Understanding reductive group representations

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Outline

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Cartan subgroups of $Sp(2n, \mathbb{R})$

Weyl groups for $Sp(2n, \mathbb{R})$

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Finding the representations of W

Thank you for allowing me to join this celebration

Slides available at
http://www-math.mit.edu/~dav/paper.html

Some history

I first met Professor Hou almost thirty years ago.

At that time, he began work on Langlands classification of representations of reductive groups.

Reductive groups are nice groups of real matrices, like

$$SL(n,\mathbb{R}) = n \times n \text{ real matrices of determinant } 1,$$

$$SO(1,1) = \left\{ \pm \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

A representation is a way to realize G as linear operators on a vector space, usually infinite-dimensional.

Langlands classification is a way to list all representations.

Professor Hou wrote papers explaining Langlands' list in many interesting cases.

Topic for today: given one term in Professor Hou's list, what does the representation look like?

What does Langlands classification look like?

Suppose K is a compact Lie group, with maximal torus T.

Define $X^*(T) =$ lattice of characters of T

Define $W = N_K(T)/T =$ Weyl group of T in K.

Example.

1. $K = U(n) = n \times n$ complex unitary matrices.
2. $T = U(1)^n = n$-dimensional torus.
3. $X^*(T) \simeq \mathbb{Z}^n$, character lattice.
4. $N_K(T) = n \times n$ permutation matrices (entries $e^{i\theta_j}$).
5. $W = S_n$ symmetric group of order $n!$.
6. W acts on $X^*(T)$ by permuting coordinates.
7. $\lambda \in X^*(T)$ regular if fixed only by $1 \in W$.

Theorem (Cartan-Weyl) Irr repns of K are parametrized by regular $\lambda \in X^*(T)/W$, $\lambda \rightsquigarrow \pi(\lambda)$.

This is Langlands classification for compact group K.

Note for experts: I am ignoring a translate of X^* by ρ.

What does one representation look like?

Listing representations of compact Lie group K is easy.

$K = U(n)$: representations are indexed by n-tuples of distinct integers up to permutation:

$$\lambda = (\lambda_1, \ldots, \lambda_n)/(\text{permutation}) \leftrightsquigarrow \pi(\lambda).$$

Examples

1. $\pi(n, n-1, \ldots, 1) =$ trivial representation, $\dim = 1$
2. $\pi(n+1, n-1, \ldots, 1) =$ representation on $\mathbb{C}^n$, $\dim = n$
3. $\pi(n+1, \ldots, n-p+2, n-p, \ldots, 1) = \bigwedge^p(\mathbb{C}^n)$, $\dim = \binom{n}{p}$
4. $\pi(n+q, n-1, \ldots, 1) = S^q(\mathbb{C}^n)$, $\dim = \binom{n+q-1}{q}$
5. $\pi(n+1, n-1, \ldots, 2, 0) =$ trace zero matrices, $\dim = n^2 - 1$

Moral of the story: it isn't easy to give a general description of the representation $\pi(\lambda)$. But there is a nice general formula for $\dim \pi(\lambda)$, a polynomial in the coordinates of λ.

How to compute $\dim \pi(\lambda)$

Case of $U(n)$: if $(\lambda_1, \dots, \lambda_n)$ distinct integers, then

$$\dim \pi(\lambda) =_{\text{def}} d_{U(n)}(\lambda) = \frac{1}{\prod_{k=1}^{n-1} k!} \prod_{i<j} (\lambda_i - \lambda_j).$$

General $K \supset T$ compact Lie: write

$$\Delta^\vee(K, T) \subset [X^*(T)]^* = \text{coroots of } T \text{ in } K.$$

Choose positive coroots $\Delta^{\vee,+}\Delta^\vee$, and set

$$d_K(\lambda) = \frac{1}{D_K} \prod_{\alpha^\vee \in \Delta^{\vee,+}} \langle \lambda, \alpha^\vee \rangle,$$

polynomial of degree $|\Delta^{\vee,+}|$ in λ.

