The orbit method for reductive groups

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Outline

Introduction: a few things I didn't learn from Bert

Commuting algebras: how representation theory works

Differential operator algebras: how orbit method works

Hamiltonian *G*-spaces: how Bert does the orbit method

Orbits for reductive groups: what else to steal from Bert

Meaning of it all

Abstract harmonic analysis

Say Lie group G acts on manifold M. Can ask about

- topology of M
- solutions of G-invariant differential equations
- special functions on M (automorphic forms, etc.)

Method step 1: LINEARIZE. Replace M by Hilbert space $L^2(M)$. Now G acts by unitary operators.

Method step 2: DIAGONALIZE. Decompose $L^2(M)$ into minimal G-invariant subspaces.

Method step 3: REPRESENTATION THEORY. Study minimal pieces: irreducible unitary repns of G.

Difficult questions: how does DIAGONALIZE work, and what do minimal pieces look like?

Plan of talk

- Outline strategy for decomposing $L^2(M)$: analogy with "double centralizers" in finite-diml algebra.
- Strategy $\rightsquigarrow$ **Philosophy of coadjoint orbits:**

irreducible unitary representations of Lie group G

$$\Updownarrow$$

(nearly) symplectic manifolds with (nearly) transitive Hamiltonian action of G

- "Strategy" and "philosophy" have a lot of wishful thinking. Describe theorems supporting $\Updownarrow$.

Decomposing a representation

Given: interesting operators $\mathcal{A}$ on Hilbert space $\mathcal{H}$.
Goal: decompose $\mathcal{H}$ in $\mathcal{A}$-invt way.

Finite-dimensional case:

$V/\mathbb{C}$ fin-diml, $\mathcal{A} \subset \mathrm{End}(V)$ cplx semisimple alg of ops.

Classical structure theorem:

$W_1, \ldots, W_r$ list of all simple $\mathcal{A}$-modules; then

$$\mathcal{A} \simeq \mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_r) \quad V \simeq m_1 W_1 + \cdots + m_r W_r.$$

Positive integer m_i is *multiplicity* of W_i in V.

Slicker version: define *multiplicity space*
$M_i = \mathrm{Hom}_{\mathcal{A}}(W_i, V)$; then $m_i = \dim M_i$, and

$$V \simeq M_1 \otimes W_1 + \cdots + M_r \otimes W_r.$$

Slickest version: COMMUTING ALGEBRAS...

Commuting algebras and all that

Theorem

Suppose $\mathcal{A}$ is semisimple algebras of operators on V as above; define $\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$, a second semisimple algebra of operators on V.

1. *Relation between $\mathcal{A}$ and $\mathcal{Z}$ is symmetric:*
$$\mathcal{A} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{Z}).$$
2. *There is a natural bijection between irr modules W_i for $\mathcal{A}$ and irr modules M_i for $\mathcal{Z}$, given by*
$$M_i \simeq \mathrm{Hom}_{\mathcal{A}}(W_i, V), \qquad W_i \simeq \mathrm{Hom}_{\mathcal{Z}}(M_i, V).$$
3. $V \simeq \sum_i M_i \otimes W_i$ *as a module for $\mathcal{A} \times \mathcal{Z}$.*

Example 1: finite G acts left and right on $\mathbb{C}[G]$.

Example 2: S_n and $GL(E)$ act on $V = T^n(E)$.

But those are stories for other days...

Infinite-dimensional representations

Need framework to study ops on inf-diml V.

Finite-diml $\leftrightarrow$ infinite-diml dictionary

finite-diml V	$\leftrightarrow$	$C^\infty(M)$
repn of G on V	$\leftrightarrow$	action of G on M
$\mathrm{End}(V)$	$\leftrightarrow$	$\mathrm{Diff}(M)$
$\mathcal{A} = \mathrm{im}(\mathbb{C}[G]) \subset \mathrm{End}(V)$	$\leftrightarrow$	$\mathcal{A} = \mathrm{im}(U(\mathfrak{g})) \subset \mathrm{Diff}(M)$
$\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$	$\leftrightarrow$	$\mathcal{Z} = G$-invt diff ops

Suggests: G-irreducible pieces of function space correspond to simple modules for G-invt diff ops.

