The size of infinite-dimensional representations

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Outline

Introduction

Lessons from finite fields

Lessons from *p*-adic fields

Repns over $\mathbb{R}$, $\mathbb{C}$

GK dimension and characters

Other real reductive groups

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic today: what's an irreducible unitary representation look like?

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Concentrate on group $G(k) = GL(V(k))$ invertible linear transformations of n-diml vector space $V(k)$.

Stay vague about (locally compact) ground field k: mostly $\mathbb{R}$ or $\mathbb{C}$, but $\mathbb{F}_q$, p-adic fields also interesting.

$G(k)$ acts on $(n-1)$-diml (over k) proj alg variety

$$X_{1,n-1}(k) = \{\text{1-diml subspaces of } V(k)\}$$

Hilbert space

$$\mathcal{H}_{1,n-1}(k) = \{L^2 \text{ half-densities on } X_{1,n-1}(k)\}$$

$k = \mathbb{R}, \mathbb{C}$, p-adic: $G(k)$ acts by irr rep $\rho(1, n-1)$.

Question for today: how big is this Hilbert space?

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To understand size of repns of $GL(V)$, need size of $GL(V)$…

The "q-analogue" of m is the polynomial

$$q^{m-1} + q^{m-2} + \cdots + q + 1 = \frac{q^m - 1}{q - 1};$$

value at $q = 1$ is m.

$$(m!)_q = (q^{m-1} + q^{m-2} \cdots + 1)(q^{m-2} + \cdots + 1) \cdots (q+1) \cdot 1$$
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(q-analogue of $m!$; poly in q, deg $= \binom{m}{2}$, val at $1 = m!$)

Geometric meaning: number of complete flags in an m-dimensional vector space over $\mathbb{F}_q$.

Cardinality of $GL(V(\mathbb{F}_q))$ is $(n!)_q(q-1)^n q^{\binom{n}{2}}$.

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$\pi = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; $G(\mathbb{F}_q)$ acts on

$$X_\pi(\mathbb{F}_q) = \{0 = S_0 \subset S_1 \subset \cdots S_m = V(\mathbb{F}_q),$$
$$\text{subspace chains, } \dim(S_j/S_{j-1} = p_j\};$$

$\mathbb{F}_q$-variety of dimension

$$d(\pi) =_{\text{def}} \binom{n}{2} - \sum_j \binom{p_j}{2}.$$

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$G(\mathbb{F}_q) = GL(V(\mathbb{F}_q)) =$ q-analogue of symm group S_n

irr rep of $G(\mathbb{F}_q) \rightsquigarrow$ partition π of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep $\approx$ functions on $X_\pi(\mathbb{F}_q)$

dim(irr rep) $=$ poly in q of degree $\dim X_\pi$

Problem: what partition is attached to each irr rep?

Dimension of representation provides a clue.

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Note: partition $\pi \leftrightsquigarrow$ irreducible rep of S_n.

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$V(k)$ n-diml vec space; fix basis $\rightsquigarrow V(k) \simeq k^n$.

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For $r \geq 0$, have open subgroups (nbhd base at I)

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Flag varieties over p-adic k

$G(k) = GL(V(k)) \supset$ compact open $G_0 \supset G_1 \supset \cdots$

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$$\mathcal{H}_\pi(k) = \{L^2 \text{ half-densities on } X_\pi(k)\}$$

carries unitary rep $\rho(\pi)$ of $G(k)$; space is incr union

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If $(\rho, \mathcal{H})$ arb irr rep of $G(k)$, then for every partition π of n there is an integer $a_\pi(\rho)$ so that for $r \geq r(\rho)$

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$$\dim \mathcal{H}^{G_r} = \text{poly in } q^r \text{ of deg } d(\pi(\rho)),$$

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Moral of the p-adic story

$G(k) = GL(V(k))$ has neighborhood base at 1 of compact open subgroups $G_0 \supset G_1 \supset \cdots \supset G_r \supset \cdots$

irr rep of $G(k) \rightsquigarrow$ partition $\pi(\rho)$ of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep on $\mathcal{H} \approx$ functions on $X_\pi(k)$

$\dim(\mathcal{H}^{G_r}) =$ poly in q^r of deg $d(\pi) = \dim X_\pi$ (large r)

Problem: what partition is attached to each irr rep?

Rate of growth of chain of subspaces

$$\mathcal{H}_\pi^{G_0} \subset \mathcal{H}_\pi^{G_1} \subset \cdots \mathcal{H}_\pi^{G_r} \subset \cdots$$

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big reps $\leftrightsquigarrow$ partitions with small parts.

