FINITE MAXIMAL TORI

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To Nolan Wallach, with respect, admiration, and very best wishes.

ABSTRACT. We examine the structure of compact Lie groups using a finite maximal abelian subgroup A in place of a maximal torus. Just as the classical notion of roots exhibits many interesting subgroups, so the notion of roots of A exhibits many (rather different) interesting subgroups.

CONTENTS

1. INTRODUCTION

Suppose G is a compact Lie group, with identity component G_0. There is a beautiful and complete structure theory for G_0, based on the notion of maximal tori and root systems introduced by Élie Cartan and Hermann Weyl. The purpose of this paper is to introduce a parallel structure theory using "finite maximal tori." A *maximal torus* is by definition a maximal *connected* abelian subgroup of G_0. We define a *finite maximal torus* to be a maximal *finite* abelian subgroup of G.

It would be etymologically more reasonable to use the term *finite maximally diagonalizable subgroup*, but this name seems not to roll easily off the tongue. A more restrictive notion is that of *Jordan subgroup*, introduced by Alekseevskiĭ in[1]; see also[6], Definition 3.18. Also very closely related is the notion of *fine grading* of a Lie algebra introduced in [14], and studied extensively by Patera and others.

The classical theory of root systems and maximal tori displays very clearly many interesting structural properties of G_0. The central point is that root systems are essentially finite combinatorial objects. Subroot systems can easily be exhibited by hand, and they correspond automatically to (compact connected) subgroups of G_0; so many subgroups can be described in a combinatorial fashion. A typical example visible in this fashion is the subgroup $U(n)\times U(m)$ of $U(n+m)$. A more exotic example is the subgroup $(E_6\times SU(3))/\mu_3$ of E_8 (with μ_3 the cyclic group of third roots of 1).

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Unfortunately, there is so far no general "converse" to this correspondence: it is not known how to relate the root system of an arbitrary (compact connected) subgroup of G_0 to the root system of G_0. (Many powerful partial results in this direction were found by Dynkin in[5].) Consequently there are interesting subgroups that are more or less invisible to the theory of root systems.

In this paper we will describe an analogue of root systems for finite maximal tori. Again these will be finite combinatorial objects, so it will be easy to describe subroot systems by hand, which must correspond to subgroups of G. The subgroups arising in this fashion are somewhat different from those revealed by classical root systems. A typical example is the subgroup $PU(n)$ of $PU(nm)$, arising from the action of $U(n)$ on $\mathbb{C}^n \otimes \mathbb{C}^m$. A more exotic example is the subgroup $F_4 \times G_2$ of E_8 (see Example 4.5).

In Section 2 we recall Grothendieck's formulation of the Cartan-Weyl theory in terms of "root data." His axiomatic characterization of root data is a model for what we seek to do with finite maximal tori.

One of the fundamental classical theorems about maximal tori is that if T is a maximal torus in G_0, and $\widetilde{G}_0$ is a finite covering of G_0, then the preimage $\widetilde{T}$ of T in $\widetilde{G}_0$ is a maximal torus in $\widetilde{G}_0$. The corresponding statement about finite maximal tori is *false* (see Example 4.1); the preimage often fails to be abelian. In order to keep this paper short, we have avoided any serious discussion of coverings.

In Section 3 we define root data and Weyl groups for finite maximal tori. We will establish analogues of Grothendieck's axioms for these finite root data, but we do not know how to prove an existence theorem like Grothendieck's (saying that every finite root datum arises from a compact group).

In Section 4 we offer a collection of examples of finite maximal tori. The examples (none of which is original) are the main point of this paper, and are what interested us in the subject. Reading this section first is an excellent way to approach the paper.

Of course it is possible and interesting to work with maximal abelian subgroups which may be neither finite nor connected. We have done nothing about this.

Grothendieck's theory of root data was introduced not for compact Lie groups but for reductive groups over algebraically closed fields. The theory of finite maximal tori can be put into that setting as well, and this seems like an excellent exercise. It is not clear to us (for example) whether one should allow p-torsion in a "finite maximal torus" for a group in characteristic p; excluding it would allow the theory to develop in a straightforward parallel to what we have written about compact groups, but allowing p-torsion could lead to more very interesting examples of finite maximal tori.

Much of the most interesting structure and representation theory for a connected reductive algebraic group G_0 (over an algebraically closed field) can be expressed in terms of (classical) root data. For example, the irreducible representations of G_0 are indexed (following Cartan and Weyl) by orbits of the Weyl group on the character lattice; and Lusztig has defined a surjective map from conjugacy classes in the Weyl group to unipotent classes in G. It would be fascinating to rewrite such results in terms of finite root data; but we have done nothing in this direction.

2. ROOT DATA

In this section we introduce Grothendieck's root data for compact connected Lie groups. As in the introduction, we begin with

$$G = \text{compact connected Lie group} \tag{2.1a}$$

The real Lie algebra of G and its complexification are written

$$\mathfrak{g}_0 = \mathrm{Lie}(G), \qquad \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.1b}$$

The conjugation action of G on itself is written Ad:

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(G), \qquad \mathrm{Ad}(g)(x) = gxg^{-1} \quad (g,x \in G). \tag{2.1c}$$

The differential (in the target variable) of this action is an action of G on $\mathfrak{g}_0$ by Lie algebra automorphisms

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}_0). \tag{2.1d}$$

This differential of this action of G is a Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g}_0 \to \mathrm{Der}(\mathfrak{g}_0), \qquad \mathrm{ad}(X)(Y) = [X,Y] \quad (X,Y \in \mathfrak{g}_0). \tag{2.1e}$$

Analogous notation will be used for arbitrary real Lie groups. The kernel of the adjoint action of G on G or on $\mathfrak{g}_0$ is the center $Z(G)$:

$$\begin{aligned} Z(G) &= \{g \in G \mid gxg^{-1} = x, \quad \text{all } x \in G\} \\ &= \{g \in G \mid \mathrm{Ad}(g)(Y) = Y, \quad \text{all } Y \in \mathfrak{g}_0\} \end{aligned} \tag{2.1f}$$

So far all of this applies to arbitrary connected Lie groups G. We will also have occasion to use the existence of a nondegenerate symmetric bilinear form B on $\mathfrak{g}_0$, with the invariance properties

$$B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X,Y) \qquad (X,Y \in \mathfrak{g}_0, g \in G). \tag{2.1g}$$

We may arrange for this form to be negative definite: if for example G is a group of unitary matrices, so that the Lie algebra consists of skew-Hermitian matrices, then

$$B(X,Y) = \mathrm{tr}(XY)$$

will serve. (Since X has purely imaginary eigenvalues, the trace of X^2 is negative.) We will also write B for the corresponding (nondegenerate) complex-linear symmetric bilinear form on $\mathfrak{g}$.

Definition 2.2. A *maximal torus* of a compact connected Lie group G is a maximal connected abelian subgroup T of G.

We now fix a maximal torus $T \subset G$. Because of Corollary 4.52 of[11], T is actually a maximal abelian subgroup of G, and therefore equal to its own centralizer in G:

$$T = Z_G(T) = G^T = \{g \in G \mid \mathrm{Ad}(t)(g) = g, \text{ (all } t \in T)\}. \tag{2.3a}$$

Because T is a compact connected abelian Lie group, it is isomorphic to a product of copies of the unit circle

$$S^1 = \{e^{2\pi i\theta} | \theta \in \mathbb{R}\}, \qquad \mathrm{Lie}(S^1) = \mathbb{R}; \tag{2.3b}$$

the identification of the Lie algebra is made using the coordinate θ. The *character lattice of* T is

$$\begin{aligned} X^*(T) &= \mathrm{Hom}(T, S^1) \\ &= \{\lambda : T \to S^1 \text{ continuous}, \lambda(st) = \lambda(s)\lambda(t) \quad (s,t \in T)\}. \end{aligned} \tag{2.3c}$$

The character lattice is a (finitely generated free) abelian group, written additively, under multiplication of characters:

$$(\lambda + \mu)(t) = \lambda(t)\mu(t) \qquad (\lambda, \mu \in X^*(T)).$$

The functor X^* is a contravariant equivalence of categories from compact abelian Lie groups to finitely generated torsion-free abelian groups. The inverse functor is given by Hom into S^1:

$$T \simeq \mathrm{Hom}(X^*(T), S^1), \qquad t \mapsto (\lambda \mapsto \lambda(t)). \tag{2.3d}$$

The *cocharacter lattice of T* is

$$\begin{aligned} X_*(T) &= \mathrm{Hom}(S^1, T) \\ &= \{\xi\colon S^1 \to T \text{ continuous}, \ \xi(zw) = \xi(z)\xi(w) \quad (z, w \in S^1)\}. \end{aligned} \tag{2.3e}$$

There are natural identifications

$$X^*(S^1) = X_*(S^1) = \mathrm{Hom}(S^1, S^1) \simeq \mathbb{Z}, \qquad \lambda_n(z) = z^n \quad (z \in S^1).$$

The composition of a character with a cocharacter is a homomorphism from S^1 to S^1, which is therefore some nth power map. In this way we get a biadditive pairing

$$\langle \cdot, \cdot \rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}, \tag{2.3f}$$

defined by

$$\langle \lambda, \xi \rangle = n \Leftrightarrow \lambda(\xi(z)) = z^n \quad (\lambda \in X^*(T), \xi \in X_*(T), z \in S^1). \tag{2.3g}$$

This pairing identifies each of the lattices as the dual of the other:

$$X_* \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*, \mathbb{Z}), \qquad X^* \simeq \mathrm{Hom}_{\mathbb{Z}}(X_*, \mathbb{Z}). \tag{2.3h}$$

