Regular polyhedra in *n* dimensions

David Vogan

Department of Mathematics
Massachusetts Institute of Technology

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Outline

The talk in one line

Want to understand the possibilities for a regular polyhedron P_n of dimension n.

Schläfli symbol is string $\{m_1, \ldots, m_{n-1}\}$.

Meaning of m_1: two-dimensional faces are regular m_1-gons.

Equivalent: m_1 edges ("1-faces") in a fixed 2-face.

Meaning of m_2: fixed vertex $\subset$ m_2 2-faces $\subset$ fixed 3-face.

$\vdots$

fixed $k-1$-face $\subset$ m_{k+1} $k+1$-faces $\subset$ fixed $k+2$-face.

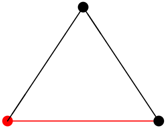
$\vdots$

What are the possible Schläfli symbols, and why do they characterize P_n?

Dimension two

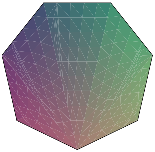
One regular m-gon for every $m \geq 3$

$m = 3$: equilateral triangle



Schläfli symbol $\{3\}$

$m = 7$: regular heptagon

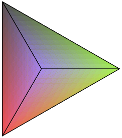


Schläfli symbol $\{7\}$

Dimension three

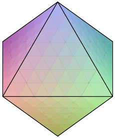
Five regular polyhedra.

tetrahedron



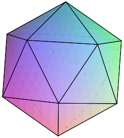
Schläfli symbol $\{3, 3\}$

octahedron



Schläfli symbol $\{3, 4\}$

icosahedron



Schläfli symbol $\{3, 5\}$

Dimensions one and zero

One regular 1-gon.

interval ———— Schläfli symbol $\{\}$

Symmetry group: two elements $\{1, s\}$

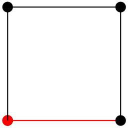
There is also just one regular 0-gon:

point • Schläfli symbol undefined

Symmetry group trivial (zero gens of order 2).

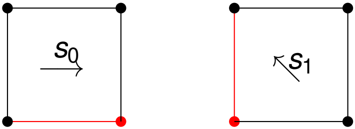
What's a regular polyhedron?

Something really symmetrical...like a square



FIX one vertex inside one edge inside square.

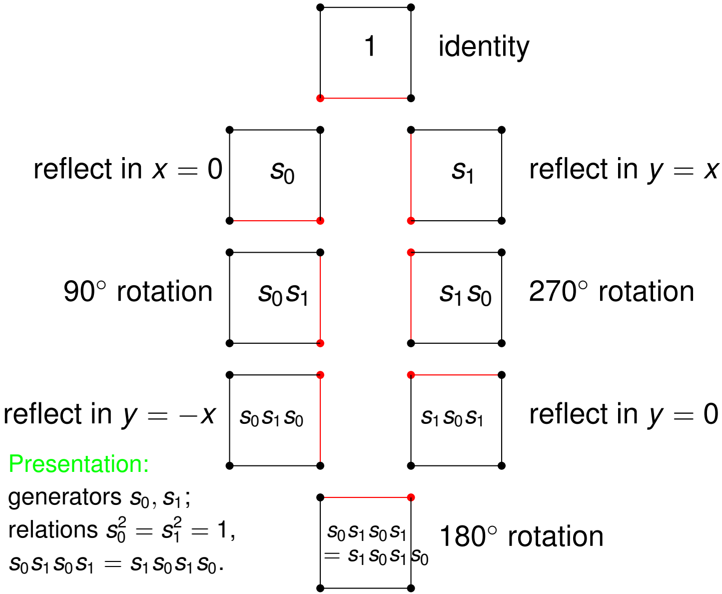
Two building block symmetries.



s_0 takes red vertex to adj vertex along red edge;

s_1 takes red edge to adj edge at red vertex.

More symmetries from building blocks



Presentation:
generators s_0, s_1;
relations $s_0^2 = s_1^2 = 1$,
$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

Understanding all regular polyhedra

Define a flag as a chain of faces like vertex $\subset$ edge.

Introduce basic symmetries like s_0, s_1 which change a flag as little as possible.

Find a presentation of the symmetry group.

Reconstruct the polyhedron from this presentation.

Decide which presentations are possible.

Most of linear algebra

V n-diml vec space $\rightsquigarrow GL(V)$ invertible linear maps.

complete flag in V is chain of subspaces $\mathcal{F}$

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V, \quad \dim V_i = i.$$

Stabilizer $B(\mathcal{F})$ called Borel subgroup of $GL(V)$.

