Quantization, the orbit method, and unitary representations

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Outline

Physics: a view from a neighboring galaxy

Classical representation theory

History of the orbit method in two slides

Hyperbolic coadjoint orbits for reductive groups

Elliptic coadjoint orbits for reductive groups

Quantum mechanics

Physical system $\longleftrightarrow$ complex Hilbert space $\mathcal{H}$

States $\longleftrightarrow$ lines in $\mathcal{H}$

Observables $\longleftrightarrow$ linear operators $\{A_j\}$ on $\mathcal{H}$

Expected value of obs $A \longleftrightarrow \langle Av, v\rangle$

Energy $\longleftrightarrow$ special skew-adjoint operator A_0

Time evolution $\longleftrightarrow$ unitary group $t \mapsto \exp(tA_0)$

Observable A conserved $\longleftrightarrow [A_0, A] = 0$

Moral of the story: quantum mechanics is about Hilbert spaces and Lie algebras.

Unitary representations of a Lie group G

Unitary repn is Hilbert space $\mathcal{H}_\pi$ with action

$$G \times \mathcal{H}_\pi \to \mathcal{H}_\pi, \qquad (g, v) \mapsto \pi(g)v$$

respecting inner product: $\langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle$.

π is irreducible if has exactly two invt subspaces.

Unitary dual problem: find $\widehat{G}_u =$ unitary irreps of G.

$X \in \mathrm{Lie}(G) \rightsquigarrow$ skew-adjoint operator $d\pi(X)$:

$$\pi(tX) = \exp(td\pi(X)).$$

Moral of the story: unitary representations are about Hilbert spaces and Lie algebras.

Here's the big idea

One of Kostant's greatest contributions was understanding the power of the analogy

unitary repns	$\leftrightsquigarrow$	quantum mech systems
Hilb space, Lie alg of ops		Hilb space, Lie alg of ops

Unitary repns are hard, but quantum mech is hard too. How does an analogy help?

Physicists have a cheat sheet!

There is an easier version of quantum mechanics called classical mechanics. Theories related by

$$\text{classical mech} \quad \underset{\longleftarrow}{\overset{\text{quantization}}{\longrightarrow}}\ \text{classical limit} \quad \text{quantum mech}$$

A little bit of background

Symplectic manifold is manifold M with Lie algebra structure $\{,\}$ on $C^\infty(M)$ satisfying

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

and a nondegeneracy condition.

Any smooth function f on M defines

Hamiltonian vector field $\xi_f = \{f, \cdot\}$.

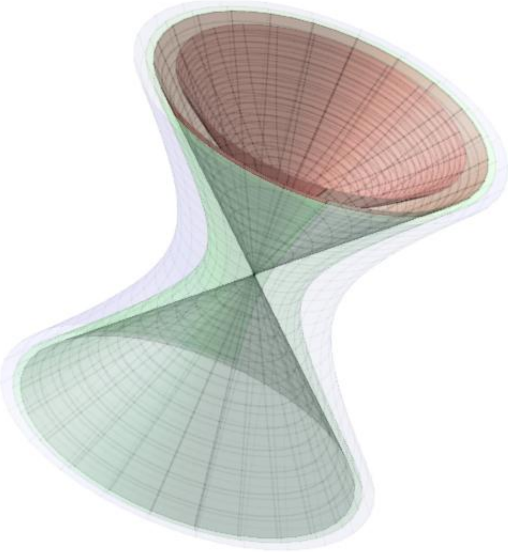
Example: M = cotangent bundle.

Example: M = Kahler manifold.

Example: M = conjugacy class of $n \times n$ matrices.

Pictures

Some conjugacy classes of 2×2 real matrices



Classical mechanics

Physical system $\longleftrightarrow$ symplectic manifold M

States $\longleftrightarrow$ points in M

Observables $\longleftrightarrow$ smooth functions $\{a_j\}$ on M

Value of obs a on state m $\longleftrightarrow$ $a(m)$

Energy $\longleftrightarrow$ special real-valued function a_0

Time evolution $\longleftrightarrow$ flow of vector field ξ_{a_0}

Observable a conserved $\longleftrightarrow$ $\{a_0, a\} = 0$

Moral of the story: classical mechanics is about symplectic manifolds and Poisson Lie algebras.

Representation theory and physics

Here's how Kostant's analogy looks now.

unitary repns		quantum mech system
Hilb space, Lie alg of ops	$\leftrightsquigarrow$	Hilb space, Lie alg of ops
quantization $\uparrow\downarrow$ classical limit		quantization $\uparrow\downarrow$ classical limit
Hamiltonian G-space		classical mech system
symplectic manifold	$\leftrightsquigarrow$	symplectic manifold
Poisson Lie alg of fns		Poisson Lie alg of fns

That is, the analogy suggests that there is a classical analogue of unitary representations.