Theorem (Weyl). Suppose $K \supset T$ compact Lie.

1. In the action of W on $S(\mathfrak{t})$ = poly functions on $X^*(T)$, d_K transforms by sign character sgn of W.
2. For an appropriate choice of the constant D_K,

$$\dim \pi(\lambda) = d_K(\lambda).$$

Weyl's dimension formula is first thing to know about $\pi(\lambda)$.

Dimension formula suggests defining $\pi(\lambda) = 0$ for singular λ.

Plan for this talk

Explain one example of a reductive group G.

$Sp(2n,\mathbb{R})$ = linear maps on $\mathbb{R}^{2n}$ preserving symplectic form.

Explain Langlands' list of representations of G (as made explicit by Professor Hou).

Indexed by characters of Cartan subgroups.

Explain version[s] of dimension for infinite-dimensional representations.

$\mathrm{Dim}\,\pi$ = Gelfand-Kirillov dimension, $m(\pi)$ = multiplicity.

Explain how to calculate these dimensions for representations on the list.

I have a marvelous explanation, but it does not fit in this margin.

$Sp(2n, \mathbb{R})$

On $\mathbb{R}^{2n}$ there is a skew-symmetric bilinear form

$$\omega((x_1, y_1), (x_2, y_2)) = x_1 \cdot y_2 - x_2 \cdot y_1 \qquad (x_i, y_i \in \mathbb{R}^n).$$

The symplectic group is linear maps preserving the form:

$$Sp(2n, \mathbb{R}) = \left\{ g \in GL(2n, \mathbb{R}) \mid \omega(g \cdot v_1, g \cdot v_2) = \omega(v_1, v_2) \quad (v_i \in \mathbb{R}^{2n}) \right\}.$$

The symplectic group is a Lie group of dimension $2n^2 + n$.

It is a great example of a reductive group: more complicated than $GL(n, \mathbb{R})$, but still allowing the use of linear algebra to calculate many things.

Easiest subgroup: $n = p + q$, $\mathbb{R}^{2n} = \mathbb{R}^{2p} \oplus \mathbb{R}^{2q} \rightsquigarrow$

$$Sp(2p, \mathbb{R}) \times Sp(2q, \mathbb{R}) \subset Sp(2n, \mathbb{R}).$$

Subgroups of $Sp(2n, \mathbb{R})$

1. It is very easy to check that

$$d\colon GL(n,\mathbb{R}) \hookrightarrow Sp(2n,\mathbb{R}), \quad d(\ell) = \begin{pmatrix} \ell & 0 \\ 0 & {}^t\ell^{-1} \end{pmatrix}.$$

2. In identification $\mathbb{C}^n \overset{j}{\simeq} \mathbb{R}^{2n}$, relate ω to (hermitian) dot product:

$$\omega(j(z_1), j(z_2)) = \mathrm{im}(z_1 \cdot z_2) \qquad (z_i \in \mathbb{C}^n).$$

   So the action of $U(n)$ on $\mathbb{C}^n$ defines

$$j\colon U(n) \hookrightarrow Sp(2n,\mathbb{R}).$$

3. The same idea as in (2) gives a natural inclusion

$$j\colon GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R}) \hookrightarrow Sp(4n,\mathbb{R}).$$

4. Suppose $n = p + 2s + q$. Then there is a natural inclusion

$$\begin{aligned}(\mathbb{R}^\times)^p \times (\mathbb{C}^\times)^s \times U(1)^q &\simeq GL(1,\mathbb{R})^p \times GL(1,\mathbb{C})^s \times U(1)^q \\ &\hookrightarrow GL(p,\mathbb{R}) \times GL(s,\mathbb{C}) \times U(q) \\ &\hookrightarrow GL(p,\mathbb{R}) \times GL(2s,\mathbb{R}) \times U(q) \\ &\hookrightarrow Sp(2p,\mathbb{R}) \times Sp(4s,\mathbb{R}) \times Sp(2q,\mathbb{R}) \\ &\hookrightarrow Sp(2n,\mathbb{R}).\end{aligned}$$

Cartan subgroups of $Sp(2n,\mathbb{R})$: list

In study of the reductive group $GL(n,\mathbb{C})$, the subgroup $GL(1,\mathbb{C})^n = (\mathbb{C}^\times)^n$ plays an important part.