Which differential operators commute with G?

Answer leads to generalizations of dictionary...

Differential operators and symbols

$$\mathrm{Diff}_n(M) = \text{diff operators of order} \le n.$$

Increasing filtration, $(\mathrm{Diff}_p)(\mathrm{Diff}_q) \subset \mathrm{Diff}_{p+q}$.

Theorem (Symbol calculus)

1. *There is an isomorphism of graded algebras*

$$\sigma\colon \operatorname{gr}\mathrm{Diff}(M) \to \mathrm{Poly}(T^*(M))$$

to fns on $T^*(M)$ that are polynomial in fibers.

2. $$\sigma_n\colon \mathrm{Diff}_n(M)/\mathrm{Diff}_{n-1}(M) \to \mathrm{Poly}^n(T^*(M)).$$

3. *Commutator of diff ops* $\rightsquigarrow$ *Poisson bracket* $\{,\}$ *on* $T^*(M)$*: for* $D \in \mathit{Diff}_p(M)$, $D' \in \mathrm{Diff}_q(M)$,

$$\sigma_{p+q-1}([D, D']) = \{\sigma_p(D), \sigma_q(D')\}.$$

Diff ops comm with $G \leftrightsquigarrow$ symbols Poisson-comm with $\mathfrak{g}$.

Poisson structure and Lie group actions

X mfld w. Poisson $\{,\}$ on fns (e.g. $T^*(M)$).

Bracket with $f \rightsquigarrow \xi_f \in \text{Vect}(X)$: $\xi_f(g) = \{f, g\}$.

Vector flds ξ_f called *Hamiltonian*; preserve $\{,\}$. Map $C^\infty(X) \to \text{Vect}(X)$, $f \mapsto \xi_f$ is Lie alg hom.

G action on $X \rightsquigarrow$ Lie alg hom $\mathfrak{g} \to \text{Vect}(X)$, $Y \mapsto \xi_Y$.

Call X *Hamiltonian G-space* if the Lie alg action *lifts*

$$\begin{array}{ccc} & & C^\infty(X) \\ & \nearrow & \downarrow \\ \mathfrak{g} & \to & \text{Vect}(X) \end{array} \qquad \begin{array}{ccc} & & f_Y \\ & \nearrow & \downarrow \\ Y & \to & \xi_Y \end{array}$$

Map $\mathfrak{g} \to C^\infty(X)$ same as moment map $\mu\colon X \to \mathfrak{g}^*$.

Example. G acts on $M \Rightarrow T^*(M)$ is Hamiltonian G-space: Lie alg elt $Y \rightsquigarrow$ vec fld ξ_Y on $M \rightsquigarrow$ function f_Y on $T^*(M)$:

$$f_Y(m, \lambda) = \lambda(\xi_Y(m)) \qquad (m \in M, \lambda \in T^*_m(M)).$$

function f on X with $\{f, \mathfrak{g}\} = 0 \Leftrightarrow f$ constant on G orbits.

G action transitive $\Rightarrow$ only $[\mathbb{C}, G] = 0 \overset{?}{\leftrightsquigarrow}$ irr repn of G

Our story so far...

G acts on $M \leftrightsquigarrow T^*(M)$ Hamiltonian G-space.

G-decomp of $C^\infty(M) \leftrightsquigarrow (\text{Diff}\, M)^G$-modules.

$(\text{Diff}\, M)^G \overset{\sigma}{\leftrightsquigarrow} C^\infty(T^*(M))^G \leftrightsquigarrow C^\infty((T^*(M))/G)$.

Hope $C^\infty(M)$ irr $\Leftrightarrow$ G has dense orbit on $T^*(M)$.

Suggests generalization...

Hamiltonian G-cone $X \rightsquigarrow$ graded alg Poly(X).

Seek filtered alg $\mathcal{D}$, symbol calc $\text{gr}\,\mathcal{D} \xrightarrow{\sigma} \text{Poly}(X)$ carrying $[,]$ on $\mathcal{D}$ to $\{,\}$ on Poly(X).