The functor X_* is a covariant equivalence of categories from compact abelian Lie groups to finitely generated torsion-free abelian groups. The inverse functor is given by $\otimes$ with S^1:

$$X_*(T) \otimes_{\mathbb{Z}} S^1 \simeq T, \qquad \xi \otimes z \mapsto \xi(z). \tag{2.3i}$$

The action by Ad of T on the complexified Lie algebra $\mathfrak{g}$ of G, like any complex representation of a compact group, decomposes into a direct sum of copies of irreducible representations; in this case, of characters of T:

$$\mathfrak{g} = \sum_{\lambda \in X^*(T)} \mathfrak{g}_\lambda, \qquad \mathfrak{g}_\lambda = \{Y \in \mathfrak{g} \mid \mathrm{Ad}(t)Y = \lambda(t)Y \ (t \in T)\}. \tag{2.3j}$$

In particular, the zero weight space is

$$\begin{aligned} \mathfrak{g}_{(0)} &= \{Y \in \mathfrak{g} \mid \mathrm{Ad}(t)Y = 0 \ (t \in T)\} \\ &= \mathrm{Lie}(G^T) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Lie}(T) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{t}, \end{aligned} \tag{2.3k}$$

the complexified Lie algebra of T; the last two equalities follow from (2.3a). We define the *roots of T in G* to be the non-trivial characters of T appearing in the decomposition of the adjoint representation:

$$R(G, T) = \{\alpha \in X^*(T) - \{0\} \mid \mathfrak{g}_\alpha \neq 0\}. \tag{2.3l}$$

Because of the description of the zero weight space in (2.3k), we have

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in R(G,T)} \mathfrak{g}_\alpha, \tag{2.3m}$$

We record two elementary facts relating the root decomposition to the Lie bracket and the invariant bilinear form B:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \tag{2.3n}$$

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \qquad \alpha + \beta \neq 0. \tag{2.3o}$$

Example 2.4. Suppose $G = SU(2)$, the group of 2×2 unitary matrices of determinant 1. We can choose as a maximal torus

$$SD(2) = \left\{ \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \simeq S^1.$$

We have given this torus a name in order to be able to formulate the definition of coroot easily. The "D" is meant to stand for "diagonal," and the "S" for "special" (meaning determinant one, as in the "special unitary group"). The last identification gives canonical identifications

$$X^*(SD(2)) \simeq \mathbb{Z}, \qquad X_*(SD(2)) \simeq \mathbb{Z}.$$

The Lie algebra of G is

$$\mathfrak{su}(2) = \{2\times 2 \text{ complex matrices } X \mid {}^t\overline{X} = -X, \mathrm{tr}(X) = 0\}.$$

The obvious map identifies

$$\mathfrak{su}(2)_{\mathbb{C}} \simeq \{2\times 2 \text{ complex matrices } Z \mid \mathrm{tr}(Z) = 0\} = \mathfrak{sl}(2,\mathbb{C}).$$

The adjoint action of $SD(2)$ on $\mathfrak{g}$ is

$$\mathrm{Ad}\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & e^{4\pi i\theta} b \\ e^{-4\pi i\theta} c & -a \end{pmatrix}.$$

This formula shows at once that the roots are

$$R(SU(2), SD(2)) = \{\pm 2\} \subset X^*(SD(2)) \simeq \mathbb{Z},$$

with root spaces

$$\mathfrak{sl}(2)_2 = \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \mid t \in \mathbb{C} \right\}, \qquad \mathfrak{sl}(2)_{-2} = \left\{ \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \mid s \in \mathbb{C} \right\}.$$

When we define coroots in a moment, it will be clear that

$$R^\vee(SU(2), SD(2)) = \{\pm 1\} \subset X_*(SD(2)) \simeq \mathbb{Z}.$$

Now we are ready to define coroots in general. For every root α, we define

$$\mathfrak{g}^{[\alpha]} = \text{Lie subalgebra generated by root spaces } \mathfrak{g}_{\pm\alpha}. \tag{2.5a}$$

It is easy to see that $\mathfrak{g}^{[\alpha]}$ is the complexification of a real Lie subalgebra $\mathfrak{g}_0^{[\alpha]}$, which in turn is the Lie algebra of a compact connected subgroup

$$G^{[\alpha]} \subset G. \tag{2.5b}$$

This subgroup meets the maximal torus T in a one-dimensional torus $T^{[\alpha]}$, which is maximal in $G^{[\alpha]}$. There is a continuous surjective group homomorphism

$$\phi_\alpha \colon SU(2) \to G^{[\alpha]} \subset G, \tag{2.5c}$$

which we may choose to have the additional properties

$$\phi_\alpha(SD(2)) = T_\alpha \subset T \quad (d\phi_\alpha)_{\mathbb{C}}(\mathfrak{sl}(2)_2) = \mathfrak{g}_\alpha. \tag{2.5d}$$

The homomorphism ϕ_α is then unique up to conjugation by T in G (or by $SD(2)$ in $SU(2)$). In particular, the restriction to $SD(2) \simeq S^1$, which we call $\alpha^\vee$, is a uniquely defined cocharacter of T:

$$\alpha^\vee \colon S^1 \to T, \qquad \alpha^\vee(e^{2\pi i\theta}) = \phi_\alpha \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}. \tag{2.5e}$$

The element $\alpha^\vee \in X_*(T)$ is called the *coroot corresponding to the root* α. We write

$$R^\vee(G,T) = \{\alpha^\vee \mid \alpha \in R(G,T)\} \subset X_*(T) - \{0\}. \tag{2.5f}$$

Essentially because the positive root in $SU(2)$ is $+2$, we see that

$$\langle \alpha, \alpha^\vee \rangle = 2 \qquad (\alpha \in R(G,T)). \tag{2.5g}$$

We turn now to a description of the Weyl group (of a maximal torus in a compact group). Here is the classical definition.

Definition 2.6. Suppose G is a compact connected Lie group, and T is a maximal torus in G. The *Weyl group of T in G* is

$$W(G,T) = N_G(T)/T,$$

the quotient of the normalizer of T by its centralizer. From this definition, it is clear that $W(G,T)$ acts faithfully on T by automorphisms (conjugation):

$$W(G,T) \hookrightarrow \mathrm{Aut}(T).$$

By acting on the range of homomorphisms, $W(G,T)$ may also be regarded as acting on cocharacters:

$$W(G,T) \hookrightarrow \mathrm{Aut}(X_*(T)), \qquad (w\cdot\xi)(z) = w\cdot(\xi(z))$$
$$(w \in W, \quad \xi \in X_*(T) = \mathrm{Hom}(S^1,T), z \in S^1).$$

Similarly,

$$W(G,T) \hookrightarrow \mathrm{Aut}(X^*(T)), \qquad (w\cdot\lambda)(t) = \lambda(w^{-1}\cdot t))$$
$$(w \in W, \quad \lambda \in X^*(T) = \mathrm{Hom}(T,S^1), t \in T).$$

The actions on the dual lattices $X_*(T)$ and $X^*(T)$ are inverse transposes of each other. Equivalently, for the pairing of (2.3g),

$$\langle w\cdot\lambda, \xi\rangle = \langle \lambda, w^{-1}\cdot\xi\rangle \qquad (\lambda \in X^*(T), \xi \in X_*(T)).$$

We recall now how to construct the Weyl group from the roots and the coroots, this is the construction that we will seek to extend to finite maximal tori. We begin with an arbitrary root $\alpha \in R(G,T)$, and ϕ_α as in (2.5c). The element

$$\sigma_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N_G(T) \tag{2.7a}$$

is well-defined (that is, independent of the choice of ϕ_α) up to conjugation by $T \cap G^{[\alpha]}$. Consequently the coset

$$s_\alpha = \sigma_\alpha T \in N_G(T)/T = W(G,T) \tag{2.7b}$$

is well-defined; it is called the *reflection in the root* α. Because it is constructed from the subgroup G_α, we see that

$$\sigma_\alpha \text{ commutes with } \ker(\alpha) \subset T, \tag{2.7c}$$

and therefore that

$$s_\alpha \text{ acts trivially on } \ker(\alpha) \subset T. \tag{2.7d}$$

A calculation in $SU(2)$ shows that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ acts by inversion on } SD(2), \tag{2.7e}$$

and therefore that

(2.7f) $$s_\alpha \text{ acts by inversion on } \mathrm{im}(\alpha^\vee) \subset T.$$

The two properties (2.7d) and (2.7f) are equivalent to

(2.7g) $$s_\alpha(t) = t \cdot \alpha^\vee(\alpha(t))^{-1} \qquad (t \in T).$$

From this formula we easily deduce

(2.7h) $$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \qquad (\lambda \in X^*(T))$$

and similarly

(2.7i) $$s_\alpha(\xi) = \xi - \langle \alpha, \xi \rangle \alpha^\vee \qquad (\xi \in X_*(T)).$$

Here is the basic theorem about the Weyl group.

Theorem 2.8. *Suppose G is a compact connected Lie group, and $T \subset G$ is a maximal torus. Then the Weyl group of T in G (Definition 2.6) is generated by the reflections described by any of the equivalent conditions* (2.7g), (2.7h), *or* (2.7i)*:*

$$W(G,T) = \langle s_\alpha \mid \alpha \in R(G,T) \rangle .$$

The automorphisms s_α of $X^*(T)$ must permute the roots $R(G,T)$, and the automorphisms s_α of $X_*(T)$ must permute the coroots $R^\vee(G,T)$.

Grothendieck's understanding of the classification of compact Lie groups by Cartan and Killing is that the combinatorial structure of roots and Weyl group determines G completely. Here is a statement.