Reason is that almost every element of $GL(n,\mathbb{C})$ is conjugate to an element of $GL(1,\mathbb{C})^n$.

The subgroup $GL(1,\mathbb{C})^n$ is called a Cartan subgroup or maximal torus of $GL(n,\mathbb{C})$.

For our group $G = Sp(2n,\mathbb{R})$, the subgroups

$$H_{p,s,q} = GL(1,\mathbb{R})^n \times GL(1,\mathbb{C})^s \times U(1)^q \qquad (p+2s+q=n)$$

(constructed on the previous slide) play the same role: almost every element of G has a conjugate in exactly one of the subgroups $H_{p,s,q}$.

Cartan subgroups of $Sp(2n, \mathbb{R})$: structure

Here is what the groups $H_{p,s,q}$ look like.

$$H_{1,0,0} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| \; t \in \mathbb{R}^\times \right\}$$

$$\simeq GL(1,\mathbb{R}) = \mathbb{R}^\times \subset Sp(2,\mathbb{R}) \quad (\text{eigenvalues } t,\, t^{-1})$$

$$H_{0,0,1} = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \; \theta \in \mathbb{R} \right\}$$

$$\simeq U(1) \subset Sp(2,\mathbb{R}) \quad (\text{eigenvalues } e^{i\theta},\, e^{-i\theta})$$

$$H_{0,1,0} = \left\{ \begin{pmatrix} r\cos\phi & r\sin\phi & 0 & 0 \\ -r\sin\phi & r\cos\phi & 0 & 0 \\ 0 & 0 & r^{-1}\cos\phi & -r^{-1}\sin\phi \\ 0 & 0 & r^{-1}\sin\phi & r^{-1}\cos\phi \end{pmatrix} \middle| \; z = re^{i\phi} \in \mathbb{C}^\times \right\}$$

$$\simeq GL(1,\mathbb{C}) = \mathbb{C}^\times \subset Sp(4,\mathbb{R}) \quad (\text{eigenvalues } z,\, z^{-1},\, \overline{z},\, \overline{z}^{-1})$$

For general (p, s, q), $H_{p,s,q}$ is a product of block-diagonal subgroups of these three forms.

Why Weyl groups matter

In study of the reductive group $GL(n, \mathbb{C})$, the Weyl group

$$W(GL(n,\mathbb{C}), GL(1,\mathbb{C})^n) = N_{GL(n,\mathbb{C})}(GL(1,\mathbb{C})^n)/GL(1,\mathbb{C})^n \simeq S_n$$

(symmetric group of order $n!$) plays an important part.

Reason is that two diagonal elements of $GL(n, \mathbb{C})$ are conjugate by $GL(n, \mathbb{C})$ if and only if they are conjugate by S_n.

That is, the list of n eigenvalues of a complex matrix is only defined up to permutation.

S_n is called the Weyl group of $GL(1, \mathbb{C})^n$ in $GL(n, \mathbb{C})$.

Similarly, we need to understand Weyl groups

$$W_{p,s,q} = W(Sp(2n, \mathbb{R}), H_{p,s,q}) = N_{Sp(2n,\mathbb{R})}(H_{p,s,q})/H_{p,s,q}.$$

These also are permutation groups of eigenvalues. . .

Weyl group of $Sp(2n,\mathbb{C})$

The complex reductive group $G(\mathbb{C}) = Sp(2n,\mathbb{C})$ has just one conjugacy class of Cartan subgroup, represented by

$$H(\mathbb{C}) = GL(1,\mathbb{C})^n \subset GL(n,\mathbb{C}) \subset Sp(2n,\mathbb{C}).$$

Its Weyl group

$$W_{\mathbb{C}} = W(G(\mathbb{C}),H(\mathbb{C})) = N_{G(\mathbb{C})}(H(\mathbb{C}))/H(\mathbb{C}) = W(BC_n)$$

is called the nth hyperoctahedral group.