Seek to lift G action on Poly(X) to G action on $\mathcal{D}$ via Lie alg hom $\mathfrak{g} \to \mathcal{D}_1$.

Seek simple $\mathcal{D}$-module $\mathcal{W}$ (analogue of $C^\infty(M)$).

Hope $\mathcal{W}$ irr for $G \Leftrightarrow G$ has dense orbit on X.

Suggests: irreducible representations of $G \leftrightsquigarrow$ homogeneous Hamiltonian G-spaces.

Method of coadjoint orbits

Recall: Hamiltonian G-space X comes with (G-equivariant) moment map $\mu\colon X \to \mathfrak{g}^*$.

Kostant's theorem: homogeneous Hamiltonian G-space $=$ covering of G-orbit on $\mathfrak{g}^*$.

Includes classification of symp homog spaces for G. (Riem homog spaces hopelessly complicated.)

Recall: commuting algebra formalism for diff operators suggests irreducible representations $\leftrightsquigarrow$ homogeneous Hamiltonian G-spaces.

Kirillov-Kostant philosophy of coadjt orbits suggests

$$\{\text{irr unitary reps of } G\} = \widehat{G} \leftrightsquigarrow \mathfrak{g}^*/G. \qquad (\star)$$

MORE PRECISELY... restrict right side to "admissible" orbits (integrality cond). Expect to find "almost all" of $\widehat{G}$: enough for interesting harmonic analysis.

Evidence for orbit method

With the caveat about restricting to admissible orbits...

$$\widehat{G} \leftrightsquigarrow \mathfrak{g}^*/G. \qquad (\star)$$

$(\star)$ is true for G simply conn nilpotent (Kirillov).

$(\star)$ is true for G type I solvable (Auslander-Kostant).

$(\star)$ for algebraic G reduces to reductive G (Duflo).

Case of reductive G is still open.

Actually $(\star)$ is false for connected nonabelian reductive G.

But there are still theorems close to $(\star)$.

Two ways to do repn theory for reductive G:

1. start with coadjt orbit, look for repn. Hard.
2. start with repn, look for coadjt orbit. Easy.

Really need to do both things at once. Having started to do mathematics in the Ford administration, I find this challenging. (Gave up chewing gum at that time.)

Structure theory for reductive Lie groups

Reductive Lie group G $=$ closed subgp of $\underbrace{GL(n,\mathbb{R})}_{\text{main ex}}$

s.t. G closed under transpose, and $\#G/G_0 < \infty$.

From now on G is reductive.

$\mathrm{Lie}(G) = \mathfrak{g} \subset n \times n$ matrices. Bilinear form

$T(X, Y) = \mathrm{tr}(XY) \Rightarrow \mathfrak{g} \overset{G\text{-eqvt}}{\simeq} \mathfrak{g}^*$

Orbits of G on $\mathfrak{g}^* \subset$ conjugacy classes of matrices.

Orbits of $GL(n,\mathbb{R})$ on $\mathfrak{g}^* =$ conj classes of matrices.

Family of orbits: for real numbers λ_1 and λ_{n-1},

$$\mathcal{O}(\lambda_1, \lambda_{n-1}) = \text{ matrices, eigenvalue } \lambda_p \text{ has mult } p.$$

Base point in family:

$$\mathcal{O}_{1,n-1} = \text{ nilp matrices, Jordan blocks } 1,\ n-1.$$

One irreducible unitary representation

V n-diml real. $G = GL(V)$ acts on $M = \mathbb{P}V$ = lines in V.

$$X = T^*(M) = \{(v, \lambda) \in (V - 0) \times V^* \mid \lambda(v) = 0\}/\sim.$$

Relation is $(v, \lambda) \sim (tv, t^{-1}\lambda)$.

Orbits of G on X: zero sec M, all else $X_{1,n-1}$.

Moment map $\mu\colon T^*(M) \to \mathfrak{gl}(V)^* \simeq \mathrm{End}(V)$,
$\mu(v, \lambda)(w) = \lambda(w)v$.

Have $\mu\colon X_{1,n-1} \xrightarrow{\sim} \mathcal{O}_{1,n-1}$: one coadjoint orbit!

