(2)_\Delta} = \{E(n), E(n+2), E(n+4) \cdots\}$

Ex: $G = SL(2)$, $K = SO(2)$, $H_c = K$, $H_s =$ (diag torus)

$T_c = K$, $\quad \Psi^{hol}, \Psi^{ahol}$; $T_s = \{\pm I\} \rightsquigarrow$ final params...

$\{(n,\Psi^{hol}) \mid n \geq 0\}$ on T_c; $\{m, \Psi^{ahol} \mid m \leq 0\}$ on T_c; (triv) on T_s.

$$I(n,\Psi^{hol})|_{SO(2)} = \{n+1, n+3, n+5 \cdots\}$$

$$I(m,\Psi^{ahol})|_{SO(2)} = \{m-1, m-3, m-5 \cdots\}$$

$$I(triv)|_{SO(2)} = \{0, \pm 2, \pm 4 \cdots\}$$

Associated varieties

$\mathcal{F}(\mathfrak{g}, K)$ category of finite length $(\mathfrak{g}, K)$-modules: $U(\mathfrak{g})$-module, alg action of $K = G^\theta$.

$\overset{\text{gr}}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, K)$ f.g. $(S(\mathfrak{g}/\mathfrak{k}), K)$-mods, supp $\subset \mathcal{N}_\theta^* \subset (\mathfrak{g}/\mathfrak{k})^*$

$\mathcal{N}^* = \{\lambda \in \mathfrak{g}^* \mid p(\lambda) = 0 \ (p \in [\mathfrak{g}S(\mathfrak{g})]^G)\}$ nilp cone

$\mathcal{N}_\theta^* = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*, \qquad \mathcal{N}_\mathbb{R}^* = \mathcal{N}^* \cap i\mathfrak{g}(\mathbb{R})^*.$

Prop (Kostant-Rallis, Sekiguchi)

1. K acts on $\mathcal{N}_\theta^*$, fin $\#$ orbs, cplx Lag in G orbit.
2. $G(\mathbb{R})$ acts on $\mathcal{N}_\theta^*$, fin $\#$ orbs, real Lag in G orbit.
3. Bij $\mathcal{N}_\theta^*/K \leftrightarrow \mathcal{N}_\mathbb{R}^*/G(\mathbb{R})$, resp G orbit, diffeo type.

Prop gr induces surjection of Groth groups

$$K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K\mathcal{C}(\mathfrak{g}, K);$$

image records restriction to K of HC module.

So restrictions to K of HC modules sit in interesting category: coherent sheaves on nilp cone in $(\mathfrak{g}/\mathfrak{k})^*$.

gr for discrete series

Recall constr of *disc ser reps* starts with maxl torus $T \subset K$ (cplx alg), reg char $\lambda \in X^*(T)/W(K,T)$.

$\mathcal{B}$ = complete flag variety of Borel subalgs $\mathfrak{b} \subset \mathfrak{g}$.

$\lambda \rightsquigarrow \mathfrak{b}(\lambda) \supset \mathfrak{t} \rightsquigarrow Z(\lambda) = K \cdot \mathfrak{b}(\lambda) \simeq K/K \cap B(\lambda) \subset \mathcal{B}$

$\mathcal{L}(\lambda + \rho) \to \mathcal{B}$ algebraic line bundle induced by $\lambda + \rho$.

$\mathcal{D}$-module picture: $I(\lambda)$ = formal conormal derivatives of hol secs of $\mathcal{L}(\lambda+\rho)$ on closed K-orbit $Z(\lambda) \subset \mathcal{B}$

Recall Schmid: Taylor exp along cpt subvar $Z(\lambda)$:

$$I(\lambda)|_K \approx H^s(Z(\lambda), \mathcal{L}(\lambda+\rho) \otimes S(\underbrace{\mathfrak{g}/[\mathfrak{k}+\mathfrak{b}(\lambda)]}_{\text{conorm to } Z \text{ in } X})^*).$$

Serre duality, etc. $\rightsquigarrow$

$$\operatorname{gr} I(\lambda) \simeq H^0\left(Z(-\lambda), \mathcal{L}(\lambda+\rho-2\rho_c) \otimes S\left(\mathfrak{g}/[\mathfrak{k}+\mathfrak{b}(-\lambda)]\right)\right)$$

$$\simeq \text{pullback of } \mathcal{L}(\lambda+\rho-2\rho_c) \text{ to } T^*_{Z(-\lambda)}\mathcal{B}$$

Conormal geometry

Recall $\mathcal{B} = \{\text{Borel subalgebras } \mathfrak{b} \subset \mathfrak{g}\}$ *flag variety*.