$W(BC_n) = S_n \rtimes (\pm 1)^n$, so has order $2^n \cdot n!$.

$h = (z_1,\ldots,z_n) \in H(\mathbb{C})$ $(z_i \in \mathbb{C}^\times)$ has eigenvalues

$$((z_1,z_1^{-1}),(z_2,z_2^{-1}),\ldots,(z_n,z_n^{-1})).$$

$W(BC_n)$ permutes these n pairs, and reverses some of them.

$N_{G(\mathbb{C})}(H(\mathbb{C}))$ is generated by

1. permutation matrices in $GL(n)$ (the S_n subgroup); and
2. elements $\sigma_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the jth coordinate $Sp(2,\mathbb{C})$ subgroups (the $(\pm 1)^n$ normal subgroup).

Each real Weyl group $W_{p,s,q}$ is a subgroup of $W(BC_n)$.

Split and compact Weyl groups of $Sp(2n, \mathbb{R})$

The split Cartan of $Sp(2n, \mathbb{R})$ is

$$H_{n,0,0} = GL(1, \mathbb{R})^n = \{(t_1, \ldots, t_n) \mid t_j \in \mathbb{R}^x.$$

An element of $H_{n,0,0}$ is diagonal, entries $(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$.

The eigenvalues are therefore the n pairs (t_j, t_j^{-1}).

Just as in the complex case, the real Weyl group $W_{n,0,0}$ permutes these n pairs and reverses some of them.

Therefore $W_{n,0,0} = W_{\mathbb{C}} = W(BC_n)$, order $2^n \cdot n!$.

The compact Cartan of $Sp(2n, \mathbb{R})$ is

$$H_{0,0,n} = U(1)^n = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta_j \in \mathbb{R}.$$

The eigenvalues are the n pairs $(e^{i\theta_j}, e^{-i\theta_j})$.

The real Weyl group $W_{0,0,n}$ permutes these n pairs.

Therefore $W_{0,0,n} = S_n$, order $n!$.

Complex Weyl groups of $Sp(2n,\mathbb{R})$

The complex Cartan of $Sp(4n,\mathbb{R})$ is

$$H_{0,n,0} = GL(1,\mathbb{C})^n = \{(z_1,\ldots,z_n) \mid z_j \in \mathbb{C}^\times.$$

The eigenvalues are n pairs of pairs $((z_j, z_j^{-1}), (\overline{z_j}, \overline{z_j}^{-1}))$.

The Weyl group $W_{0,n,0}$ has a subgroup $W(BC_n)$ which acts by permuting the n pairs of pairs and (on some of them) interchanging each of the two internal pairs:

$$((z_j, z_j^{-1}), (\overline{z_j}, \overline{z_j}^{-1})) \mapsto ((z_j^{-1}, z_j), (\overline{z_j}^{-1}, \overline{z_j})).$$

There is another (normal) subgroup $\{\pm 1\}^n$ which on a subset of the pairs of pairs interchanges the outer pair:

$$((z_j, z_j^{-1}), (\overline{z_j}, \overline{z_j}^{-1})) \mapsto ((\overline{z_j}, \overline{z_j}^{-1}), (z_j, z_j^{-1})).$$

Real Weyl group is $W_{0,n,0} = W(BC_n) \ltimes \{\pm 1\}^n$, order $2^{2n} \cdot n!$.

$$\begin{aligned} W_{p,s,q} &= W_{p,0,0} \times W_{0,s,0} \times W_{0,0,q} \\ &= W(BC_p) \times [W(BC_s)_\Delta \ltimes \{\pm 1\}^s] \times S_q \\ &\subset W(BC_p) \times W(BC_{2s}) \times W(BC_q) \subset W(BC_{p+2s+q}). \end{aligned}$$

Characters of Cartan subgroups

Suppose $H \subset G$ is Cartan subgroup of reductive G.