$X_{1,n-1}$ dense $\Rightarrow C^\infty(T^*(M))^G = \mathbb{C} \overset{\text{hope}}{\Rightarrow} C^\infty(M)$ irr.

This hope *does* disappoint us: $C^\infty(M) \supset$ constants, so rep is reducible. Also there's no G-invt msre on M, so no unitary Hilbert space version $L^2(M)$.

Fix both problems: $\delta^{1/2}$ = half-density bdle on $\mathbb{P}V$.

Smooth half densities $C^\infty(M, \delta^{1/2})$ are irr rep of $GL(n, \mathbb{R})$, $\rightsquigarrow$ irr unitary rep $\pi_{1,n-1}$ on $L^2(M, \delta^{1/2})$.

Family of irr unitary representations

Natural generalization: replace functions on $M = \mathbb{P}V$ by sections of Hermitian line bundle.

Two natural ($GL(V)$-eqvt) real bdles on $\mathbb{P}V$: tautological line bdle $\mathcal{L}$ (fiber at line L is L); and $\mathcal{Q}$ ($(n-1)$ -diml real bundle, fiber at L is V/L).

Given real parameters λ_1 and λ_{n-1}, get Hermitian line bundle $\mathcal{H}(\lambda_1, \lambda_{n-1}) = \mathcal{L}^{i\lambda_1} \otimes (\wedge^{n-1}\mathcal{Q})^{i\lambda_{n-1}}$.

Define

$$\pi_{1,n-1}(\lambda_1, \lambda_{n-1}) = \text{ rep on } L^2(M, \delta^{1/2} \otimes \mathcal{H}(\lambda_1, \lambda_{n-1})).$$

These are irr unitary representations of $GL(V)$; naturally assoc to coadjt orbits $\mathcal{O}(\lambda_1, \lambda_{n-1})$.

Same techniques (still for reductive G) deal with all hyperbolic coadjt orbits (that is, orbits of matrices diagonalizable over $\mathbb{R}$.

And now for something completely different...

V $2m$-dimensional real vector space, $G = GL(V)$. Fix real $t_m \geq 0$, real s_m, define coadjt orbit

$$\mathcal{O}(s_m + it_m, s_m - it_m) = \{A \in \mathrm{End}(V) \mid \text{ eigval } s_m \pm it_m \text{ mult } m\}.$$

Base point in family:

$$\mathcal{O}_{m,m} = \{A \in \mathrm{End}(V) \text{ nilpotent, Jordan blocks } m,\ m\}.$$

Parameter s_m corresponds to twisting by one-diml char of $GL(V)$: cumbersome and dull. So pretend it doesn't exist.

Corresponding repns related to cplx alg variety

$$X = \text{ complex structures on } V, \qquad \dim X = m^2.$$

Have K-invariant projective subvariety

$$Z = \text{ orthogonal cplx structures} \quad \dim Z = (m^2 - m)/2 = s.$$

Turns out (Schmid, Wolf) X is $(s+1)$-complete, which means Stein away from Z.

X has G-invt indef Kähler structure, signature $((m^2 - m)/2, (m^2 + m)/2)$; underlying real symplectic mfld is $\mathcal{O}(it_m, -it_m)$ (any $t_m > 0$).

Representations attached to $\mathcal{O}(it_m, -it_m)$

Brought to you by Birgit Speh.

$$\dim V = 2m, \quad X = \text{ space of cplx structures on } V.$$
$$n = \dim_{\mathbb{C}}(X) = m^2, \qquad s = \dim_{\mathbb{C}}(\text{maxl cpt subvar}) = (m^2 - m)/2.$$

Point $x \in X$ interprets V as m-diml complex vector space V_x. Defines (tautological) holomorphic vector bundle $\mathcal{V}$ on X. Top exterior power of $\mathcal{V}$ is a holomorphic line bundle $\mathcal{L}$.

Every eqvt hol line bdle on X is $\mathcal{L}^p$, some $p \in \mathbb{Z}$.
Canonical bdle is $\omega_X = \mathcal{L}^{-2m}$.