Definition 2.9 (Root datum; see[16], 7.4)**.** An *abstract (reduced) root datum* is a quadruple $\Psi = (X^*, R, X_*, R^\vee)$, subject to the requirements

a) X^* and X_* are lattices (finitely generated torsion-free abelian groups), dual to each other (cf. (2.3h)) by a specified pairing
$$\langle , \rangle \colon X^* \times X_* \to \mathbb{Z};$$
b) $R \subset X^*$ and $R^\vee \subset X_*$ are finite subsets, with a specified bijection $\alpha \mapsto \alpha^\vee$ of R onto $R^\vee$.

These data define lattice endomorphisms (for every root $\alpha \in R$)

$$s_\alpha \colon X^* \to X^*, \qquad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha,$$
$$s_\alpha \colon X_* \to X_*, \qquad s_\alpha(\xi) = \xi - \langle \alpha, \xi \rangle \alpha^\vee,$$

called *root reflections*. It is easy to check that each of these endomorphisms is the transpose of the other with respect to the pairing $\langle , \rangle$. We impose the axioms

RD 0 if $\alpha \in R$, then $2\alpha \notin R$;
RD 1 $\langle \alpha, \alpha^\vee \rangle = 2 \qquad (\alpha \in R)$; and
RD 2 $s_\alpha(R) = R, \quad s_\alpha(R^\vee) = R^\vee \qquad (\alpha \in R)$.

Axiom (RD 0) is what makes the root datum *reduced*. Axiom (RD 1) implies that $s_\alpha^2 = 1$, so s_α is invertible. The *Weyl group of the root datum* is the group generated by the root reflections:

$$W(\Psi) = \langle s_\alpha \ \ (\alpha \in R) \rangle \subset \mathrm{Aut}(X^*).$$

The definition of root datum is symmetric in the two lattices: the *dual root datum* is

$$\Psi^\vee = (X_*, R^\vee, X^*, R).$$

The inverse transpose isomorphism identifies the Weyl group with

$$W(\Psi^\vee) = \langle s_\alpha \quad (\alpha^\vee \in R^\vee) \rangle \simeq W(\Psi).$$

Theorem 2.8 (and the material leading to its formulation) show that if T is a maximal torus in a compact connected Lie group G, then

$$\Psi(G,T) = (X^*(T), R(G,T), X_*(T), R^\vee(G,T)) \tag{2.10}$$

is an abstract reduced root datum. The amazing fact—originating in the work of Cartan and Killing, but most beautifully and perfectly formulated by Grothendieck—is that the root datum determines the group, and that every root datum arises in this way. Here is a statement.

Theorem 2.11 ([4], exp. XXV; see also[16], Theorems 9.6.2 and 10.1.1)**.** *Suppose* $\Psi = (X^*, R, X_*, R^\vee)$ *is an abstract reduced root datum. Then there is a maximal torus in a compact connected Lie group* $T \subset G$ *so that*

$$\Psi(G,T) \simeq \Psi$$

(notation (2.10)*). The pair* (G,T) *is determined by these requirements up to an inner automorphism from* T*. We have*

$$W(G,T) = N_G(T)/T \simeq W(\Psi).$$

Sketch of proof. What is proved in[4] is that to Ψ there corresponds a complex connected reductive algebraic group $\mathbf{G}(\Psi)$. There is a correspondence between complex connected reductive algebraic groups and compact connected Lie groups obtained by passage to a compact real form (see for example[13], Theorem 5.12 (page 247)). Combining these two facts proves the theorem. □

3. Finite maximal tori

Throughout this section we write

$$\begin{aligned} G &= \text{(possibly disconnected) compact Lie group} \\ G_0 &= \text{identity component of } G. \end{aligned} \tag{3.1}$$

We use the notation of (2.1), especially for the identity component G_0.

Definition 3.2. A *finite maximal torus* for G is a finite maximal abelian subgroup

$$A \subset G. \tag{3.3}$$

The definition means that the centralizer in G of A is equal to A:

$$Z_G(A) = A. \tag{3.3a}$$

The differentiated version of this equation is

$$Z_{\mathfrak{g}}(A) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(a)X = X, \text{all } a \in A\} = \mathrm{Lie}(A) = \{0\}; \tag{3.3b}$$

the last equality is because A is finite. We define the *large Weyl group of A in G_0* to be

$$W_{\mathrm{large}}(G_0, A) = N_{G_0}(A)/Z_{G_0}(A) = N_{G_0}(A)/(A \cap G_0). \tag{3.3c}$$

Clearly

$$W_{\mathrm{large}}(G_0, A) \subset \mathrm{Aut}(A), \tag{3.3d}$$

a finite group.

The term "large" should be thought of as temporary. We will introduce in Definition 3.8 a subgroup $W_{\mathrm{small}}(G,A)$, given by generators analogous to the root reflections in a classical Weyl group. We believe that the two groups are equal; but until that is proved, we need terminology to talk about them separately.

In contrast to classical maximal tori, finite maximal tori need not exist. For example, if $G = U(n)$, then any abelian subgroup must (after change of basis) consist entirely of diagonal matrices; so the only maximal abelian subgroups are the connected maximal tori, none of which is finite.

For the rest of this section we fix a (possibly disconnected) compact Lie group G, and a finite maximal torus

$$A \subset G. \tag{3.4a}$$

Our goal in this section is to introduce roots, coroots, and root transvections, all by analogy with the classical case described in Section 2.

The *character group of* A is

$$\begin{aligned} X^*(A) &= \mathrm{Hom}(A,S^1) \\ &= \{\lambda : A \to S^1\ \lambda(ab) = \lambda(a)\lambda(b) \quad (a,b \in A)\}. \end{aligned} \tag{3.4b}$$

The character group is a finite abelian group, written additively, under multiplication of characters:

$$(\lambda+\mu)(a) = \lambda(a)\mu(a) \qquad (\lambda,\mu \in X^*(A)).$$

In particular, we write 0 for the trivial character of A. We can recover A from $X^*(A)$ by a natural isomorphism

$$A \simeq \mathrm{Hom}(X^*(A),S^1), \qquad a \mapsto [\lambda \mapsto \lambda(a)]. \tag{3.4c}$$

As a consequence, the functor X^* is a contravariant exact functor from the category of finite abelian groups to itself. The group $X^*(A)$ is always isomorphic to A, but not canonically.

For any positive integer n, define

$$\mu_n = \{z \in \mathbb{C} \mid z^n = 1\}, \tag{3.4d}$$

the group of *nth roots of unity in* $\mathbb{C}$. We identify

$$X^*(\mathbb{Z}/n\mathbb{Z}) \simeq \mu_n, \qquad \lambda_\omega(m) = \omega^m \quad (\omega \in \mu_n, m \in \mathbb{Z}/n\mathbb{Z}). \tag{3.4e}$$

Similarly we identify

$$X^*(\mu_n) \simeq \mathbb{Z}/n\mathbb{Z}, \qquad \lambda_m(\omega) = \omega^m \quad (m \in \mathbb{Z}/n\mathbb{Z}, \omega \in \mu_n). \tag{3.4f}$$

Of course we can write

$$\mu_n = \{e^{2\pi i\theta} \mid \theta \in \mathbb{Z}/n\mathbb{Z}\} \simeq \mathbb{Z}/n\mathbb{Z},$$

and so identify a particular generator of the cyclic group μ_n; but (partly with the idea of working with reductive groups over other fields, and partly to to see what is most natural) we prefer to avoid using this identification.

A character $\lambda \in X^*(A)$ is said to be *of order dividing n* if $n\lambda = 0$; equivalently, if

$$\lambda : A \to \mu_n. \tag{3.4g}$$

We will say "character of order n" to mean a character of order dividing n. We write

$$\begin{aligned} X^*(A)(n) &= \{\lambda \in X^*(A) \mid n\lambda = 0\} \\ &= \mathrm{Hom}(A,\mu_n), \end{aligned} \tag{3.4h}$$

for the *group of characters of order n*. Therefore

$$X^*(A) = \bigcup_{n \geq 1} X^*(A)(n). \tag{3.4i}$$

We say that λ has *order exactly n* if n is the smallest positive integer such that $n\lambda = 0$; equivalently, if

$$\lambda : A \twoheadrightarrow \mu_n \tag{3.4j}$$

is surjective.

The action by Ad of A on the complexified Lie algebra $\mathfrak{g}$ of G decomposes into a direct sum of characters:

$$\mathfrak{g} = \sum_{\lambda \in X^*(A)} \mathfrak{g}_\lambda, \qquad \mathfrak{g}_\lambda = \{Y \in \mathfrak{g} \mid \mathrm{Ad}(a)Y = \lambda(a)Y \ (a \in A)\}. \tag{3.4k}$$

According to (3.3b), the trivial character does not appear in this decomposition; that is, $\mathfrak{g}_0 = 0$. We define the *roots of A in G* to be the characters of A that do appear:

$$R(G,A) = \{\alpha \in X^*(A)\} \mid \mathfrak{g}_\alpha \neq 0\} \subset X^*(A) - \{0\}. \tag{3.4l}$$

The analogue of the root decomposition (2.3m) has no term like the Lie algebra of the maximal torus:

$$\mathfrak{g} = \sum_{\alpha \in R(G,A)} \mathfrak{g}_\alpha. \tag{3.4m}$$

Just as for classical roots, we see immediately

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \tag{3.4n}$$

and

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \qquad \alpha + \beta \neq 0. \tag{3.4o}$$

Fix a positive integer n. A *cocharacter of order dividing n* is a homomorphism

$$\xi : \mu_n \to A. \tag{3.5a}$$

We will say "cocharacter of order n" to mean a cocharacter of order dividing n. The cocharacter has order exactly n if and only if ξ is injective.