Character of H = homom. $\lambda\colon H \to \mathbb{C}^\times$; $\widehat{H}$ = all chars of H.

Examples

1. $H = T \subset K$ maximal torus; $\widehat{T} = X^*(T)$ lattice in $\mathfrak{t}^*$.
2. $H = H_{1,0,0} = \mathbb{R}^\times \subset Sp(2,\mathbb{R})$;
$$\widehat{H_{1,0,0}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}, \quad \lambda_{\epsilon,\nu}(t) = \mathrm{sgn}(t)^\epsilon \cdot |t|^\nu.$$
3. $H = H_{0,0,1} = U(1) \subset Sp(2,\mathbb{R})$;
$$\widehat{H_{0,0,1}} = \mathbb{Z}, \quad \lambda_m(e^{i\theta}) = e^{im\theta}.$$
4. $H = H_{0,1,0} = \mathbb{C}^\times \subset Sp(4,\mathbb{R})$;
$$\widehat{H_{0,1,0}} = \{(\nu_1,\nu_2) \mid \nu_i \in \mathbb{C},\ \nu_1 - \nu_2 \in \mathbb{Z}\},$$
$$\lambda_{\nu_1,\nu_2}(re^{i\phi}) = r^{\nu_1+\nu_2} e^{i(\nu_1-\nu_2)\phi}.$$

Summary: character ξ of $H_{p,r,q} \subset Sp(2n,\mathbb{R})$ $\leftrightsquigarrow$

1. $\nu \in \mathbb{C}^n$ subject to
2. integrality conditions
   a) $\nu_i \in \mathbb{Z}$ for last q coordinates;
   b) $\nu_j - \nu_{j+1} \in \mathbb{Z}$ for middle r pairs of coordinates; PLUS
3. p choices of parity $\epsilon_k \in \mathbb{Z}/2\mathbb{Z}$.

Langlands classification

Theorem (Langlands) Irreducible representations of a real reductive group G are in one-to-one correspondence

$$(H, \lambda)/(G \text{ conjugacy}) \leftrightsquigarrow \pi(H, \lambda)$$

subject to

1. $H \subset G$ is a Cartan subgroup, $\lambda \in \widehat{H}$ a character;
2. λ nontrivial on each compact imaginary simple coroot; and
3. λ nontrivial on each simple real coroot.

(2) is the "regularity" condition in Langlands classification for K;

(3) excludes the reducible tempered principal series of $SL(2, \mathbb{R})$

Means reps of $Sp(2n, \mathbb{R})$ appear in families, one for each expression $n = p + 2r + q$. Representation is indexed by

1. p pairs $(\nu_i, \epsilon_i) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$;
2. r pairs $(\nu_j, \nu_{j+1}) \in \mathbb{C} \times \mathbb{C}$, with $\nu_j - \nu_{j+1} \in \mathbb{Z}$; and
3. q integers $\nu_k \in \mathbb{Z}$.

Experts: omitted translate of $\widehat{H}$ by ρ, choice of pos singular imag roots.

Infinitesimal characters

Promised to get dimension of $\pi(\lambda)$ from λ.

First: more technical but easier infinitesimal character.

$H \subset G$ Cartan subgroup $\rightsquigarrow \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ complexified Lie algs.

$W_{\mathbb{C}} = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ = complex Weyl group.

$\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ = center of enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$.

Theorem (Harish-Chandra)

1. $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ acts by scalars on each irr representation π of G, $\rightsquigarrow$ infinitesimal character of π: $\xi_\pi \colon \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$.
2. $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \simeq S(\mathfrak{h}_{\mathbb{C}})^{W_{\mathbb{C}}}$.
3. Homomorphisms $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ are indexed by $\mathfrak{h}_{\mathbb{C}}^*/W_{\mathbb{C}}$:
$$\nu \in \mathfrak{h}_{\mathbb{C}}^* \mapsto (\xi_\nu \colon \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C})$$
4. IF $\lambda \in \widehat{H}$, $\quad d\lambda = \nu \in \mathfrak{h}_{\mathbb{C}}^*$, THEN $\pi(\lambda)$ has infl char ξ_ν.