Very rough idea: $\mathcal{O}(it_m, -it_m) \leftrightsquigarrow$ repn $\Gamma(\mathcal{L}^p)$. Fin-diml, not unitary.
Better: $\mathcal{O}(it_m, -it_m) \leftrightsquigarrow$ repn $H^{0,s}(X, \mathcal{L}^p)$. Inf unit for $p \le -m$.
Better: $\mathcal{O}(it_m, -it_m) \leftrightsquigarrow$ repn $H^{0,s}(X, \mathcal{L}^{-t_m} \otimes \omega_X^{1/2})$. Inf unit for $t_m \ge 0$.
Best: $\mathcal{O}(it_m, -it_m) \leftrightsquigarrow$ repn $H_c^{n,n-s}(X, \mathcal{L}^{t_m} \otimes \omega_X^{1/2})$. Pre-unit for $t_m \ge 0$.
Call this (last) representation $\pi(t_m) \quad (t_m = 0, 1, 2, \ldots)$.

Inclusion of compact subvariety Z gives lowest $O(V)$-type:
$(t_m + 1)$-Cartan power of $\bigwedge^m(V)$. (Shift $+1$ since $\omega_Z = \omega_X^{1/2} \otimes \mathcal{L}^{-1}$.)

Parallel techniques deal with elliptic coadjt orbits (that is, orbits of semisimple matrices with purely imaginary eigenvalues.

Quantizing nilpotent orbits

For $GL(n,\mathbb{R})$, nilp coadjt orbits $=$ special points in families of semisimp orbits $\rightsquigarrow$ quantize by continuity (see below).

For other reductive groups, not true: many nilpotent orbits have no deformation to semsimple orbits.

Kostant-Rallis idea: nilp coadjt orbit $\mathcal{O}_{\mathbb{R}}$ has natural K-invt cplx structure $\mathcal{O}_\theta \rightsquigarrow$ holomorphic action of $K_{\mathbb{C}}$.

Get repn of K that quantizes $\mathcal{O}_\theta$; look for a way to extend it to G. Carried out by Brylinski and Kostant for minimal coadjt orbit in many cases.

Rossi-Vergne idea: Given semisimple orbits, quantizations

$$\{\mathcal{O}(\lambda) \mid \lambda \text{ dom reg}\} \rightsquigarrow \{\pi(\lambda) \mid \lambda \text{ dom reg adm}\}.$$

Repns make sense (but may not be unitary) for "all" admissible λ (not dominant or regular).

Continuity above means limiting nilpotent orbit $\mathcal{O}(0)$ quantized by $\pi(0)$.

Rossi-Vergne idea: smaller nilpotent orbits $\mathcal{O}'$ (contained in $\mathcal{O}(0)$) should be quantized by smaller constituents of representations $\pi(\lambda')$, with λ' admissible, not dominant.

Meaning of it all

My first class from Bert began February 4, 1975. In the first hour he defined symplectic forms; symplectic manifolds; symplectic structure on cotangent bundles; Lagrangian submanifolds; and proved coadjoint orbits were symplectic.

After that the class picked up speed.

Many years later I took another class from Bert. At some point he needed differential operators. So he gave an introduction: "You form the algebra generated by the derivations of C^∞."

That's Bert: mathematics at Mach 2, always exciting, and the explanations are always complete; you'll figure them out eventually. The first third of a century has been fantastic, and I hope to keep listening for a very long time.

HAPPY BIRTHDAY BERT!

References

W. Graham and D. Vogan, “Geometric quantization for nilpotent coadjoint orbits,” in *Geometry and Representation Theory of real and p-adic groups*. Birkhauser, Boston-Basel-Berlin, 1998.

B. Kostant, “Quantization and unitary representations. I. Prequantization,” pp. 87–208 in *Lectures in modern analysis and applications, III*. Lecture Notes in Math., **170**. Springer, Berlin, 1970.

D. Vogan, “The method of coadjoint orbits for real reductive groups,” in *Representation Theory of Lie Groups*. IAS/Park City Mathematics Series **8** (1999), 179–238.

D. Vogan, “Unitary representations and complex analysis,” in*Representation Theory and Complex Analysis (CIME 2004)*, Andrea D’Agnolo, editor. Springer, 2008.