Example

$V = k^n$, $V_i = \{(x_1, \ldots, x_i, 0, \ldots, 0) \mid x_i \in k\} \simeq k^i$.
Stabilizer of this flag is upper triangular matrices.

Theorem

1. *$GL(V)$ acts transitively on flags.*
2. *Stabilizer of one flag is isomorphic to group of invertible upper triangular matrices.*

Rest of linear algebra

Fix integers $\mathbf{d} = (0 = d_0 < d_1 < \cdots < d_r = n)$

Partial flag of type $\mathbf{d}$ is chain of subspaces $\mathcal{G}$

$$W_0 \subset W_1 \subset \cdots \subset V_{r-1} \subset W_r, \qquad \dim W_j = d_j.$$

Stabilizer $P(\mathcal{G})$ is a parabolic subgroup of $GL(V)$.

Theorem

Fix a complete flag $(0 = V_0 \subset \cdots \subset V_n = V)$*, and consider the* $n-1$ *partial flags*

$$\mathcal{G}_p = (V_0 \subset \cdots \subset \widehat{V}_p \subset \cdots \subset V_n) \quad 1 \le p \le n-1$$

obtained by omitting one proper subspace.

1. $GL(V)$ *is generated by the* $n-1$ *subgroups* $P(\mathcal{G}_p)$*.*
2. $P(\mathcal{G}_p)$ *is isomorphic to block upper-triangular matrices with a single* 2×2 *block.*

So build all linear transformations from two by two matrices and upper triangular matrices.

Flags

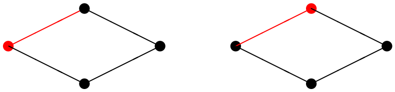
Suppose P_n compact n-diml convex polyhedron.

A (complete) flag $\mathcal{F}$ in P is a chain

$$P_0 \subset P_1 \subset \cdots \subset P_n, \qquad \dim P_i = i$$

with P_{i-1} a face of P_i.

Example



Two flags in two-diml P. Symmetry group (generated by reflections in x and y axes) is transitive on edges, not transitive on flags.

Definition

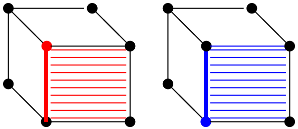
P regular if symmetry group acts transitively on flags.

Adjacent flags

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \qquad \dim P_i = i$$

complete flag in n-diml compact convex polyhedron.

A flag $\mathcal{F}' = (P'_0 \subset P'_1 \subset \cdots \subset P'_n)$ is i-adjacent to $\mathcal{F}$ if $P_j = P'_j$ for all $j \neq i$, and $P_i \neq P'_i$.



Three flags adjacent to $\mathcal{F}$, $i = 0, 1, 2$.
$\mathcal{F}'_0$: move vertex P_0 only. $\mathcal{F}'_1$: move edge P_1 only.
$\mathcal{F}'_2$: move face P_2 only.

There is exactly one $\mathcal{F}'$ i-adjacent to $\mathcal{F}$ (each $i = 0, 1, \ldots, n-1$).

Symmetry doesn't matter for this!

Stabilizing a flag

Lemma

Suppose $\mathcal{F} = (P_0 \subset P_1 \subset \cdots)$ is a complete flag in n-dimensional compact convex polyhedron P_n. Any affine map T preserving $\mathcal{F}$ acts trivially on P_n.

Proof. Induction on n. If $n = -1$, $P_n = \emptyset$ and result is true.

Suppose $n \geq 0$ and the the result is known for $n - 1$.

Write p_n = center of mass of P_n. Since center of mass is preserved by affine transformations, $Tp_n = p_n$.

By inductive hypothesis, T acts trivially on $n - 1$-diml affine span(P_{n-1}) spanned by P_{n-1}.

Easy to see that $p_n \notin$ span(P_{n-1}), so p_n and $(n-1)$-diml span(P_{n-1}) must generate n-diml span(P_n).

Since T trivial on gens, trivial on span(P_n). Q.E.D.

Compactness matters; result fails for $P_1 = [0, \infty)$.

Symmetries and flags

From now on P_n is a compact convex regular polyhedron with fixed flag

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \qquad \dim P_i = i$$

Write $p_i =$ center of mass of P_i.

Theorem

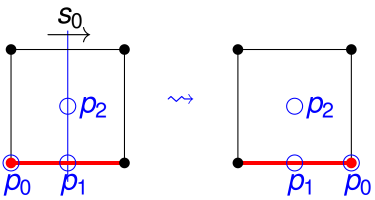
There is exactly one symmetry w of P_n for each complete flag $\mathcal{G}$, characterized by $w\mathcal{F} = \mathcal{G}$.