Says representation $\pi(\lambda)$ can be realized inside eigenspaces for G-invariant differential operators, eigenvalues $\leftsquigarrow \nu = d\lambda$.

Technically valuable information, but not very concrete.

Counting representations (A)

So far we know (for $G = Sp(2n,\mathbb{R})$) how to

1. list Cartan subgroups $H_{p,s,q}$ ($p + 2s + q = n$);
2. describe Weyl groups $W_{p,s,q}$;
3. describe characters $\widehat{H_{p,s,q}}$;
4. parametrize G-representations using characters of Cartans;
5. find infinitesimal characters of G-representations.

Recall: infl character for $Sp(2n,\mathbb{R})$ is $\nu \in \mathbb{C}^n/W(BC_n)$.

This is n-tuple from $\mathbb{C}$ mod permutation, sign changes.

infinitesimal character of $\pi(\lambda)$ is $d\lambda$.

infinitesimal character of trivial rep is $[n, n-1, \ldots, 1]$.

$$\#\left\{\lambda \in \widehat{H_{p,s,q}},\ d\lambda \sim [n,\ldots,1]\right\} = |W(BC_n)| \cdot 2^p = 2^{n+p} \cdot n!$$

$$\#\left\{\text{reps} \leftrightarrow H_{p,s,q},\ \text{triv infl char}\right\} = |W(BC_n)/W_{p,s,q}| \cdot 2^p = \frac{2^{p+q} \cdot n!}{p!s!q!}.$$

Counting representations (B)

Last slide sketched a formula for $Sp(2n,\mathbb{R})$

$$\#\{\text{reps of triv infl char}\} = \sum_{n=p+2s+q} |W(BC_n)/W_{p,s,q}| \cdot 2^p$$

Here $W_{p,s,q} = W(BC_p) \times W_{0,s,0} \times S_q$.

Factor 2^p comes from characters of $H_{p,s,q}/(H_{p,s,q})_0$.

Such a character corresponds to subset of $\{1,\ldots p\}$.

$W(BC_p)$ acts; stabilizer of p_1-element subset is $W(BC_{p_0}) \times W(BC_{p_1})$ $(p = p_0 + p_1)$.

Conclude that

$$\#\{\text{reps of triv infl char}\} = \sum_{n=p_0+p_1+2s+q} |W(BC_n)/W_{p_0,p_1,s,q}|.$$

Here $W_{p_0,p_1,s,q} = W(BC_{p_0}) \times W(BC_{p_1}) \times W_{0,s,0} \times S_q$.

Such counting problems are addressed by Professor Hou's work.

We'll refine this to get detailed info about G reps.

Gelfand-Kirillov dimension

Suppose G real reductive and π irreducible rep of G.

Can attach to π two integers

$$\mathrm{Dim}(\pi) = \text{GK dimension}, \quad 0 \le \mathrm{Dim}(\pi) \le (\dim(G/H))/2$$
$$m(\pi) = \text{multiplicity}, \qquad 1 \le m(\pi) < \infty.$$

Definitions are complicated, so just give EXAMPLES...

1. $\mathrm{Dim}(\pi) = 0$ if and only if π is finite-dimensional.
2. If $\mathrm{Dim}(\pi) = 0$, then $m(\pi) = \dim \pi$.
3. Suppose $P = LU$ parabolic subgroup of G, and π_L finite-dimensional rep of L. Define

   $$\pi = \mathrm{Ind}_P^G \pi_L = \text{secs of vec bdle on } G/P \text{ with fiber } \pi_L.$$

   Then $\mathrm{Dim}(\pi) = \dim(G/P)$ and $m(\pi) = \dim \pi_L$.

Examples say: if π = sections of rank m vec bdle on d-diml manifold, then $\mathrm{Dim}(\pi) = d$ and $m(\pi) = m$.

Value of GK dim and mult: use intuition from manifolds and vector bundles even when they don't exist.