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Representations of $GL(V(\mathbb{R}))$

$G(\mathbb{R}) = GL(V(\mathbb{R})) \simeq GL(n, \mathbb{R})$.

$G(\mathbb{R})$ acts on $(n-1)$-diml compact manifold

$$X_{1,n-1}(\mathbb{R}) = \{\text{1-diml subspaces of } V(\mathbb{R})\} \simeq \mathbb{RP}^{n-1}$$

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Hilbert space carrying irr unitary rep of $G(\mathbb{R})$.

Question for today: how big is this Hilbert space? Can we extract $n-1$ from it?

Difficulty: all inf-diml separable Hilbert spaces are isomorphic (as Hilbert spaces).

Same problem for other function spaces:

$$C^\infty(\mathbb{RP}^{n-1}) \simeq C^\infty(\mathbb{RP}^{m-1}) \text{ as topological vec space}$$

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Lessons from p-adic fields

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$$C^\infty(\mathbb{RP}^{n-1}) \simeq C^\infty(\mathbb{RP}^{m-1}) \text{ as topological vec space}$$

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Distinguish using exhaustion by fin-diml subspaces.

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$G(\mathbb{R}) = GL(V(\mathbb{R})) \simeq GL(n, \mathbb{R})$.

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X compact d-diml Riemannian, Δ_X Laplacian

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Theorem (Weyl)

If $\mathcal{H}(N) = \sum_{\lambda \le N^2} \mathcal{H}_\lambda$, then $\dim \mathcal{H}(N) \sim c_X N^d$.

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Choice of basis defines compact subgroup

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Casimir $\Omega_{O(n)} = -\sum X_i^2$, $\{X_i\}$ orth basis of Lie $O(n)$.

$\pi = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; $G(\mathbb{R})$ acts on cpt Riemannian

$$X_\pi(\mathbb{R}) = \text{subspace chains of type } \pi$$

$$d(\pi) =_{\text{def}} \binom{n}{2} - \sum_m \binom{p_m}{2} = \dim X_\pi$$

$O(n)$ transitive on $X_\pi(\mathbb{R})$, $\Delta_{X_\pi} =$ action of $\Omega_{O(n)}$; isotropy

$$O(\pi) =_{\mathit{def}} O(p_1) \times \cdots O(p_m) \subset O(n).$$

Unitary rep $\rho(\pi)$ on $\mathcal{H}_\pi(\mathbb{R}) = L^2(X_\pi(\mathbb{R}))$; res to $O(n)$ is

$$\mathrm{Ind}_{O(\pi)}^{O(n)}(\mathbb{C}) = \sum_{\mu \in \widehat{O(n)}} (\dim \mu^{O(\pi)})\mu$$

Therefore compute Laplacian eigenvalue distribution

$$\mathcal{H}_\pi(N) = \sum_{\mu(\Omega) \le N^2} (\dim \mu^{O(\pi)})\mu.$$

$\dim \mathcal{H}_\pi(N) \sim a(\pi) N^{d(\pi)}$: res to $O(n)$ computes $d(\pi)$.

General representations over $\mathbb{R}$

$(\rho, \mathcal{H})$ arbitrary irr rep of $G(\mathbb{R}) \simeq GL(n, \mathbb{R})$.
Restriction to cpt subgp $O(n)$ decomposes

$$\mathcal{H} \simeq \sum_{\mu \in \widehat{O(n)}} m_\rho(\mu)\mu \qquad (m_\rho(\mu) \text{ non-neg integer}).$$

Example of $\mathcal{H}_\pi = L^2(X_\pi)$ suggests defining

$$\mathcal{H}(N) =_{\text{def}} \sum_{\mu(\Omega) \le N^2} m_\rho(\mu)\mu.$$

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There is partition $\pi(\rho)$ of n, pos integer $c(\rho)$ so that

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Recall that $\dim \mathcal{H}_\pi(N) \sim a(\pi) N^{d(\pi)}$.

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irr rep of $G(\mathbb{R}) \rightsquigarrow$ partition $\pi(\rho)$ of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep on $\mathcal{H} \approx$ functions on $X_\pi(\mathbb{R})$, cpt homog space for $G(\mathbb{R})$ and for $O(n)$. Precisely:

asymp distn of eigenvalues of Casimir $\Omega_{O(n)}$ in $\rho \rightsquigarrow$ eigenvals of Laplacian on $X_\pi(\mathbb{R})$.

Problems: what partition is attached to each irr rep?

what else does partition tell you about irr rep?