If we fix a primitive nth root ω, then a cocharacter ξ of order n is the same thing as an element $x \in A$ of order n, by the correspondence

$$x = \xi(\omega). \tag{3.5b}$$

We write

$$X_*(A)(n) = \mathrm{Hom}(\mu_n, A) \tag{3.5c}$$

for the group of cocharacters of order n. The natural surjection

$$\mu_{mn} \twoheadrightarrow \mu_n, \qquad \omega \mapsto \omega^m$$

gives rise to a natural inclusion

$$\mathrm{Hom}(\mu_n, A) \hookrightarrow \mathrm{Hom}(\mu_{mn}, A), \qquad X_*(A)(n) \hookrightarrow X_*(A)(mn). \tag{3.5d}$$

Using these inclusions, we can define the *cocharacter group of A*

$$\bigcup_n X_*(A)(n). \tag{3.5e}$$

The functor X_* is a covariant functor equivalence of categories from the category of finite abelian groups to itself; but the choice of a functorial isomorphism $A \simeq X_*(A)$ requires compatible choices of primitive nth roots ω_n for every n. (The compatibility requirement is $\omega_n = (\omega_{mn}^m$.) Partly to maintain the analogy with cocharacters of connected tori, and partly for naturality, we prefer not to make such choices, and to keep $X_*(A)$ as a group distinct from A.

Suppose $\lambda \in X^*(A)$ is any character and $\xi \in X_*(A)(n)$ is an order n cocharacter. The composition $\lambda \circ \xi$ is a homomorphism $\mu_n \to S^1$. Such a homomorphism must take values in μ_n, and is necessarily raising to the mth power for a unique $m \in \mathbb{Z}/n\mathbb{Z} \simeq (1/n)\mathbb{Z}/\mathbb{Z}$. In this way we get a natural pairing

$$X^*(A) \times X_*(A)(n) \to (1/n)\mathbb{Z}/\mathbb{Z},$$
$$\lambda(\xi(\omega)) = \omega^{n\langle\lambda,\xi\rangle} \qquad (\lambda \in X^*(A)(n),\ \xi \in X_*(A)(n),\ \omega \in \mu_n). \tag{3.5f}$$

Taking the union over n defines a biadditive pairing

$$X^*(A) \times X_*(A) \to \mathbb{Q}/\mathbb{Z} \tag{3.5g}$$

which identifies

$$X_*(A) \simeq \mathrm{Hom}(X^*(A),\mathbb{Q}/\mathbb{Z}), \qquad X^*(A) \simeq \mathrm{Hom}(X_*(A),\mathbb{Q}/\mathbb{Z}). \tag{3.5h}$$

Before defining coroots in general, we need an analog of Example 2.4.

Example 3.6. Suppose $A \subset H$ is a finite maximal torus, in a compact Lie group H of strictly positive dimension N. Assume that the roots of A in H lie on a single line; that is, that there is a character α so that

$$R(H,A) \subset \{m\alpha \mid m \in \mathbb{Z}\} \subset X^*(A). \tag{3.6a}$$

(We do not assume that α itself is a root.) Since H is assumed to have positive dimension, there must be *some* (necessarily nonzero) roots; so α must have some order exactly $n > 0$:

$$\alpha \colon A \twoheadrightarrow \mu_n. \tag{3.6b}$$

Fix now a primitive nth root of unity ω, and an element $y \in A$ so that

$$\mu(y) = \omega. \tag{3.6c}$$

Then

$$\mathfrak{h}_{m\alpha} = \{X \in \mathfrak{h} \mid \mathrm{Ad}(y)X = \omega^m X\}. \tag{3.6d}$$

From this description (or indeed from (3.4m) and (3.4n)) it is clear that $\mathfrak{h}[m] = \mathfrak{h}_{m\alpha}$ is a $\mathbb{Z}/n\mathbb{Z}$-grading of the complex reductive Lie algebra $\mathfrak{h}$, and that $\mathfrak{h}[0] = 0$. According to the Kač classification of automorphisms of finite order (see for example[10], pp. 490–515; what we need is Lemma 10.5.3 on page 492)

$$\mathfrak{h} \text{ is necessarily abelian,} \tag{3.6e}$$

so the identity component H_0 is a compact torus.

We chose y so that $\alpha(y)$ generates the image of α. From this it follows immediately that A is generated by y and the kernel of α:

$$A = \langle \ker(\alpha), y\rangle. \tag{3.6f}$$

Because of the definition of roots and (3.6a), $\mathrm{Ad}(\ker(\alpha))$ must act trivially on $\mathfrak{h}$, and therefore on H_0. It follows that

$$Z_{H_0}(A) = H_0^y,$$

the fixed points of the automorphism $\mathrm{Ad}(y)$ on the torus H_0. Since A is assumed to be maximal abelian, we deduce

$$A \cap H_0 = H_0^y. \tag{3.6g}$$

We are therefore going to analyze this fixed point group.

Because

$$\operatorname{Aut}(H_0) = \operatorname{Aut}(X_*(H_0)), \tag{3.6h}$$

the automorphism $\operatorname{Ad}(y)$ is represented by a lattice automorphism

$$y_* \in \operatorname{Aut}(X_*(H_0)), \qquad (y_*)^n = 1 \tag{3.6i}$$

and therefore (after choice of a lattice basis) by an invertible $N \times N$ integer matrix

$$Y_* \in GL(N,\mathbb{Z}), \qquad (Y_*)^n = I. \tag{3.6j}$$

Because 0 is not a root, the matrix Y_* does not have one as an eigenvalue. Every eigenvalue must be a primitive dth root of 1 for some d dividing n (and not equal to 1). Define

$$\begin{aligned} m_y(d) &= \text{multiplicity of primitive } d\text{th roots as eigenvalues of } Y_* \\ &= \dim(\mathfrak{h}_{m\alpha}), \qquad \text{all } m \text{ such that } \gcd(m,n) = n/d. \end{aligned} \tag{3.6k}$$

Then the characteristic polynomial of the matrix Y_* is

$$\det(xI - Y_*) = \prod_{d|n,\ d>1} \Phi_d(x)^{m_y(d)}. \tag{3.6l}$$

The number of fixed points of $\operatorname{Ad}(y)$ is easily computed to be

$$\begin{aligned} |H_0^y| &= |\det(I - Y_*)| \\ &= \prod_{d|n,\ d>1} \Phi_d(1)^{m_y(d)} \end{aligned} \tag{3.6m}$$

Here Φ_d is the dth cyclotomic polynomial

$$\Phi_d(x) = \prod_{\omega\in\mu_d \text{ primitive}} (x - \omega). \tag{3.6n}$$

Evaluating cyclotomic polynomials at 1 is standard and easy:

$$\Phi_d(1) = \begin{cases} 0 & d = 1 \\ p & d = p^m,\ (p \text{ prime}, m \geq 1) \\ 1 & d \text{ divisible by at least two primes.} \end{cases} \tag{3.6o}$$

Inserting these values (3.6m) gives

$$|H_0^y| = \prod_{\substack{p^m|n \\ p \text{ prime},\ m\geq 1}} p^{\dim \mathfrak{h}_{(n/p^m)\alpha}}. \tag{3.6p}$$

It is easy to see that every element of H_0^y has order dividing n.

Definition 3.7. Suppose A is a finite maximal torus in the compact Lie group G, and $\alpha \in R(G,A)$ is a root of order exactly n:

$$1 \longrightarrow \ker\alpha \longrightarrow A \xrightarrow{\alpha} \mu_n \longrightarrow 1. \tag{3.7a}$$

The characters of A that are trivial on $\ker\alpha$ are precisely the multiples of α. If we define

$$G^{[\alpha]} = Z_G(\ker\alpha), \tag{3.7b}$$

(a compact subgroup of G) then its complexified Lie algebra is

$$\mathfrak{g}^{[\alpha]} = \sum_{m\in\mathbb{Z}} \mathfrak{g}_{m\alpha}. \tag{3.7c}$$

That is, the pair $(G^{[\alpha]}, A)$ is of the sort considered in Example 3.6. We now use the notation of that example, choosing in particular a primitive nth root $\omega \in \mu_n$, and an element $y \in A$ so that

$$\alpha(y) = \omega. \tag{3.7d}$$

As we saw in the example, $(G^{[\alpha]})_0$ is a (connected) torus, on which y acts as an automorphism of order n; and

$$A \cap (G^{[\alpha]})_0 = ((G^{[\alpha]})_0)^y. \tag{3.7e}$$

The outer parentheses are included for clarity: first take the identity component, then compute the fixed points of $\mathrm{Ad}(y)$. Reversing this order would give $(G^{[\alpha]})^y = A$, which has trivial identity component. But we will omit them henceforth. We define the group of *coroots for* α to be

$$R^\vee(\alpha) = \{\xi\colon \mu_n \to (G^{[\alpha]})_0^y \subset \ker\alpha \subset A\} \subset X_*(\ker\alpha)(n) \subset X_*(n), \tag{3.7f}$$

the cocharacters taking values in the group of fixed points of $\mathrm{Ad}(y)$ on $(G^{[\alpha]})_0$. Choosing a primitive nth root of 1 identifies $R^\vee(\alpha)$ with $(G^{[\alpha]})_0^y$. Its cardinality may therefore be computed in terms of root multiplicities using (3.6p):

$$\begin{aligned} |R^\vee(\alpha)| &= |(G^{[\alpha]})_0^y| \\ &= \prod_{\substack{p^m | n \\ p \text{ prime},\, m \geq 1}} p^{\dim \mathfrak{g}_{(n/p^m)\alpha}}. \end{aligned} \tag{3.7g}$$

There are nontrivial coroots for α if and only if there is a nontrivial prime power p^m dividing n so that $(n/p^m)\alpha$ is a root.