Theoretical facts about GK dimension

$G_{\mathbb{C}}$ complex reductive, Lie algebra $\mathfrak{g}_{\mathbb{C}}$, $W_{\mathbb{C}}$ Weyl group.

$\mathcal{N}^* \subset \mathfrak{g}_{\mathbb{C}}^* =$ complex nilpotent cone $= \bigcup_{i=0}^m \mathcal{O}_i^*$ $G_{\mathbb{C}}$ orbits.

$\mathcal{O}_0^* = \{0\}$ zero orbit, unique of dimension 0.

$\mathcal{O}_m^* =$ principal orbit, unique of dimension $\dim(G_{\mathbb{C}}/H_{\mathbb{C}})$.

$0 < \dim \mathcal{O}_i^* < \dim(G_{\mathbb{C}}/H_{\mathbb{C}}), \quad (0 < i < m)$.

Theorem (Barbasch-V) Attached to any $\pi \in \widehat{G}$ is nilpotent orbit $\mathcal{O}^*(\pi)$, such that $\mathrm{Dim}(\pi) = (\dim \mathcal{O}^*(\pi))/2$.

A refinement of how do you compute $\mathrm{Dim}(\pi(\lambda))$ from λ? is how do you compute $\mathcal{O}^*(\pi(\lambda))$ from λ?

Nilpotents and Weyl group representations

Theorem (Springer)

1. There is an inclusion (nilpotent orbits) $\hookrightarrow$ (Weyl group reps)
$$\mathcal{N}^*/G_{\mathbb{C}} \hookrightarrow \widehat{W_{\mathbb{C}}}, \quad \mathcal{O}^* \mapsto \sigma(\mathcal{O}^*).$$
2. Define $d(\mathcal{O}^*) = (\dim(G_{\mathbb{C}}/H_{\mathbb{C}}) - \dim(\mathcal{O}^*))/2$, half the codimension of $\mathcal{O}^*$ in $\mathcal{N}^*$. Then $\sigma(\mathcal{O}^*)$ has multiplicity one in $S^{d(\mathcal{O}^*)}(\mathfrak{h}_{\mathbb{C}})$, and multiplicity zero in all lower degrees.

Any irreducible $\sigma \in \widehat{W_{\mathbb{C}}}$ occurs in $S(\mathfrak{h}_{\mathbb{C}})$, so we can define
$$d(\sigma) = \text{smallest } d \text{ so } \sigma \subset S^d(\mathfrak{h}_{\mathbb{C}}).$$

Springer's theorem says that $d(\sigma(\mathcal{O}^*)) = d(\mathcal{O}^*)$, or
$$\dim(\mathcal{O}^*) = \dim(G_{\mathbb{C}}/H_{\mathbb{C}}) - 2 \cdot d(\sigma(\mathcal{O}^*)).$$

Approximately this means the more complicated the representation $\sigma(\mathcal{O}^*)$, the smaller the orbit.

Theoretical facts about multiplicity

Suppose $\lambda \in \widehat{H}$ Langlands parameter, so $\pi(\lambda) \in \widehat{G}$.

$$\begin{aligned} X^*(H) &= \text{lattice of rational characters of } H \subset \widehat{H} \\ &= \text{weights of finite-dimensional algebraic reps of } G \end{aligned}$$

$$\lambda\text{-dominant weights} = \left\{ \gamma \in X^*(H) \mid \begin{array}{rcl} d\lambda(\alpha^\vee) & = & \text{pos int} \\ \Longrightarrow & \gamma(\alpha^\vee) \geq 0 & \end{array} \right\}$$

$$=_{\text{def}} X^{*,+}(\lambda),$$

a cone in the lattice $X^*(H)$.