To address these questions, use characters of reps. . .

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Distribution characters

Idea of Gelfand-Kirillov dimension began with *dimension* for fin-diml irr rep $(\rho, \mathcal{H})$ of G.

Can write $\dim \rho = \operatorname{tr} \mathrm{Id}_{\mathcal{H}} = \operatorname{tr} \rho(1)$.

Useful to consider character of ρ, function on G:

$$\Theta_\rho(g) =_{\text{def}} \operatorname{tr} \rho(g),$$

because character of ρ determines ρ up to equiv.

Inf-diml irr $(\rho, \mathcal{H})$: $\rho(g)$ never trace class. *Regularize*. . .

$G(\mathbb{R}) = GL(V(\mathbb{R}))$, δ cptly supp test density on $G(\mathbb{R})$,

$$\rho(\delta) = \int_{G(\mathbb{R})} \rho(g)\delta(g) \in \mathrm{End}(\mathcal{H})$$

is trace class operator (Harish-Chandra).

Map $\Theta_\rho(\delta) = \operatorname{tr} \rho(\delta)$ is generalized function on $G(\mathbb{R})$.

GK dim of $\rho \leftrightsquigarrow$ singularity of Θ_ρ at $1 \in G(\mathbb{R})$.

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More lessons from real analysis

f smooth on vec space $W(\mathbb{R})$, $f_t(w) = f(tw)$; $\overset{\text{Taylor}}{\rightsquigarrow}$
$f_t \sim \sum_{k=0}^{\infty} t^k P_k, \qquad (t \to \infty)$, P_k homog deg k poly.

Seek analogous expansion for non-smooth gen fns.

Theorem (Barbasch-V)

Θ_ρ distn char of irr rep ρ of $G(\mathbb{R})$, $\overset{\exp}{\rightsquigarrow}$ gen fn θ_ρ on

$$\mathfrak{g}(\mathbb{R}) = \text{Lie}(G(\mathbb{R})) = n \times n \text{ real matrices}$$

Then θ_ρ has asymptotic expansion

$$\theta_{\rho,t} \sim \sum_{k=-d(\rho)}^{\infty} t^k T_k(\rho),$$

$T_k(\rho)$ tempered gen fn homog of deg k.

Leading terms match: $T_{-d(\rho)}(\rho) = c(\rho) T_{-d(\pi)}(\rho(\pi(\rho)))$.

Conclusion: char Θ_ρ near $1 \in G(\mathbb{R})$ equal to $c(\rho) \cdot \Theta_{\rho(\pi)}$ modulo lower order terms.

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More about character leading terms

Looked at expansion $\theta_{\rho,t} \sim \sum_{k=-d(\rho)}^{\infty} t^k T_k(\rho)$.

Fin-diml rep: $d(\rho) = 0$, leading term $T_0(\rho) = \dim \rho$.

Leading term $T_{-d(\rho)} \leftrightsquigarrow$ analogue of dimension

Example: $G(\mathbb{R})$ action on $X_\pi(\mathbb{R}) \rightsquigarrow$ moment map

$$\mu_\pi \colon T^* X_\pi(\mathbb{R}) \to \mathfrak{g}(\mathbb{R})^*.$$

μ_π is birational onto closure of nilpotent conj class

$$\mathcal{O}_{\pi^t} \subset \mathfrak{g}(\mathbb{R})^* \simeq n \times n \text{ real matrices;}$$

Natural measure on $T^* X_\pi(\mathbb{R}) \overset{\mu_\pi}{\rightsquigarrow}$ measure on $\mathcal{O}_{\pi^t}$ $\overset{\text{Fourier}}{\rightsquigarrow}$ generalized function on $\mathfrak{g}(\mathbb{R})$.

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Fin-diml rep: $d(\rho) = 0$, leading term $T_0(\rho) = \dim \rho$.

Leading term $T_{-d(\rho)}$ $\leftrightsquigarrow$ analogue of dimension

Example: $G(\mathbb{R})$ action on $X_\pi(\mathbb{R}) \rightsquigarrow$ moment map

$$\mu_\pi \colon T^* X_\pi(\mathbb{R}) \to \mathfrak{g}(\mathbb{R})^*.$$

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Example: $Sp(2n, \mathbb{R})$, $\mathbb{R}$-linear transf of $\mathbb{C}^n$ preserving symplectic form

$$\omega(v, w) = \mathrm{Im}\langle v, w\rangle$$

(imag part of std Herm form); $K(\mathbb{R}) = U(n)$.

