Signatures for finite-dimensional representations of real reductive Lie groups

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1 Introduction

Suppose G is a complex connected reductive algebraic group defined over $\mathbb{R}$, and $G(\mathbb{R})$ the group of real points. Suppose that

$$(\pi, V), \qquad \pi \colon G(\mathbb{R}) \to GL(V) \simeq GL(\dim(\pi), \mathbb{C}) \tag{1.1a}$$

is an irreducible finite-dimensional complex representation of $G(\mathbb{R})$. Of course Weyl's dimension formula provides a simple closed formula for $\dim(\pi)$. It often happens that V admits a non-zero $G(\mathbb{R})$-invariant Hermitian form

$$\langle , \rangle_\pi \colon V \times V \to \mathbb{C}, \qquad \langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle. \tag{1.1b}$$

In this case Schur's lemma guarantees that the form is non-degenerate, and unique up to a nonzero real factor. Sylvester's law says that the form has a signature

$$(p(\pi), q(\pi)), \quad p(\pi) + q(\pi) = \dim(\pi), \tag{1.1c}$$

$$\pi \colon G \to U(V, \langle \cdot, \cdot \rangle) \simeq U(p(\pi), q(\pi)). \tag{1.1d}$$

Changing the form by a positive factor does not change $p(\pi)$ and $q(\pi)$, and changing it by a negative factor interchanges them. Therefore both the absolute value of the difference and the unordered pair

$$\mathrm{Sig}(\pi) =_{\mathrm{def}} |p(\pi) - q(\pi)|, \qquad \Sigma(\pi) =_{\mathrm{def}} \{p(\pi), q(\pi)\} \tag{1.1e}$$

are well-defined whenever π is finite-dimensional irreducible, and admits a non-zero invariant form. Because $\dim(\pi)$ is computable, calculating $\Sigma(\pi)$ is equivalent to calculating the non-negative integer $\mathrm{Sig}(\pi)$. That calculation is the main result of this paper (Theorem 5.12), with a formula nearly as easy to calculate as Weyl's dimension formula. Because the general case involves a number of slightly subtle technicalities, we will in this introduction state only a special case.

Theorem 1.2. *Suppose $G = GL(n,\mathbb{R})$, and*

$$\lambda = (\lambda_1, \ldots, \lambda_n)$$

is a decreasing sequence of integers. Write

$$(\pi_{\mathbb{C}}(\lambda), V(\lambda)) = \textit{algebraic representation of } GL(n,\mathbb{C}) \textit{ of highest weight } \lambda.$$

Write

$$n = 2m + \epsilon, \qquad m = [n/2], \qquad \epsilon = 0 \textit{ or } 1.$$

1. *The restriction $\pi(\lambda)$ to $GL(n,\mathbb{R})$ is still irreducible.*
2. *The representation $\pi(\lambda)$ of $GL(n,\mathbb{R})$ admits an invariant Hermitian form if and only if*
$$\lambda = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1);$$
equivalently, if there is a decreasing sequence of nonnegative integers
$$\mu = (\mu_1, \ldots, \mu_m)$$
so that
$$\lambda = \begin{cases} (\mu_1, \ldots, \mu_m, -\mu_m, \ldots, -\mu_1) & (\epsilon = 0) \\ (\mu_1, \ldots, \mu_m, 0, -\mu_m, \ldots, -\mu_1) & (\epsilon = 1) \end{cases}$$
3. *Suppose $\pi(\lambda)$ admits an invariant Hermitian form. Define $\sigma(\mu)$ to be the irreducible representation of* Spin(n) *of highest weight $\mu + (1/2, \ldots, 1/2)$. Then*
$$\mathrm{Sig}(\pi(\lambda)) = \dim(\sigma(\mu))/2^{m-1+\epsilon}.$$

The denominator in the last formula is the dimension of an irreducible (half) spin representation of Spin(n), of highest weight $(1/2, \ldots, 1/2)$. That it always divides the numerator is a classical fact about representations of spin groups. Of course the division is needed to make the formula give the correct signature of $+1$ in case $\lambda = 0$.