Should make irreducible unitary correspond to homogeneous Hamiltonian.

Must make sense of $\uparrow\downarrow$. Physics $\uparrow\downarrow$ not our problem.

What's a Hamiltonian G-space?

M manifold with Poisson bracket $\{,\}$ on smooth functions

$$\{f, *\} \rightsquigarrow \xi_f \in \text{Vect}(M) \qquad \text{Hamiltonian vector field}$$

G action on $X \rightsquigarrow$ Lie alg hom $\mathfrak{g} \to \text{Vect}(M)$, $Y \mapsto \xi_Y$.

M is a Hamiltonian G-space if this Lie algebra map lifts

$$\begin{array}{ccc} & & C^\infty(M) \\ & \nearrow & \downarrow \\ \mathfrak{g} & \to & \text{Vect}(M) \end{array} \qquad \begin{array}{ccc} & & f_Y \\ & \nearrow & \downarrow \\ Y & \to & \xi_Y \end{array}$$

Map $\mathfrak{g} \to C^\infty(M)$ same as moment map $\mu \colon M \to \mathfrak{g}^*$.

Theorem (Kostant)

Homogeneous Hamiltonian G-space is the same thing (by moment map) as covering of an orbit of G on $\mathfrak{g}^*$.

Method of coadjoint orbits

Recall: Hamiltonian G-space X comes with (G-equivariant) moment map $\mu\colon X \to \mathfrak{g}^*$.

Kostant's theorem: homogeneous Hamiltonian G-space $=$ covering of G-orbit on $\mathfrak{g}^*$.

Kostant's rep theory $\leftrightsquigarrow$ physics analogy now leads to

Kirillov-Kostant philosophy of coadjt orbits:

$$\{\text{irr unitary reps of } G\} =_{\text{def}} \widehat{G} \overset{?}{\leftrightsquigarrow} \mathfrak{g}^*/G. \qquad (\star)$$

MORE PRECISELY. . . restrict right side to "admissible" orbits (integrality cond). Expect to find "almost all" of $\widehat{G}$: enough for interesting harmonic analysis.

Evidence for orbit method

With the caveat about restricting to admissible orbits. . .

$$\widehat{G} \overset{?}{\leftrightsquigarrow} \mathfrak{g}^*/G. \qquad (\star)$$

$(\star)$ true for G simply connected nilpotent (Kirillov)

General idea $(\star)$, without physics motivation, due to Kirillov.

$(\star)$ true for G type I solvable (Auslander-Kostant).

$(\star)$ for algebraic G reduces to reductive G (Duflo).

Case of reductive G is still open.

Actually $(\star)$ is false for connected nonabelian reductive G.

But there are still theorems close to $(\star)$.

So concentrate on reductive groups. . .

Two ways to study representations for reductive G:

1. start with coadjt orbit, seek representation. Hard.
2. start with representation, seek coadjt orbit. Easy.

Really need to do both things at once. Having started to do mathematics in the Ford administration, I find this challenging. (Gave up chewing gum at that time.)

Reductive Lie group $G =$ closed subgp of $GL(n, \mathbb{R})$ which is closed under transpose, and $\#G/G_0 < \infty$.

From now on G is reductive.

$\text{Lie}(G) = \mathfrak{g} \subset n \times n$ matrices. Bilinear form

$$T(X, Y) = \text{tr}(XY) \Rightarrow \mathfrak{g} \overset{G\text{-eqvt}}{\simeq} \mathfrak{g}^*$$

Orbits of G on $\mathfrak{g}^* \subset$ conjugacy classes of matrices.

Orbits of $GL(n, \mathbb{R})$ on $\mathfrak{g}^* =$ conj classes of matrices.

First example: hyperbolic orbits

$G = GL(n, \mathbb{R})$, $n = p + q$, $x > y$ real numbers

$\mathcal{O}_{p,q}(x, y) =_{\text{def}}$ diagonalizable matrices with eigvalues x (mult p) and y (mult q).

Define Gr(p, n) = Grassmann variety of p-dimensional subspaces of $\mathbb{R}^n$.

$\mathcal{O}_{p,q}$ is Hamiltonian G-space of dimension $2pq$.

$$\mathcal{O}_{p,q}(x, y) \to \text{Gr}(p, n), \qquad \lambda \mapsto x \text{ eigenspace}$$

exhibits $\mathcal{O}_{p,q}(x, y)$ as affine bundle over Gr(p, n)

General reductive G: $\mathcal{O} \subset \mathfrak{g}^*$ hyperbolic if elements are diagonalizable with real eigenvalues.