Theorem (Jantzen-Zuckerman, Joseph, Barbasch-Vogan)

1. For every $\gamma \in X^{*,+}(\lambda)$, $\lambda + \gamma$ is a Langlands parameter.
2. Reps $\pi(\lambda + \gamma)$ (a translation family) give the same nilp orbit:
$$\mathcal{O}^*(\pi(\lambda)) = \mathcal{O}^*(\pi(\lambda + \gamma)) \quad (\gamma \in X^{*,+}(\lambda)),$$
so the Gelfand-Kirillov dimension is constant on the family.
3. Multiplicity varies on the family by a polynomial $\mu \in S^{d(\mathcal{O}^*)}(\mathfrak{h}_{\mathbb{C}})$:
$$m(\pi(\lambda + \gamma)) = \mu(d\lambda + \gamma).$$
4. The polynomial μ is in the (unique) copy $\sigma(\mathcal{O}^*) \subset S^{d(\mathcal{O}^*)}(\mathfrak{h}_{\mathbb{C}})$.

Summary about GK dimension

Irreducible $\pi = \pi(\lambda)$ of G gives translation family

$$\pi(\lambda + \gamma) \qquad (\gamma \in X^{*,+}(\lambda)).$$

Multiplicity varies in family by polynomial $\mu(\pi) \in S^{d(\pi)}(\mathfrak{h}_{\mathbb{C}})$.

Space $W_{\mathbb{C}} \cdot \mu$ is irr rep $\sigma(\pi) \in \widehat{W_{\mathbb{C}}}$; $\quad d(\sigma(\pi)) = d(\pi)$.

$\sigma =$ Springer representation of nilpotent orbit $\mathcal{O}^*(\pi)$.

$\mathrm{Dim}(\pi) = (\dim(\mathcal{O}^*(\pi))/2 = (\dim G/H)/2 - d(\pi)$.

$\mathrm{Dim}(\pi)$, $m(\pi)$, $\sigma(\pi)$, $d(\pi)$, $\mathcal{O}^*(\pi)$ are tied together.

Will discuss how to compute $\sigma(\pi)$.

Identifying the W representation

Recall from Counting representations the $Sp(2n, \mathbb{R})$ formula

$$\#\{\text{reps of triv infl char}\} = \sum_{n=p_0+p_1+2s+q} |W(BC_n)/W_{p_0,p_1,s,q}|.$$

We want to refine number to representation of $W(BC_n)$.

Langlands classification required pos imag roots for $H_{p,s,q}$.

Gives natural one-diml character $\epsilon_{p,s,q} \colon W_{p,s,q} \to \{\pm 1\}$,

$$\epsilon_{p,s,q}(w) = (-1)^{\#\text{pos imag roots changing sign under } w}.$$

Attach to $H_{p,s,q}$ the $W(BC_n)$ representations

$$\tau(p_0, p_1, s, q) = \mathrm{Ind}_{W_{p_0,p_1,s,q}}^{W(BC_n)} \epsilon_{p,s,q}.$$

Counting formula becomes

$$\#\{\text{reps of triv infl char}\} = \sum_{n=p_0+p_1+2s+q} \dim \tau_{p_0,p_1,s,q}.$$

Stopping just when it's interesting

Explained irr G-rep $\pi(\lambda) \rightsquigarrow$ irreducible $W_{\mathbb{C}}$-rep $\sigma(\pi)$ encoding $\mathrm{Dim}(\pi(\lambda))$.

Computing $\sigma(\pi)$ is hard: Kazhdan-Lusztig theory.

Explained structure of $G \rightsquigarrow$ reducible $W_{\mathbb{C}}$-reps $\tau_{p_0,p_1,s,q}$.

Computing $\tau_{p_0,p_1,s,q}$ is easy: symmetric group representation theory, combinatorics of partitions.

Would be nice to relate easy problem to hard one.

Theorem Suppose $G = Sp(2n, \mathbb{R})$, and $\mathcal{O}^*$ is a nilpotent orbit with Springer rep $\sigma(\mathcal{O}^*$. Then the number of G-reps π of trivial infinitesimal character with $\mathcal{O}^*(\pi) = \mathcal{O}^*$ is related to the multiplicity of $\sigma(\mathcal{O}^*)$ in $\tau_{p_0,p_1,s,q}$.

HAPPY BIRTHDAY

PROFESSOR HOU!