This formulation is a bit misleading. The general result Theorem 5.12 involves for $GL(n)$ a rather different representation of Spin(n), of highest weight

$$2\mu + (m - 1 + \epsilon/2, m - 3 + \epsilon/2, \ldots, \epsilon/2).$$

The proof of Theorem 1.2 will then follow by a formal manipulation of the Weyl dimension formula. We carry out the details at the end of Section 5.

Nevertheless we can see in this special case some interesting behavior of the signature. In what follows we use the notation of the theorem, always assuming that

$$\pi(\lambda) = \text{finite-dimensional Hermitian irreducible of } GL(n,\mathbb{R}). \tag{1.3a}$$

Because $SL(n,\mathbb{R})$ is noncompact and simple, it cannot admit nontrivial finite-dimensional unitary representations; that is, there can be no nontrivial homomorphism from $SL(n,\mathbb{R})$ to $U(N)$. Consequently $\mathrm{Sig}(\pi(\lambda)) > 0$ whenever $\lambda \neq 0$. It is not difficult to prove (for example, using the structure of maximal tori in $U(p,q)$) a little more: a nontrivial homomorphism from $SL(n,\mathbb{R})$ to $U(p,q)$ can exist only if $|p-q| \geqslant n-1$. That is,

$$\mathrm{Sig}(\pi(\lambda)) \geqslant n-1 \qquad (\lambda \neq 0).$$

This estimate is the best possible absolute bound, because

$$\mathrm{Sig}(\pi(1,0,\ldots,0,-1)) = n-1$$

either by Theorem 1.2 or by direct calculation of the invariant Hermitian form

$$\langle X, Y\rangle = \mathrm{tr}(X\overline{Y})$$

on the complexified adjoint representation (on $n \times n$ complex matrices of trace zero).

One thing that Theorem 1.2 shows is the "typical" behavior of signatures. The Weyl dimension formula is a polynomial in λ:

$$\deg_\lambda(\dim(\pi(\lambda))) = \left(n^2-n\right)/2 = \binom{n}{2} = 2m^2 + m(2\epsilon - 1). \tag{1.3b}$$

(The number of positive roots for G is $(\dim G - \operatorname{rank} G)/2$.) The signature formula in the theorem is also a polynomial in λ, but now of degree

$$\deg_\lambda(\mathrm{Sig}(\pi(\lambda))) = \left(\binom{n}{2} - [n/2]\right)/2 = m^2 + m(\epsilon - 1) \leqslant \deg_\lambda(\dim)/2. \tag{1.3c}$$

The conclusion is that for "generic" λ,

$$\text{signature grows more slowly than square root of dimension:} \tag{1.3d}$$

the invariant Hermitian form is close to being maximally isotropic. There is a similar statement for any real reductive $G(\mathbb{R})$, with square root replaced by

$$(\dim K - \mathrm{rank}(K))/(\dim G - \mathrm{rank}(G)).$$

Here $K(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$.

For $GL(n,\mathbb{R})$, the formulas are so simple and explicit that we can calculate

$$\dim(\pi(\lambda)) = \mathrm{Sig}(\pi(\lambda))^2 \cdot \prod_{i=1}^{m} \frac{2\lambda_i + n - 2i + 1}{n - 2i + 1} \tag{1.3e}$$

This is a much stronger version of (1.3d). It would be fascinating to find a direct representation-theoretic interpretation of this formula. (A difficulty is that for $n \geqslant 4$, the last product (which is always at least 1) need not be an integer.)

Here is how the rest of the paper is organized. Section 2 recalls the highest weight parametrization of finite-dimensional representations (Proposition 2.6) and the identification of Hermitian representations (Proposition 2.8). Section 3 concerns the structure of the "restricted Weyl group;" it can be omitted in the (very common) case $\operatorname{rank} G = \operatorname{rank} K$. Section 4 calculates the signature of an invariant Hermitian form on extremal weight spaces (Corollary 3.9); this is easy, amounting to a calculation in $SL(2,\mathbb{R})$. Section 5 recalls from [4] and [3, Theorem 4.2] elementary facts (Proposition 5.9) about the eigenvalues of the Dirac operator on finite-dimensional representations. A simple linear algebra result (Lemma 5.11), based on the self-adjointness of the Dirac operator, then implies that the signature of an invariant Hermitian form is essentially equal to the signature on the kernel of the Dirac operator (Corollary 5.10). Finally, we use the result from Section 4 to calculate the signature on the kernel of the Dirac operator, and deduce our main result Theorem 5.12 calculating signatures for finite-dimensional representations of arbitrary real reductive groups.