Always affine bundle over a compact real flag variety.

We pause for a word from our sponsor...

Classical physics example:

configuration space X = manifold of positions.

State space $T^*(X) =$ symplectic manifold of positions and momenta.

Quantization

$$\mathcal{H} = L^2(X)$$
$$= \text{square-integrable half-densities on } X$$
$$= \text{wave functions for quantum system.}$$

Size of wave function $\leftrightsquigarrow$ probability of configuration.
oscillation of wave function $\leftrightsquigarrow$ velocity.

Kostant-Kirillov idea:

Hamiltonian G-space $M \approx T^*(X) \implies$

unitary representation $\approx L^2(X) =$ square-integrable half-densities on X.

Hyperbolic representations

Two $GL(n, \mathbb{R})$-equivariant real line bdles on $\mathrm{Gr}(p, n)$:

1. $\mathcal{L}_1$: fiber at p-diml $S \subset \mathbb{R}^n$ is $\bigwedge^p S$;
2. $\mathcal{L}_2$: fiber at S is $\bigwedge^{n-p}(\mathbb{R}^n/S)$.

Real numbers x and y $\rightsquigarrow$ Hermitian line bundle

$$\mathcal{L}(x, y) = \mathcal{L}_1^{ix} \otimes \mathcal{L}_2^{iy}.$$

Unitary representations of $GL(n, \mathbb{R})$ associated to coadjoint orbits $\mathcal{O}_{p,q}(x, y)$ are

$$\pi_{p,q}(x, y) = L^2(\mathrm{Gr}(p, n), \mathcal{L}(x, y)).$$

Same techniques (still for reductive G) deal with all hyperbolic coadjoint orbits.

Second example: elliptic orbits

$G = GL(2n, \mathbb{R})$, $x > 0$ real number

$$\mathcal{O}_e(x) =_{\text{def}} \text{real matrices } \lambda \text{ with } \lambda^2 = -x^2 I$$
$$= \text{diagonalizable } \lambda \text{ with eigenvalues } \pm xi.$$

$\mathcal{O}_e(x)$ is Hamiltonian G-space of dimension $2n^2$.

Define a complex manifold

$$X = \text{complex structures on } \mathbb{R}^{2n}$$
$$\simeq n\text{-dimensional complex subspaces}$$
$$S \subset \mathbb{C}^{2n} \text{ such that } S + \overline{S} = \mathbb{C}^{2n}$$

Last condition is open, so X open in $\text{Gr}_{\mathbb{C}}(n, 2n)$.

$$\mathcal{O}_e(x) \to X, \qquad \lambda \mapsto ix \text{ eigenspace}$$

is isomorphism $\mathcal{O}_e(x) \simeq X$

General reductive G: $\mathcal{O} \subset \mathfrak{g}^*$ elliptic if elements are diagonalizable with purely imaginary eigenvalues.

Always $\simeq$ open orbit X on cplx flag variety: Kähler.

Remind me, what was a Kähler manifold?

First, complex manifold X: real space $T_x X$ has complex structure j_x: real linear aut, $j_x^2 = -I$.

Second, symplectic: $T_x X$ has symp form ω_x.

Third, structures compatible: $\omega_x(j_x u, j_x v) = \omega_x(u, v)$.

These structures define indefinite Riemannian structure $g_x(u, v) = \omega_x(u, j_x v)$.

Kähler structure is positive if all g_x are positive; signature (p, q) if all g_x have signature (p, q)

Example: $X =$ complex structures on $\mathbb{R}^{2n}$ has signature $\left(\binom{n}{2}, \binom{n+1}{2}\right)$ or $\left(\binom{n+1}{2}, \binom{n}{2}\right)$.

Positive Kähler structures are better, but here we can't have them. Need direction...

Dealing with indefinite Kähler

Example: $U(n)/U(1)^n$ has $n!$ equivariant Kähler structures. Here's how...

1. Distinct reals $\ell = (\ell_1, \ldots, \ell_n) \rightsquigarrow U(n)$ coadjt orbit
$$\mathcal{O}_e(\ell) = U(n) \cdot \mathrm{diag}(i\ell_1, \ldots, i\ell_n);$$
with natural symplectic structure.
2. Isomorphic to complex X = complete flags in $\mathbb{C}^n$ by
$$\lambda \in \mathcal{O}_e(\ell) \mapsto \left(\{0\} \subset \mathbb{C}^n_{i\ell_n}(\lambda) \subset \mathbb{C}^n_{i\ell_n}(\lambda) + \mathbb{C}^n_{i\ell_{n-1}}(\lambda) \subset \cdots\right);$$
here $\mathbb{C}^n_{i\ell_j}(\lambda)$ = (one-diml) $i\ell_j$-eigenspace of λ.
3. Define σ = permutation putting ℓ in decreasing order.
4. Isomorphism with $X \rightsquigarrow$ Kähler structure of signature $\left(\binom{n}{2} - \ell(\sigma), \ell(\sigma)\right)$.