Example: $O(p, q)$ linear transf of $\mathbb{R} \times \mathbb{R}^q$ preserving symmetric form

$$\langle (v_1, v_2), (w_1, w_2)\rangle_{p,q} = \langle v_1, w_1\rangle - \langle v_2, w_2\rangle;$$

$K(\mathbb{R}) = O(p) \times O(q)$.

(Al)most general example: $G(\mathbb{R}) \subset GL(N, \mathbb{R})$ closed subgp preserved by transpose, $K(\mathbb{R}) = G(\mathbb{R}) \cap O(N)$.

Big idea:

$G(\mathbb{R})$ rep "size" $\leftrightsquigarrow$ restriction to $K(\mathbb{R})$ asymptotics

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$$\mathcal{H} \simeq \sum_{\mu \in \widehat{K(\mathbb{R})}} m_\rho(\mu)\mu, \qquad (m_\rho(\mu) \text{ non-neg integer}).$$

As for $GL(n)$, can define

$$\mathcal{H}(N) =_{\text{def}} \sum_{\mu(\Omega_{K(\mathbb{R})}) \leq N^2} m_\rho(\mu)\mu.$$

Theorem

There is a non-negative integer $d(\rho)$ and a positive constant $b(\rho)$ so that

$$\dim \mathcal{H}(N) \sim b(\rho) N^{d(\rho)}.$$

Call $d(\rho)$ the Gelfand-Kirillov dimension of ρ.

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What's wrong with GK dimension for other G

Case of $GL(n)$: have special homog spaces $X_\pi(\mathbb{R})$ (partial flag variety) so that reps $L^2(X_\pi(\mathbb{R}))$ "approximately model" any irr rep.

Other $G(\mathbb{R})$: have analogues of X_π (real flag varieties); but they no longer model *all* irr reps.

Example: $G(\mathbb{R}) = Mp(4, \mathbb{R})$ nonlinear double cover of symplectic group. Four possible spaces "X_π":

- point $X_\emptyset$ (dim $= 0$)
- (isotropic) lines $X_1 = \{L_1 \subset \mathbb{R}^4\} = \mathbb{RP}^3$ (dim 3)
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Get GK dims 0, 3, 4; metaplectic repn has GK dim 2.

But asymptotic expansion of characters still works. . .

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Get GK dims 0, 3, 4; metaplectic repn has GK dim 2.

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What's wrong with GK dimension for other G

Case of $GL(n)$: have special homog spaces $X_\pi(\mathbb{R})$ (partial flag variety) so that reps $L^2(X_\pi(\mathbb{R}))$ "approximately model" any irr rep.

Other $G(\mathbb{R})$: have analogues of X_π (real flag varieties); but they no longer model *all* irr reps.

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Character expansions for real groups

$G(\mathbb{R})$ real reductive group, $(\rho, \mathbb{H})$ irr rep

δ cptly supp test density on $G(\mathbb{R}) \rightsquigarrow$ trace class op

$$\rho(\delta) = \int_{G(\mathbb{R})} \rho(g)\delta(g) \in \mathrm{End}(\mathcal{H})$$

Map $\Theta_\rho(\delta) = \mathrm{tr}\, \rho(\delta)$ is generalized function on $G(\mathbb{R})$.

Lift via exp to gen fn θ_ρ on $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G(\mathbb{R}))$

Theorem (Barbasch-V)

θ_ρ has asymptotic expansion $\theta_{\rho,t} \sim \sum_{k=-d(\rho)}^{\infty} t^k T_k(\rho)$,
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Leading term $T_{-d(\rho)}$ is finite linear comb of Fourier transforms of invt measures on nilp orbits in $\mathfrak{g}(\mathbb{R})^*$:

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θ_ρ has asymptotic expansion [illegible]

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(Third) moral of the real story

$G(\mathbb{R})$ real reductive

$$\text{irr rep } \rho \text{ of } G(\mathbb{R}) \xrightarrow{\text{trace...support}} \text{non-neg integer comb } T_{-d(\rho)} = \sum_{\dim \mathcal{O}=2d(\rho)} c(\rho,\mathcal{O})\widehat{\mathcal{O}}.$$

of several nilpotent orbits of $G(\mathbb{R})$ on $\mathfrak{g}(\mathbb{R})^*$

More to do. . .

Can (approx) describe $\rho|_{K(\mathbb{R})}$ with orbits $\mathcal{O}$.

Relate unitarity of ρ to expansion; not understood.

Seek to compute constants $c(\rho,\mathcal{O})$ using KL calculation of character Θ_ρ; not understood.

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