Definition 3.8. Suppose A is a finite maximal torus in the compact Lie group G, and $\alpha \in R(G, A)$ is a root of order n, and

$$\xi\colon \mu_n \to (G^{[\alpha]})_0^y \subset \ker\alpha \subset A \tag{3.8a}$$

is a coroot for α (Definition 3.7). A *transvection generator for* (α, ξ) is an element

$$\sigma(\alpha, \xi) \in (G^{[\alpha]})_0 \tag{3.8b}$$

with the property that

$$\mathrm{Ad}(y^{-1})(\sigma(\alpha, \xi)) = \sigma(\alpha, \xi)\xi(\omega). \tag{3.8c}$$

We claim that there is a transvection generator for each coroot. To see this, write the abelian group $(G^{[\alpha]})_0$ additively. Then the equation we want to solve looks like

$$[\mathrm{Ad}(y^{-1}) - I]\sigma = \xi(\omega).$$

Because the determinant of the Lie algebra action is

$$|\det(\mathrm{Ad}(y^{-1}) - I)| = |\det(I - \mathrm{Ad}(y))| = |\det(I - Y_*)|$$

which is equal to the number of coroots (see (3.6m)), we see that (3.8c) has a solution $\sigma(\alpha, \xi)$, and that in fact $\sigma(\alpha, \xi)$ is unique up to a factor from $(G^{[\alpha]})_0^y$.

The defining equation for a transvection generator may be rewritten as

$$\sigma(\alpha, \xi)\, y\, \sigma(\alpha, \xi)^{-1} = y\,\xi(\alpha(y)). \tag{3.8d}$$

Because $\sigma(\alpha, \xi)$ is built from exponentials of root vectors for roots that are multiples of α, $\sigma(\alpha, \xi)$ must commute with $\ker\alpha$:

$$\sigma(\alpha, \xi)\, a_0\, \sigma(\alpha, \xi)^{-1} = a_0 \qquad (a_0 \in \ker\alpha \subset A). \tag{3.8e}$$

Combining the last two formulas, and the fact that A is generated by $\ker\alpha$ and y, we find

$$\sigma(\alpha,\xi)\,a\,\sigma(\alpha,\xi)^{-1} = a\,\xi(\alpha(a)) \qquad (a\in A). \tag{3.8f}$$

In particular,

$$\sigma(\alpha,\xi)\in N_{G_0}(A), \tag{3.8g}$$

and the *root transvection* is the coset

$$s(\alpha,\xi) = \sigma(\alpha,\xi)(A\cap G_0)\in N_{G_0}(A)/(A\cap G_0) = W_{\text{large}}(G,A). \tag{3.8h}$$

We define the *small Weyl group of A in G* to be the subgroup

$$W_{\text{small}}(G,A) = \left\langle s(\alpha,\xi) \quad (\alpha\in R(G,A), \xi\in R^\vee(\alpha)\right\rangle. \tag{3.8i}$$

generated by root transvections.

Conjecture 3.9. *If A is a finite maximal torus in a compact Lie group G (Definition 3.2), the normalizer of A in G_0 is generated by $A\cap G_0$ and the transvection generators $\sigma(\alpha,\xi)$ described in Definition 3.8. Equivalently,*

$$W_{small}(G,A) = W_{large}(G,A)$$

(Definitions 3.2 and 3.8).

In case G is the projective unitary group $PU(n)$, then it is shown in[9] that A must be one of the subgroups described in (4.3) below. In these cases the conjecture is established in[8].

We want to record explicitly one of the conclusions of Example 3.6.

Proposition 3.10. *Suppose A is a finite maximal torus in a compact Lie group G (Definition 3.2), and that α and β are roots of A in G.*

(1) If α and β are both multiples of the same root γ, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=0$.
(2) If α and β have relatively prime orders, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=0$.

Proof. Part (1) is (3.6e) (together with the argument used in Definition 3.7 to get into the setting of Example 3.6). If α and β have orders m and n, then the hypothesis of (2) produces integers x and y so that $mx+ny=1$. Consequently

$$\beta = (mx+ny)\beta = mx\beta = mx(\alpha+\beta),$$

and similarly

$$\alpha = ny(\alpha+\beta).$$

So (2) follows from (1) (with $\gamma=\alpha+\beta$). □

We conclude this section with a (tentative and preliminary) analogue of Grothendieck's notion of root datum.

Definition 3.11 (Finite root datum)**.** An *abstract finite root datum* is a quadruple $\Psi = (X^*,R,X_*,R^\vee)$, subject to the requirements

a) X^* and X_* are finite abelian groups, dual to each other (cf. (3.5h)) by a specified pairing

$$\langle,\rangle\colon X^*\times X_*\to \mathbb{Q}/\mathbb{Z};$$

b) $R\subset X^* - \{0\}$
c) $R^\vee$ is a map from R to subgroups of X_*; we call $R^\vee(\alpha)$ the group of *coroots for α*.

We impose first the axioms

FRD 0 If α has order n, and k is relatively prime to n, then $k\alpha$ is also a root and $R^\vee(\alpha) = R^\vee(k\alpha)\subset X_*(n)$; and

FRD 1 If $\xi \in R^\vee(\alpha)$, then $\langle \alpha, \xi \rangle = 0$.

(The condition about k is a rationality hypothesis, corresponding to some automorphism being defined over $\mathbb{Q}$. The rest of (FRD)(0) says that the order of a coroot must divide the order of the corresponding root. Axiom (FRD)(1) says that a coroot must take values in the kernel of the corresponding root.)

For each root α of order n and coroot $\xi \in R^\vee(\alpha)$ we get homomorphisms of abelian groups

$$s(\alpha,\xi)\colon X^* \to X^*, \qquad s(\alpha,\xi)(\lambda) = \lambda - n\langle \lambda, \xi \rangle \alpha,$$
$$s(\alpha,\xi)\colon X_* \to X_*, \qquad s(\alpha,\xi)(\tau) = \tau - n\langle \alpha, \tau \rangle \xi,$$

called *root transvections*. The coefficients of α and of ξ in these formulas are integers because of axiom (FRD)(0); so the formulas make sense. It is easy to check that each of these endomorphisms is the transpose of the other with respect to the pairing $\langle , \rangle$. The axiom (FRD)(1) means that $s(\alpha,\xi)$ is the identity on multiples of α, and clearly $s(\alpha,\xi)$ induces the identity on the quotient $X^*/\langle \alpha \rangle$. Therefore $s(\alpha,\xi)$ is a transvection, and

$$s(\alpha,\cdot)\colon R^\vee(\alpha) \hookrightarrow \mathrm{Aut}(X^*)$$

is a group homomorphism.

We impose in addition the axioms

FRD 2 If $\alpha \in R$ and $\xi \in R^\vee(\alpha)$, then $s(\alpha,\xi)(R) = R$.
FRD 3 If $\alpha \in R$ and $\xi \in R^\vee(\alpha)$, then $s(\alpha,\xi)(R^\vee(\beta)) = R^\vee(s(\alpha,\xi)(\beta))$.

The *Weyl group of the root datum* is the group generated by the root transvections:

$$W(\Psi) = \langle s(\alpha,\xi) \ \ (\alpha \in R,\ \xi \in R^\vee(\alpha)) \rangle \subset \mathrm{Aut}(X^*).$$

The inverse transpose isomorphism identifies the Weyl group with a group of automorphisms of X_*.

We have shown in this section that the root datum

$$\Psi(G,A) = (X^*(A), R(G,A), X_*(A), R^\vee) \tag{3.12}$$

of a finite maximal A torus in a compact Lie group G is an abstract finite root datum. The point of making these observations is the hope of finding and proving a result analogous to Theorem 2.11: that an abstract finite root datum determines a pair (G,A) uniquely.

We do not yet understand precisely how to formulate a reasonable conjecture along these lines. First, in order to avoid silly counterexamples from finite groups, we should assume

$$G = G_0 A; \tag{3.13}$$

that is, that A meets every component of G.

To see a more serious failure of the finite root datum to determine G, consider the finite root datum

$$(\mathbb{Z}/6\mathbb{Z}, \{1,5\}, (1/6)\mathbb{Z}/\mathbb{Z}, R^\vee), \tag{3.14a}$$

in which $R^\vee(1) = R^\vee(5) = \{0\}$. Write

$$\mathbb{A} = \mathbb{Z}[x]/\langle \Phi_6(x) \rangle = \mathbb{Z}[x]/\langle x^2 - x + 1 \rangle, \tag{3.14b}$$

the ring of integers of the cyclotomic field $\mathbb{Q}[\omega_6]$, with ω_6 a primitive sixth root of unity. The choice of ω_6 defines an inclusion $\mu_6 \hookrightarrow \mathbb{A}$ sending ω_6 to the image of x. Therefore the rank two free abelian group $\mathbb{A}$ acquires an action of $A = \mu_6$. If we write T^1 for the two-dimensional torus with

$$X_*(T^1) = \mathbb{A}, \tag{3.14c}$$

then the equivalence of categories (2.3i) provides an action of μ_6 on T^1. Explicitly, the action of ω_6 on T^1 is

$$\omega_6 \cdot (z, w) = (w^{-1}, zw) \qquad (z, w \in S^1). \tag{3.14d}$$

The roots for this action are 1 and 5. According to (3.6p) (or by inspection of (3.14d)) the action of A on T^1 has no fixed points. It follows that A is a maximal abelian subgroup of

$$G^1 = T^1 \rtimes A, \tag{3.14e}$$

and that the corresponding finite root datum is exactly the one described by (3.14a).