Corollary

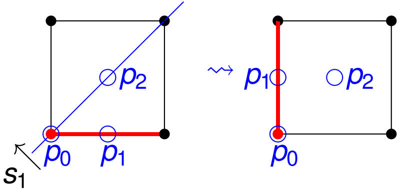
Define $\mathcal{F}'_i =$ unique flag (i)-adj to $\mathcal{F}$ $(0 \le i < n)$. There is a unique symmetry s_i of P_n characterized by $s_i(\mathcal{F}) = \mathcal{F}'_i$. It satisfies

1. $s_i(\mathcal{F}'_i) = \mathcal{F}$, $s_i^2 = 1$.
2. *s_i fixes the $(n-1)$-diml hyperplane through the n points $\{p_0, \ldots, p_{i-1}, \widehat{p_i}, p_{i+1}, \ldots, p_n\}$.*

Examples of basic symmetries s_i



This is s_0, which changes $\mathcal{F}$ only in P_0, so acts trivially on the line through p_1 and p_2.



This is s_1, which changes $\mathcal{F}$ only in P_1, so acts trivially on the line through p_0 and p_2.

What's a reflection?

On vector space V (characteristic not 2), a linear map s with $s^2 = 1$, $\dim(-1 \text{ eigenspace}) = 1$.

-1 eigenspace $L_s =$ span of nonzero vector $\alpha^\vee \in V$

$$L_s = \{v \in V \mid sv = -v\} = \text{span}(\alpha^\vee).$$

$+1$ eigenspace $H_s =$ kernel of nonzero $\alpha \in V^*$

$$H_s = \{v \in V \mid sv = v\} = \ker(\alpha).$$

$$sv = s_{(\alpha,\alpha^\vee)}(v) = v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha, \alpha^\vee\rangle}\alpha^\vee.$$

Definition of reflection does not mention "orthogonal."

If V has quadratic form $\langle , \rangle$ identifying $V \simeq V^*$, then

s is orthogonal $\iff$ α is proportional to $\alpha^\vee$.

Two reflections

$$sv = v - 2\frac{\langle \alpha_s, v\rangle}{\langle \alpha_s, \alpha_s^\vee\rangle}\alpha_s^\vee, \qquad tv = v - 2\frac{\langle \alpha_t, v\rangle}{\langle \alpha_t, \alpha_t^\vee\rangle}\alpha_t^\vee.$$

Assume $V = L_s \oplus L_t \oplus (H_s \cap H_t)$.

Subspace $L_s \oplus L_t$ has basis $\{\alpha_s^\vee, \alpha_t^\vee\}$, $c_{st} = 2\langle \alpha_s, \alpha_t^\vee\rangle / \langle \alpha_s, \alpha_s^\vee\rangle$;

$$s = \begin{pmatrix} -1 & c_{st} \\ 0 & 1 \end{pmatrix},\ t = \begin{pmatrix} 1 & 0 \\ c_{ts} & -1 \end{pmatrix}, \quad st = \begin{pmatrix} -1 + c_{st}c_{ts} & c_{st} \\ c_{ts} & -1 \end{pmatrix}.$$

$$\det(st) = 1, \qquad \mathrm{tr}(st) = -2 + c_{st}c_{ts},$$

st has eigenvalues z, z^{-1}, $\quad z + z^{-1} = c_{st}c_{ts} - 2$.

$$z, z^{-1} = e^{\pm i\theta}, \qquad \theta = \cos^{-1}(-1 + c_{st}c_{ts}/2)).$$

Proposition

Suppose $-1 + c_{st}c_{ts}/2 = \zeta + \zeta^{-1}$ for a primitive m^{th} root ζ. Then st is a rotation of order m in the plane $L_s \oplus L_t$. Otherwise st has infinite order. So

1. *$m = 2$ if and only if $c_{st} = c_{ts} = 0$;*
2. *$m = 3$ if and only if $c_{st}c_{ts} = 1$;*
3. *$m = 4$ if and only if $c_{st}c_{ts} = 2$;*
4. *$m = 6$ if and only if $c_{st}c_{ts} = 3$;*

Reflection symmetries

P_n cpt cvx reg polyhedron in $\mathbb{R}^n$, flag

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_k = k, \quad p_k = \text{ctr of mass}(P_k).$$

$s_k =$ symmetry preserving all P_j except P_k $(0 \leq k < n)$.

s_k must be orthogonal reflection in hyperplane

$$H_k = \text{span}(p_0, p_1, \ldots, p_{k-1}, \widehat{p_k}, p_{k+1}, \ldots, p_n)$$

(unique affine hyperplane through these n points).