How do you quantize a Kähler manifold?

Kostant-Auslander idea:

Hamiltonian G-space X positive Kähler $\Longrightarrow$

unitary representation $=$ L^2 holomorphic sections of holomorphic line bdle on X

But Kähler structures on

$$\mathcal{O}_e(x) = 2n \times 2n \text{ real } \lambda,\ \lambda^2 = -x^2$$

are both indefinite.

New idea comes from Borel-Weil-Bott theorem about compact groups (proved algebraically by Kostant)

Quantizing $U(n) \cdot \mathrm{diag}(i\ell_1, \ldots, i\ell_n) \subset \mathfrak{u}(n)^*$

ℓ_j distinct real; now assume $\ell_j \equiv (n-1)/2 \pmod{\mathbb{Z}}$. Put

$$\rho = ((n-1)/2, (n-3)/2, \ldots, (-n+1)/2) \in \mathbb{R}^n$$

$$\ell - \rho = (\ell_1 - (n-1)/2, \ldots, \ell_n + (n-1)/2) \in \mathbb{Z}^n.$$

Get $\mathcal{L}_{\ell-\rho}$ hol line bdle on X = flags in $\mathbb{C}^n$.
Recall $\sigma \cdot \ell$ decr; so $\mu = \sigma\ell - \rho$ = dom wt for $U(n)$.
Write E_μ = irr rep of $U(n)$ of highest weight μ.

Theorem (Borel-Weil-Bott-Kostant)

Write $\mathcal{O}_{\ell-\rho}$ = sheaf of germs of hol secs of $\mathcal{L}_{\ell-\rho}$. Then

$$H^p(X, \mathcal{O}_{\ell-\rho}) = \begin{cases} E_\mu & p = \ell(\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Moral of the story: look for representations not in holomorphic sections, but in cohomological degree given by signature of Kähler metric.

Let's try that: $\mathcal{O}_e(x)$ for $GL(2n,\mathbb{R})$

Brought to you by Birgit Speh.

$$X = \text{ space of cplx structures on } \mathbb{R}^{2n}.$$

$$n^2 = \dim_{\mathbb{C}}(X), \qquad s = \dim_{\mathbb{C}}(\text{maxl cpt subvar}) = \binom{n}{2}.$$

Point $x \in X$ interprets $\mathbb{R}^{2n}$ as n-diml complex vector space $\mathbb{R}^{2n}_x$. Defines (tautological) holomorphic vector bundle $\mathcal{V}$ on X; $\bigwedge^n(\mathcal{V}) = \mathcal{L}$ holomorphic line bundle on X.

Every eqvt hol line bdle on X is $\mathcal{L}^p$, some $p \in \mathbb{Z}$.

Canonical bdle is $\omega_X = \mathcal{L}^{-2n}$.

Better: $\mathcal{O}_e(x) \leftrightsquigarrow$ repn $H^{0,s}(X,\mathcal{L}^p)$. Inf unit for $p \le -n$.

Better: $\mathcal{O}_e(x) \leftrightsquigarrow$ repn $H^{0,s}(X,\mathcal{L}^{-x} \otimes \omega_X^{1/2})$. Inf unit for $x \ge 0$.

Moral: interesting orbits $\leftrightsquigarrow$, $x + n \in \mathbb{Z}$: Duflo's admissible orbits.

Best: $\mathcal{O}_e(x) \leftrightsquigarrow$ repn $H_c^{n^2, n^2-s}(X,\mathcal{L}^{x} \otimes \omega_X^{1/2})$. Pre-unit for $x \ge 0$.

This is Serre duality plus analytic results of Hon-Wai Wong.

Call this (last) representation $\pi(x) \quad (x = 0, 1, 2, \ldots)$.

Inclusion of compact subvariety Z gives lowest $O(V)$-type: $(x+1)$-Cartan power of $\bigwedge^n(\mathbb{C}^{2n})$. (Shift $+1$ since $\omega_Z = \omega_X^{1/2} \otimes \mathcal{L}^{-1}$.)

Parallel techniques deal with elliptic coadjt orbits (that is, orbits of semisimple matrices with purely imaginary eigenvalues.