So far so good. But we could equally well use the $2m$-dimensional torus

$$T^m = T^1 \times \cdots \times T^1$$

with the diagonal action of μ_6, and define

$$G^m = T^m \rtimes A, \tag{3.14f}$$

Again $A = \mu_6$ is a maximal abelian subgroup, and the root datum is exactly (3.14a). So in this case there are many different G, of different dimensions, with the same finite root datum.

The most obvious way to address this particular family of counterexamples is to include root multiplicities as part of the finite root datum, and to require that they compute the cardinalities of the coroot groups $R^\vee(\alpha)$ by (3.7g). (If A is an elementary abelian p-group, then the coroot groups determine the root multiplicities: if α has multiplicity m, then $|R^\vee(\alpha)| = p^m$. That is why we needed A of order 6 to have make an easy example where many multiplicities are possible.) But the root multiplicities alone do not determine G; one can make counterexamples with G_0 a torus and A cyclic using cyclotomic fields of class number greater than one. Perhaps the finite root datum should be enlarged to include the tori $G_0^{[\alpha]}$ (or rather the corresponding lattices), equipped with the action of μ_n constructed in Definition 3.7.

4. EXAMPLES

Example 4.1. The simplest example of a finite maximal torus is in the three-dimensional compact group

$$G = SO(3)$$

of three by three real orthogonal matrices. We can choose

$$A = \{ \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \mid \varepsilon_i = \pm 1, \ \prod_i \varepsilon_i = 1 \}.$$

This is the "Klein four-group," the four-element group in which each non-identity element has order 2. We can identify characters with subsets of $S \subset \{1, 2, 3\}$, modulo the equivalence relation that each subset is equivalent to its complement: $S \sim S^c$. The formula is

$$\lambda_S(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \prod_{i \in S} \varepsilon_i.$$

Thus the trivial character of A is $\lambda_\emptyset = \lambda_{\{1,2,3\}}$, and the three non-trivial characters correspond to the three two-element subsets $\{i,j\}$ (or equivalently to their three one-element complements):

$$\lambda_{\{i,j\}}(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \varepsilon_i\varepsilon_j.$$

The Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ consists of 3×3 skew-symmetric matrices. The root spaces of A are one-dimensional:

$$\mathfrak{g}_{\lambda_{\{i,j\}}} = \mathbb{C}(e_{ij} - e_{ji}) \qquad (1 \le i \ne j \le 3),$$

the most natural and obvious lines of skew-symmetric matrices. Therefore

$$R(G,A) = \{\lambda_{\{i,j\}} \in X^*(A) \mid (1 \le i \ne j \le 3),$$

the set of all three non-zero characters of A.

Each root space is the Lie algebra of one of the three obvious $SO(2)$ subgroups of $SO(3)$, and these are the tori $G_0^{[\alpha]}$ used in Definition 3.7. The automorphism y of each torus is inversion, so the coroots are the two elements of order (1 or) 2 in each torus. If ξ is the nontrivial coroot attached to the root (i,j), then the transvection $s(\alpha,\xi)$ acts on $\{1,2,3\}$ by transposition of i and j. The (small) Weyl group is therefore

$$W_{\text{small}}(G,A) = S_3.$$

Since this is the full automorphism group of A, it is also equal to the large Weyl group.

It is a simple and instructive matter to make a similar definition for $O(n)$, taking for A the group of 2^n diagonal matrices. The whole calculation is exactly parallel to that for the root system of $U(n)$, with the role of the complex units S^1 played by the real units $\{\pm 1\}$; or, on the level of X^*, with $\mathbb{Z}$ replaced by $\mathbb{Z}/2\mathbb{Z}$.

Example 4.2. We begin with the unitary group

$$\widetilde{G} = U(n) = n\times n \text{ unitary matrices.} \tag{4.2a}$$

The center of $\widetilde{G}$ consists of the scalar matrices

$$Z(n) = \{zI \mid z \in S^1\} \simeq S^1 \tag{4.2b}$$

(notation (2.3b)). We are going to construct a finite maximal torus A inside the projective unitary group

$$G = PU(n) = U(n)/Z(n). \tag{4.2c}$$

It is convenient to construct a preimage $\widetilde{A} \subset \widetilde{G} = U(n)$.

The Lie algebra of $U(n)$ consists of skew-Hermitian $n\times n$ complex matrices:

$$\widetilde{\mathfrak{g}}_0 = \mathfrak{u}(n) = \{X \in M_n(\mathbb{C} \mid {}^t\overline{X} = -X\}. \tag{4.2d}$$

An obvious map identifies its complexification with all $n\times n$ matrices:

$$\widetilde{\mathfrak{g}} = M_n(\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C}). \tag{4.2e}$$

The adjoint action is given by conjugation of matrices. Dividing by the center gives

$$\mathfrak{g}_0 = \mathfrak{pu}(n) = \mathfrak{u}(n)/i\mathbb{R}I, \tag{4.2f}$$

$$\mathfrak{g} = \mathfrak{pgl}(n,\mathbb{C}) = M_n(\mathbb{C})/\mathbb{C}I. \tag{4.2g}$$

It will be convenient to think of $U(n)$ as acting on the vector space

$$\mathbb{C}^n = \text{functions on } \mathbb{Z}/n\mathbb{Z},$$

functions on the cyclic group of order n. We will call the standard basis

$$e_0, e_1, \dots, e_{n-1}$$

with e_i the delta function at the group element $i + n\mathbb{Z}$. It is therefore often convenient to regard the indices as belonging to $\mathbb{Z}/n\mathbb{Z}$.

We are going to define two cyclic subgroups

$$\tau\colon \mathbb{Z}/n\mathbb{Z} \to U(n), \qquad \sigma\colon \mu_n \to U(n) \tag{4.2h}$$

of $U(n)$. We will also be interested in

$$\zeta\colon \mu_n \to Z(n), \qquad \zeta(\omega) = \omega I. \tag{4.2i}$$

The map τ comes from the action of $\mathbb{Z}/n\mathbb{Z}$ on itself by translation; the generator $1 = 1 + n\mathbb{Z}$ acts by

$$\tau(1) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.2j}$$

Often it is convenient to compute with the action on basis vectors:

$$\tau(m)e_i = e_{i-m},$$

as usual with the subscripts interpreted modulo n. The map σ is from the character group of $\mathbb{Z}/n\mathbb{Z}$. The element $\omega \in \mu_n$ is realized as multiplication by the character $m \mapsto \omega^m$:

$$\sigma(\omega) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}. \tag{4.2k}$$

This time the formula on basis vectors is

$$\sigma(\omega)e_i = \omega^i e_i.$$

Each of σ and τ has order n, and their commutator is

$$\sigma(\omega)\tau(m)\sigma(\omega^{-1})\tau(-m) = \omega^m I = \zeta(\omega^m) \in Z(G).$$

The three cyclic groups σ, τ, and ζ generate a group

$$\widetilde{A} = \langle \tau(\mathbb{Z}/n\mathbb{Z}), \sigma(\mu_n), \zeta(\mu_n) \rangle \tag{4.2l}$$

of order n^3, with defining relations

$$\sigma(\omega)\tau(m)\sigma(\omega^{-1})\tau(-m) = \zeta(\omega^m), \qquad \sigma\zeta = \zeta\sigma, \qquad \tau\zeta = \zeta\tau; \tag{4.2m}$$

this group is a *finite Heisenberg group* of order n^3. (One early appearance of such groups is in[12], pp. 294–297. There is an elementary account of their representation theory in [17], Chapter 19.)

The "finite maximal torus" we consider is

$$\begin{aligned} A &= \text{image of } \widetilde{A} \text{ in } PU(n) \\ &= \widetilde{A}/\zeta(\mu_n) \simeq (\mathbb{Z}/n\mathbb{Z}) \times \mu_n. \end{aligned} \tag{4.2n}$$

We will explain in a moment why A is *maximal* abelian. The adjoint action of A on the Lie algebra is easily calculated to be

$$\mathrm{Ad}(\tau(m))(e_{rs}) = e_{r-m,s-m}, \tag{4.2o}$$

with the subscripts interpreted modulo n. Similarly

$$\mathrm{Ad}(\sigma(\omega))(e_{rs}) = \omega^{r-s}e_{rs}. \tag{4.2p}$$

The character group of A is

$$X^*(A) \simeq \mu_n \times \mathbb{Z}/n\mathbb{Z} \qquad \lambda_{\phi,j}(\tau(m)\sigma(\omega)) = \phi^m\omega^j.$$

We now describe the roots of A in the Lie algebra $\mathfrak{g} = M_n(\mathbb{C})/\mathbb{C}I$. Fix

$$j \in \mathbb{Z}/n\mathbb{Z}, \qquad \phi \in \mu_n,$$

and define

$$X_{\phi,j} = \sum_{r-s=j} \phi^r e_{rs} = \sigma(\phi)\tau(-j). \tag{4.2q}$$

(That is, the root vectors as matrices can be taken equal to the group elements as matrices.) It follows from (4.2o) and (4.2p) that

$$\mathrm{Ad}(\tau(m))(X_{\phi,j}) = \phi^m X_{\phi,j}, \qquad \mathrm{Ad}(\sigma(\omega))(X_{\phi,j}) = \omega^j X_{\phi,j}, \tag{4.2r}$$

That is, $X_{\phi,j}$ is a weight vector for the character $(\phi, j) \in X^*(A)$. The weight vector $X_{1,0}$ is the identity matrix, by which we are dividing to get $\mathfrak{g}$; so $(1,0)$ is not a root. Therefore

$$R(G,A) = \{(\phi, j) \neq (1,0) \in X^*(A)\} \simeq [\mu_n \times \mathbb{Z}/n\mathbb{Z}] - (1,0), \tag{4.2s}$$

the set of n^2-1 non-trivial characters of $\mathbb{Z}/n\mathbb{Z} \times \mu_n$.