Write equation of H_k

$$H_k = \{v \in \mathbb{R}^n \mid \langle \alpha_k, v \rangle = c_k\}.$$

α_k characterized up to positive scalar multiple by

$$\langle \alpha_k, p_j - p_n \rangle = 0 \quad (j \neq k), \qquad \langle \alpha_k, p_k - p_n \rangle > 0.$$

$$s_k(v) = v - \frac{2\langle \alpha_k, v - p_n \rangle}{\langle \alpha_k, \alpha_k \rangle} \alpha_k.$$

Two reflection symmetries

P_n cpt cvx reg polyhedron in $\mathbb{R}^n$, flag

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_k = k, \quad p_k = \text{ctr of mass}(P_k).$$

$$s_k = \text{orthogonal reflection in hyperplane}$$
$$H_k = \text{span}(p_0, p_1, \ldots, p_{k-1}, \widehat{p_k}, p_{k+1}, \ldots, p_n)$$

For $0 \leq k \leq n-2$, have seen that $s_k s_{k+1}$ must be rotation of some order m_{k+1} in a plane inside span(P_{k+2}), fixing P_{k-1}.

Proposition

Suppose P_n is an n-dimensional regular polyhedron. Then the rotation $s_k s_{k+1}$ acts transitively on the k-dimensional faces of P_n that are contained between P_{k-1} and P_{k+2}. Therefore the Schläfli symbol of P_n is $\{m_1, m_2, \ldots, m_{n-1}\}$.

We turn next to computing m_{k+1}.

Good coordinates

P_n cpt cvx reg polyhedron in $\mathbb{R}^n$, flag

$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_i = i, \quad p_i = \text{ctr of mass}(P_i)$.

Seek to relate coordinates for P_n to geometry...

Translate so center of mass is at the origin: $p_n = 0$.

Rotate so center of mass of $n-1$-face is on x-axis:

$p_{n-1} = (a_n, 0, \ldots), \quad a_n > 0$.

Now P_{n-1} is perp. to x-axis: $\text{span}(P_{n-1}) = \{x_1 = a_n\}$.

Rotate around the x axis so center of mass of $(n-2)$-face is in the $x-y$ plane: $p_{n-2} = (a_n, a_{n-1}, 0\ldots), \;\; a_{n-1} > 0$.

Now $\text{span}(P_{n-2}) = \{x_1 = a_n, x_2 = a_{n-1}\}$.

$$\vdots$$

$p_{n-k} = (a_n, \ldots, a_{n-k+1}, 0\ldots), \quad a_{n-k+1} > 0$.

$\text{span}(P_{n-k}) = \{x_1 = a_n,\; x_2 = a_{n-1} \ldots x_k = a_{n-k+1}\}$.

Reflections in good coordinates

P_n cpt cvx reg polyhedron in $\mathbb{R}^n$, flag

$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_i = i, \quad p_i = \text{ctr of mass}(P_i).$

$$p_k = (a_n, \ldots, a_{k+1}, 0 \ldots), a_{k+1} > 0.$$

Reflection symmetry s_k $(0 \le k < n)$ preserves all P_j *except* P_k, so fixes all p_j except p_k.

Fixes $p_n = 0$, so a reflection through the origin: $s_k = s_{\alpha_k}$, α_k orthogonal to all p_j except p_k.

Solve: $\alpha_k = (0, \ldots, 0, a_k, -a_{k+1}, 0, \ldots, 0)$ (entries in coordinates $n-k$ and $n-k+1$; $\alpha_0 = (0, \ldots, 0, 1)$.

$$s_{k_1} s_{k_2} = s_{k_2} s_{k_1}, \qquad |k_1 - k_2| > 1.$$

$$s_0 s_1 = \text{rotation by } \cos^{-1}\left(\frac{-a_1^2 + a_2^2}{a_1^2 + a_2^2}\right).$$

$$s_k s_{k+1} = \text{rot by } \cos^{-1}\left(\frac{-a_{k+1}^4 - a_k^2 a_{k+1}^2 - a_{k+1}^2 a_{k+2}^2 + a_k^2 a_{k+2}^2}{a_{k+1}^4 + a_k^2 a_{k+1}^2 + a_{k+1}^2 a_{k+2}^2 + a_k^2 a_{k+2}^2}\right).$$