Deduce $T^*\mathcal{B} = \{(\mathfrak{b}, \lambda) \mid \mathfrak{b} \in \mathcal{B}, \quad \lambda \in [\mathfrak{g}/\mathfrak{b}]^*\}$.

Recall $\mathcal{N}^* = \{\lambda \in \mathfrak{g}^* \mid \lambda|_{\mathfrak{b}} = 0, \text{some } \mathfrak{b} \in \mathcal{B}\}$, *nilp cone*.

Get *moment map* / *Springer resol* $\mu\colon T^*\mathcal{B} \to \mathcal{N}^*, \quad \mu(\mathfrak{b}, \lambda) = \lambda$.

$$\begin{array}{ccccc} & \text{affine} & T^*\mathcal{B} & \text{proper} & \\ & \overset{\pi}{\swarrow} & & \overset{\mu}{\searrow} & \\ \mathcal{B} & & & & \mathcal{N}^* \end{array}$$

Recall $\mathcal{N}^*_\theta = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$, *nilp cone in* $(\mathfrak{g}/\mathfrak{k})^*$.

Here Z = any K-orbit on $\mathcal{B}$.
$\mu(T^*_Z\mathcal{B})$ is irr, K-stab in $\mathcal{N}^*_\theta$,
so dense in K-orb closure.

Surjection $K\backslash\mathcal{B} \twoheadrightarrow K\backslash\mathcal{N}^*_\theta$.

Problem: understand corr.

$GL(n)$: Robinson-Schensted.

$$\begin{array}{ccccc} & & T^*\mathcal{B} & & \\ & \overset{\pi}{\swarrow} & \cup & \overset{\mu}{\searrow} & \\ \mathcal{B} & & & & \mathcal{N}^* \\ \cup & & T^*_Z\mathcal{B} & & \cup \\ & \overset{\pi}{\swarrow} & & \overset{\mu}{\searrow} & \\ Z & & & & \mathcal{N}^*_\theta \end{array}$$

Assoc varieties for std reps

Recall gr $I(\lambda) \simeq$ pullback of $\mathcal{L}(\lambda + \rho - 2\rho_c)$ to $T^*_{Z(-\lambda)}\mathcal{B}$.

affine $\qquad$ proper

$$\begin{array}{ccc} & T^*_{Z(-\lambda)}\mathcal{B} & \\ \pi \swarrow & & \searrow \mu \\ Z(-\lambda) & & \mathcal{N}^*_\theta \end{array}$$

gr $I(\lambda) \simeq \mu_* \pi^* \mathcal{L}(\lambda + \rho - 2\rho_c)$, coh sheaf on $\mu(T^*_{Z(-\lambda)})$.

Similarly gr $I(\gamma, \Psi) \rightsquigarrow$ conorm geom $\rightsquigarrow K\backslash\mathcal{N}^*_\theta$.

Two bases for $K\mathcal{C}(\mathfrak{g}, K)$:

Langlands $\left[(\lambda, \Psi) \text{ final}, \lambda \text{ char of } H^\theta\right]/K \rightsquigarrow$ gr $I(\lambda, \Psi)$

geometric orbit $K \cdot \xi \subset \mathcal{N}^*_\theta$, irr rep τ of $K^\xi \rightsquigarrow \Gamma\left[K \times_{K^\xi} E_\tau\right]$

Prob 1: *calculate* chg-bas-mtrx; KL $\rightsquigarrow$ assoc var(irr).

Prob 2: *understand* chg-bas-mtrx; prove (nearly) triang, $\rightsquigarrow$ (nearly) bijection between bases.

Problem 2 due to Lusztig in case $G(\mathbb{R})$ complex; resolved by Bezrukavnikov, Ostrik.

Being guided through representation theory