There remains the question of why A is *maximal* abelian in $PU(n)$. Suppose $g \in PU(n)^A$. Choose a preimage $\widetilde{g} \in U(n) \subset M_n(\mathbb{C})$. Then the fact that g commutes with the images of τ and σ in $PU(n)$ means that

$$\tau(m)\widetilde{g}\tau(-m) = b(m)\widetilde{g}, \qquad \sigma(\omega)\widetilde{g}\sigma(\omega^{-1}) = c(\omega)\widetilde{g}.$$

If we write $\widetilde{g}$ in the matrix basis $X_{\phi,j}$ of (4.2q), the conclusion is that $b(m) = \phi$ is an nth root of unity, that $c(\omega) = \omega^j$, and that

$$\widetilde{g} = zX_{\phi,j} = z\sigma(\phi)\tau(-j). \tag{4.2t}$$

Therefore $g = \sigma(\phi)\tau(-j) \in A$, as we wished to show.

We want to understand, or at least to count, the coroots corresponding to each root α of order d. Since every (nontrivial) character of A has multiplicity one as a root, we conclude from (3.7g) that there are precisely d coroots ξ attached to α. In particular, if p^m is the largest power of some prime dividing n, and α has order p^m, then (since $A \simeq (\mathbb{Z}/n\mathbb{Z})^2$)

$$\ker\alpha \simeq \mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z}/(n/p^m)\mathbb{Z}.$$

So in this case there are exactly p^m homomorphisms from μ_{p^m} into $\ker\alpha$, and all of them must be coroots. We conclude that the root transvections include all transvections of A attached to characters of order exactly p^m. One can show that these transvections generate $SL(2,\mathbb{Z}/n\mathbb{Z})$, so

$$W_{\mathrm{small}}(G,A) \simeq SL(2,\mathbb{Z}/n\mathbb{Z}). \tag{4.2u}$$

We conclude this example by calculating the structure constants of $\mathfrak{g}$ in the root basis. Using the relation $X_{\phi,j} = \sigma(\phi)\tau(-j)$ from (4.2q), and the commutation relation (4.2m), we find

$$\begin{aligned} X_{\phi,j}X_{\psi,k} &= \sigma(\phi)\tau(-j)\sigma(\psi)\tau(-k) \\ &= \psi^j\sigma(\phi)\sigma(\psi)\tau(-j)\tau(-k) \\ &= \psi^j\sigma(\phi\psi)\tau(-j-k) = \psi^j X_{\phi\psi,j+k}. \end{aligned}$$

Similarly

$$X_{\psi,k}X_{\phi,j} = \phi^k X_{\phi\psi,j+k}.$$

Therefore

$$[X_{\phi,j},X_{\psi,k}] = (\psi^j - \phi^k)X_{\phi\psi,j+k}. \tag{4.2v}$$

A fundamental fact about classical roots in reductive Lie algebras (critical to Chevalley's construction of reductive groups over arbitrary fields; see[16], Chapter 10) is that the structure constants may be chosen to be integers. Here we see that the structure constants are integers in the cyclotomic field $\mathbb{Q}[\mu_n]$.

The preceding example can be generalized by replacing the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with any abelian group F of order n, and A by the "symplectic space" $A = F \times X^*(F)$ ([18], page 148). If D is the largest order of an element of F, then the symplectic form takes values in μ_D:

$$\Sigma((f_1,\lambda_1),(f_2,\lambda_2)) = \lambda_1(f_2)[\lambda_2(f_1)]^{-1}. \tag{4.3a}$$

Characters of A may be indexed by elements of A using the symplectic form:

$$\alpha_x(a) = \Sigma(a,x).$$

The transvection generators are precisely the symplectic transvections

$$a \mapsto a + \xi(\langle a,x\rangle) \tag{4.3b}$$

on A. Here x is any element of A of order d, and

$$\xi\colon \mu_d \to \langle x\rangle$$

is any homomorphism. They generate the full symplectic group

$$W_{\text{small}}(G,A) = Sp(A,\Sigma); \tag{4.3c}$$

a proof may be found in[8], Theorem 3.16. (When F is a product of elementary abelian p groups for various primes p, the assertion that symplectic transvections generate the full symplectic group comes down to the (finite) field case, and there it is well known.)

Here is a very different example.

Example 4.4. We begin with the compact connected Lie group G of type E_8; this is a simple group of dimension 248, with trivial center. We are going to describe a finite maximal torus

$$A = \mathbb{Z}/5\mathbb{Z} \times \mu_5 \times \mu_5. \tag{4.4a}$$

The roots will be the 124 nontrivial characters of A, each occurring with multiplicity 2. The group A is described in detail in[7], Lemma 10.3. We present here another description, taken from[2], p. 231.

An element of the maximal torus T of G may be specified by specifying its eigenvalue γ_i (a complex number of absolute value 1) on each of the eight simple roots α_i (the white

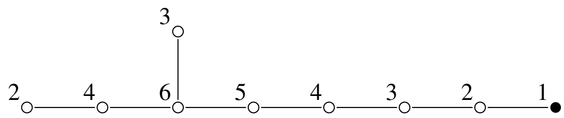

FIGURE 1. extended Dynkin diagram for E_8

vertices in Figure 4.4. Then the eigenvalue γ_0 on the lowest root α_0 (the black vertex) is specified by the requirement

$$\gamma_0^{-1} = \prod_{i=1}^{8} \gamma_i^{n_i}, \tag{4.4b}$$

with n_i the coefficient of α_i in the highest root (the vertex labels in the figure). Equivalently, we require

$$\prod_{i=0}^{8} \gamma_i^{n_i} = 1. \tag{4.4c}$$

There is a map

$$\rho\colon \mu_5 \to T \tag{4.4d}$$

in which the element $\rho(\omega)$ corresponds to the diagram of Figure 4.4. The eight roots

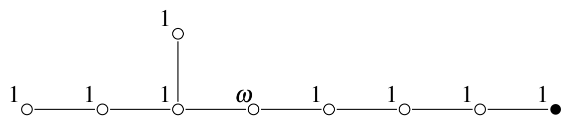

FIGURE 2. Toral subgroup $\mu_5 \subset E_8$

labeled 1 in this diagram are simple roots for a subsystem of type $A_4 \times A_4$. As is explained in[3], page 219, this subsystem corresponds to a subgroup

$$H = (SU(5) \times SU(5))/(\mu_5)_\Delta, \tag{4.4e}$$

the quotient by the diagonal copy of μ_5 in the center.

We have

$$G^{\rho(\mu_5)} = H, \qquad \rho(\mu_5) = Z(H). \tag{4.4f}$$

Because of this, the rest of the calculations we want to do can be performed inside H. It is convenient to label the two $SU(5)$ factors as L and R (for "left" and "right")

We now recall the maps σ, τ, and ζ of Example 4.2. Because 5 is odd, they are actually maps into $SU(5)$ (rather than just $U(5)$). We use subscripts L and R to denote the maps into the two factors of H, so that for example

$$\sigma_L \times \sigma_R\colon \mu_5 \times \mu_5 \to H.$$

Taking the diagonal copies of these maps gives

$$\sigma_\Delta\colon \mu_5 \to SU(5) \times SU(5), \quad \tau_\Delta\colon \mathbb{Z}/5\mathbb{Z} \to SU(5) \times SU(5). \tag{4.4g}$$

The diagonal map ζ_Δ is trivial. According to (4.2m), we have

$$\sigma_\Delta(\omega)\tau_\Delta(m)\sigma_\Delta(\omega^{-1})\tau_\Delta(-m) = \zeta_\Delta(\omega^m) \tag{4.4h}$$

in $SU(5)\times SU(5)$; so in the quotient group H, we get

$$\tau_\Delta \times \sigma_\Delta \times \rho\colon \mathbb{Z}/5\mathbb{Z} \times \mu_5 \times \mu_5 \to H \subset G; \tag{4.4i}$$

the image is our abelian group A of order 125. Because of (4.4f), we have

$$G^A = H^{\tau_\Delta,\sigma_\Delta}, \tag{4.4j}$$

and an easy calculation in $SU(5)\times SU(5)$ (parallel to the one leading to (4.2t)) shows that this is exactly A. So A is indeed a finite maximal torus.

We turn next to calculation of the roots. The character group of A is

$$X^*(A) = \mu_5 \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}.$$

The roots $(\phi, j, 0)$ are those in the centralizer H of $\rho(\mu_5)$; so they are the roots of A in

$$\mathfrak{h} = \mathfrak{sl}(5,\mathbb{C})_L \times \mathfrak{sl}(5,\mathbb{C})_R.$$

These were essentially calculated in Example 4.2. We have

$$\mathfrak{g}_{\phi,j,0} = \langle X^L_{\phi,j}, X^R_{\phi,j}\rangle \qquad ((\phi,j)\neq(1,0).$$

Here the root vectors are the ones defined in (4.2q). In particular, each of these 24 roots has multiplicity two.

To study the root vectors for the 100 roots (ϕ, j, k) with $k \neq 0$ modulo 5, one can analyze the representation of H on $\mathfrak{g}/\mathfrak{h}$, which has dimension 200. We will not do this here; the conclusion is that every root space has dimension two.