Example: n-cube

$$P_n = \{x \in \mathbb{R}^n \mid -1 \le x_i \le 1 \quad (1 \le i \le n)\}.$$

Choose flag $P_k = \{x \in P_n \mid x_1 = \cdots = x_{n-k} = 1\}$, ctr of mass $p_k = (1, \ldots, 1, 0 \ldots, 0) \quad (n-k \text{ 1s})$.

$$s_k = \text{refl in } \alpha_k = (0, \ldots, 1, -1, \ldots, 0) = e_{n-k} - e_{n-k+1}$$

$$= \text{transpos of coords } n-k, n-k+1 \quad (1 \le k < n).$$

$$s_0 = \text{refl in } \alpha_0 = (0, \ldots, 0, 1) = e_n$$

$$= \text{sign change of coord } n.$$

$$s_k s_{k+1} = \text{rot by } \cos^{-1}\left(\frac{-1^4 - 1^4 - 1^4 + 1^4}{1^4 + 1^4 + 1^4 + 1^4}\right) = 2\pi/3 \quad (1 \le k)$$

$$s_0 s_1 = \text{rotation by } \cos^{-1}\left(\frac{-1^4 + 1^4}{1^4 + 1^4}\right) = 2\pi/4$$

Symmetry grp = permutations, sign changes of coords

$$= \langle s_0, \ldots s_{n-1} \rangle / \langle s_k^2 = 1, (s_k s_{k+1})^3 = 1, (s_0 s_1)^4 = 1 \rangle$$

$$(0 \le j < n,\ 1 \le k < n-1)$$

Angles and coordinates

$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_i = i, \quad p_i = \text{ctr of mass}(P_i).$

$$p_k = (a_n, \ldots, a_{k+1}, 0 \ldots), \quad a_{k+1} > 0.$$

Geom given by $n-1$ (strictly) positive reals $r_k = (a_{k+1}/a_k)^2$.

$$s_k s_{k+1} = \text{rotation by } \theta_{k+1} \in (0, \pi),$$

$$\cos(\theta_{k+1}) = \left(\frac{-1 + r_k - r_{k+1} - r_k r_{k+1}}{1 + r_k + r_{k+1} + r_k r_{k+1}} \right).$$

When $k = 0$, some terms disappear:

$$\cos(\theta_1) = \frac{-1 + r_1}{1 + r_1}, \qquad r_1 = \frac{1 + \cos(\theta_1)}{1 - \cos(\theta_1)}.$$

These recursion formulas give all r_k in terms of all θ_k.

Next formula is

$$r_2 = -\frac{\cos(\theta_1) + \cos(\theta_2)}{1 + \cos(\theta_2)}.$$

Formula makes sense (defines strictly positive r_2) iff $\cos(\theta_1) + \cos(\theta_2) < 0$.

Coxeter graphs

Regular polyhedron given by $n-1$ pos ratios $r_k = (a_{k+1}/a_k)^2$.

Symmetry group has n generators $s_0, \ldots, s_{n-1}$,

$$s_k^2 = 1, \quad s_k s_{k'} = s_{k'} s_k \ (|k-k'| > 1), \quad (s_k s_{k+1})^{m_{k+1}} = 1.$$

Here $m_{k+1} \geq 3$. Rotation angle for $s_k s_{k+1}$ must be

$$\theta_{k+1} = 2\pi/m_{k+1} \in \{120^\circ, 90^\circ, 72^\circ, 60^\circ \ldots\},$$

$$\cos(\theta_k) \in \left\{-\frac{1}{2},\ 0,\ \frac{\sqrt{5}-1}{4},\ \frac{1}{2}, \ldots\right\},$$

Group-theoretic information recorded in Coxeter graph

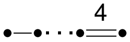
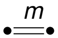
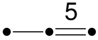
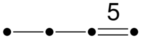
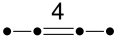
Recursion formulas give r_k from $\cos(\theta_k) = \cos(2\pi/m_k)$.

Condition $\cos(\theta_2) + \cos(\theta_1) < 0$ says

one of m_{k+1}, m_k must be 3; other at most 5.

Finite Coxeter groups with one line

Same ideas lead (Coxeter) to classification of all graphs for which recursion gives positive r_k.

type	diagram	G	$\|G\|$	reg poly
A_n		symm gp S_{n+1}	$n!$	n-simplex
BC_n		cube group	$2^n \cdot n!$	hypercube hyperoctahedron
$I_2(m)$		dihedral gp	$2m$	m-gon
H_3		H_3	120	icosahedron dodecahedron
H_4		H_4	14400	600-cell 120-cell
F_4		F_4	1152	24-cell

For much more, see Bill Casselman's amazing website

`http://www.math.ubc.ca/~cass/coxeter/crm.html`