Since all 124 characters of A have multiplicity two, it follows from (3.7g) that each root α has exactly 25 coroots; these are all the homomorphisms

$$\xi\colon \mu_5 \to \ker\alpha \subset A.$$

The corresponding transvections

$$s(\alpha,\xi)(\lambda) = \lambda - \langle\lambda,\xi\rangle\alpha \tag{4.4k}$$

are all the transvections moving λ by a multiple of α; the multiple is given by the linear functional ξ which is required only to vanish on α. The (small) Weyl group generated by all of these transvections is therefore

$$W_{\text{small}}(G,A) = SL(A) \simeq SL(3,\mathbb{F}_5), \tag{4.4l}$$

the special linear group over the field with five elements. Its cardinality is

$$|W_{\text{small}}(G,A)| = (5^2+5+1)(5+1)(1)(5^3)(5-1)^2 = 372000.$$

Precisely parallel discussions can be given for $G = F_4$, $A = \mathbb{Z}_3 \times \mu_3 \times \mu_3$, and for $G = G_2$, $A = \mathbb{Z}_2 \times \mu_2 \times \mu_2$. We omit the details. The next example, however is sufficiently different to warrant independent discussion.

Example 4.5. We begin as in Example 4.4 with G a compact connected group of type E_8. We are going to describe a finite maximal torus

$$A = \mathbb{Z}/6\mathbb{Z} \times \mu_6 \times \mu_6. \tag{4.5a}$$

The roots will be the 215 nontrivial characters of A. The 7 characters of order 2 will have multiplicity two; the 26 characters of order 3 will have multiplicity two; and the 182 characters of order 6 will have multiplicity one. To begin, we define

$$\rho\colon \mu_6 \to T \tag{4.5b}$$

so that the element $\rho(\omega)$ corresponds to the diagram of Figure 3. This time the eight roots

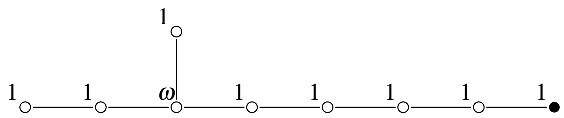

FIGURE 3. Toral subgroup $\mu_6 \subset E_8$

labeled 1 are those for a subsystem of type $A_5 \times A_2 \times A_1$. As we learn in[3], page 220, the corresponding subgroup of G is

$$H = (SU(6) \times SU(3) \times SU(2))/\zeta_\Delta(\mu_6); \tag{4.5c}$$

here

$$\zeta_\Delta \colon \mu_6 \to \mu_6 \times \mu_3 \times \mu_2 = Z(SU(6) \times SU(3) \times SU(2)), \qquad \zeta_\Delta(\omega) = (\omega, \omega^2, \omega^3).$$

Because the centralizer of a single element of a compact simply connected Lie group is connected, we conclude that

$$G^{\rho(\mu_6)} = H, \qquad \rho(\mu_6) = Z(H). \tag{4.5d}$$

Again we want to make use of the maps defined in Example 4.2. The first difficulty is that, because 6 and 2 are even, the maps $\sigma_{SU(2)}$, $\tau_{SU(2)}$, $\sigma_{SU(6)}$, and $\tau_{SU(6)}$ take some of their values in matrices of determinant -1. In order to correct this, we fix a primitive twelfth root γ of 1, and define

$$\begin{aligned} \widetilde{\tau}_{SU(6)} &\colon \mathbb{Z}/12\mathbb{Z} \to SU(6), \quad \widetilde{\tau}_{SU(6)}(m) = \gamma^m \cdot \tau_{U(6)}(2m), \\ \widetilde{\tau}_{SU(3)} &\colon \mathbb{Z}/12\mathbb{Z} \to SU(3), \quad \widetilde{\tau}_{SU(3)}(m) = \tau_{U(3)}(4m), \\ \widetilde{\tau}_{SU(2)} &\colon \mathbb{Z}/12\mathbb{Z} \to SU(2), \quad \widetilde{\tau}_{SU(2)}(m) = \gamma^{3m} \cdot \tau_{U(2)}(6m). \end{aligned} \tag{4.5e}$$

It is easy to check that these three maps are well-defined. If we form the diagonal

$$\widetilde{\tau}_\Delta \colon \mathbb{Z}/12\mathbb{Z} \to SU(6) \times SU(3) \times SU(2), \tag{4.5f}$$

then

$$\widetilde{\tau}_\Delta(6) = (\gamma^6, 1, \gamma^{18}) = (-1, 1, -1) = \zeta_\Delta(-1).$$

The image in H of this element is trivial; so $\widetilde{\tau}_\Delta$ descends to

$$\tau_\Delta \colon \mathbb{Z}/6\mathbb{Z} \to H. \tag{4.5g}$$

In exactly the same way we can define

$$\widetilde{\sigma}_\Delta \colon \mu_{12} \to SU(6) \times SU(3) \times SU(2), \tag{4.5h}$$

descending to

$$\sigma_\Delta \colon \mu_6 \to H. \tag{4.5i}$$

Just as in Example 4.4, we find a group homomorphism

$$\tau_\Delta \times \sigma_\Delta \times \rho \colon \mathbb{Z}/6\mathbb{Z} \times \mu_6 \times \mu_6 \to H \subset G; \tag{4.5j}$$

the image is our abelian group A of order 216. Because of (4.5d), we have

$$G^A = H^{\tau_\Delta, \sigma_\Delta}, \tag{4.5k}$$

and a calculation in $SU(6)\times SU(3)\times SU(2)$ (parallel to the one leading to (4.2t)) shows that this is exactly A. So A is indeed a finite maximal torus.

We turn next to the roots. The character group of A is

$$X^*(A) = \mu_6 \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}. \tag{4.5l}$$

The roots $(\phi, j, 0)$ are those in the centralizer H of $\rho(\mu_6)$; so they are the roots of A in

$$\mathfrak{h} = \mathfrak{sl}(6,\mathbb{C})\times\mathfrak{sl}(3,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C}).$$

These were essentially calculated in Example 4.2. We have

$$\mathfrak{g}_{\phi,j,0} = \left\langle X^{\mathfrak{sl}(6)}_{\phi,j}, X^{\mathfrak{sl}(3)}_{\phi,j}, X^{\mathfrak{sl}(2)}_{\phi,j}\right\rangle \qquad (\phi,j)\neq(1,0). \tag{4.5m}$$

The meaning of the first of these root vectors (defined in (4.2q) for $n=6$) is clear. The second root vector makes sense if both ϕ and j have order 3; that is, if ϕ is the square of a sixth root of 1, and j is twice an integer modulo 6. Similarly, the third root vector makes sense if ϕ and j both have order 2. The second and third root vectors cannot both make sense, for in that case (ϕ, j) would be trivial.

We have therefore shown that, among the 35 roots α vanishing on $\rho(\mu_6)$,

$$\dim\mathfrak{g}_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ has order } 6\\ 2 & \text{if } \alpha \text{ has order } 3\\ 2 & \text{if } \alpha \text{ has order } 2.\end{cases} \tag{4.5n}$$

By analyzing the action of A in the 202-dimensional representation of H on $\mathfrak{g}/\mathfrak{h}$, one can see that the same statements hold for all 215 roots.

We now calculate the coroots. If α is a root of order 6, then (3.6p) says that the number of coroots is

$$2^{\dim\mathfrak{g}_{(6/2)\alpha}}\cdot 3^{\dim\mathfrak{g}_{(6/3)\alpha}} = 2^2\cdot 3^2 = 36;$$

so the coroots are all the 36 homomorphisms

$$\xi\colon \mu_6 \to \ker\alpha \simeq (\mathbb{Z}/6\mathbb{Z})^2. \tag{4.5o}$$

The corresponding root transvections are all the transvections associated to the character α.

If β is a root of order 3, then (3.6p) says that the number of coroots is

$$3^{\dim\mathfrak{g}_{(3/3)\beta}} = 3^2 = 9;$$

so the coroots are all the 9 homomorphisms

$$\xi\colon \mu_3 \to \ker\beta \simeq (\mathbb{Z}/3\mathbb{Z})^2\times(\mathbb{Z}/2\mathbb{Z})^3. \tag{4.5p}$$

The corresponding root transvections are all the transvections associated to the character β.

Similarly, if γ is a root of order 2, there are 4 coroots, and the root transvections are all of the 4 transvections associated to γ.

We see therefore that the (small) Weyl group of A in G contains *all* the transvection automorphisms of $A\simeq(\mathbb{Z}/6\mathbb{Z})^3$; so

$$W_{\text{small}}(G,A)\simeq SL(3,\mathbb{Z}/6\mathbb{Z})\simeq SL(3,\mathbb{Z}/2\mathbb{Z})\times SL(3,\mathbb{Z}/3\mathbb{Z}), \tag{4.5q}$$

a group of order $168\cdot 13392 = 2249856$.

The 27 characters of A of order 3 are exactly the characters trivial on the 8-element subgroup $A[2]$ of elements of order 2 in A; so $G[3] = G^{A[2]}$ has Lie algebra

$$\mathfrak{g}[3] = \sum_{3\beta=0} \mathfrak{g}_\beta. \tag{4.5r}$$

The 26 roots here all have multiplicity 2, so $G[3]$ has dimension 52. It turns out that

$$G[3] \simeq F_4 \times A[2]. \tag{4.5s}$$

Similarly, if we define $G[2] = G^{A[3]}$, then

$$G[2] \simeq G_2 \times A[3]. \tag{4.5t}$$

Now Proposition 3.10 guarantees that $G[2]$ and $G[3]$ commute with each other, so we get a subgroup

$$G[2] \times G[3] \subset G, \qquad G_2 \times F_4 \subset E_8. \tag{4.5u}$$

Perhaps most strikingly

$$W(G,A) = W(G[2],A) \times W(G[3],A); \tag{4.5v}$$

this is just the product decomposition noted in (4.5q).

The construction also shows (since $A[2] \subset G_2$ and $A[3] \subset F_4$) that each of the subgroups G_2 and F_4 is the centralizer of the other in E_8. The existence of these subgroups has been known for a long time (going back at least to[5], Table 39 on page 233; see also[15], pages 62–65); but it is not easy to deduce from the classical theory of root systems alone.

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