We thank Jeffrey Adams for pointing out to us the interesting behavior of signatures of forms on finite-dimensional representations. The third author, who is an MIT undergraduate student, embarked on an exploration of this behavior using the `atlas` software from [8] as a summer research project in 2018, under the guidance of the first author, an MIT graduate student. He discovered experimentally the polynomial dependence on λ in Theorem 1.2. The first author found a way to bound signatures from above, which for $GL(n,\mathbb{R})$ gave the formula in Theorem 1.2 as an upper bound for $\operatorname{Sig}(\pi(\lambda))$. (This method of the first author is a version of Lemma 5.11.) At this point the second author, who was old enough to remember [4], was able to join the race at Hereford Street.

2 Weights and Hermitian representations

We continue as in (1.1) with

$$\begin{aligned} G &\quad \text{complex connected reductive algebraic group} \\ \sigma_{\mathbb{R}}\colon G \to G &\quad \text{antiholomorphic involutive automorphism} \\ G(\mathbb{R}) = G^{\sigma_{\mathbb{R}}} &\quad \text{real form of } G. \end{aligned} \tag{2.1a}$$

We will make constant use of a fixed *Cartan involution*

$$\theta\colon G \to G \quad \text{algebraic involutive automorphism;} \tag{2.1b}$$

the characteristic requirement of θ is that the antiholomorphic automorphism

$$\sigma_c =_{\text{def}} \theta \circ \sigma_{\mathbb{R}} \tag{2.1c}$$

is a compact real form of G. Then automatically

$$K =_{\text{def}} G^\theta \tag{2.1d}$$

is a (possibly disconnected) reductive subgroup of G, preserved by $\sigma_{\mathbb{R}}$, and

$$K(\mathbb{R}) = K^{\sigma_{\mathbb{R}}} = K^{\sigma_c} \tag{2.1e}$$

is a maximal compact subgroup of $G(\mathbb{R})$. The *Cartan decomposition* of the Lie algebra is the eigenspace decomposition under θ:

$$\begin{aligned} \mathfrak{s} &=_{\text{def}} -1 \text{ eigenspace of } \theta, \\ \mathfrak{g} &= \mathfrak{k} + \mathfrak{s}, \\ \mathfrak{g}(\mathbb{R}) &= \mathfrak{k}(\mathbb{R}) + \mathfrak{s}(\mathbb{R}). \end{aligned} \tag{2.1f}$$

What is much deeper and more powerful and is the Cartan decomposition of the group:

$$G(\mathbb{R}) = K(\mathbb{R}) \cdot \exp(\mathfrak{s}(\mathbb{R})); \tag{2.1g}$$

the map from right to left is a diffeomorphism.

Every $\sigma_{\mathbb{R}}$-stable maximal torus $H \subset G$ has a $G(\mathbb{R})$-conjugate which is preserved by θ. We therefore consider

$$\begin{aligned} H &\subset G \quad \text{maximal torus} \\ \sigma_{\mathbb{R}}(H) &= H, \qquad \theta(H) = H \\ H(\mathbb{R}) &=_{\text{def}} H^{\sigma_{\mathbb{R}}} \quad \text{real points of } H \\ T &=_{\text{def}} H^\theta = H \cap K \\ T(\mathbb{R}) &=_{\text{def}} H(\mathbb{R})^\theta = H(\mathbb{R}) \cap K(\mathbb{R}) \\ &= \text{ maximal compact subgroup of } H(\mathbb{R}. \end{aligned} \tag{2.2a}$$

Notice that T is a reductive abelian algebraic group, and $T(\mathbb{R})$ its (unique) compact real form. If we define

$$\mathfrak{a} = \mathfrak{h}(\mathbb{R}) \cap \mathfrak{s}, \qquad A = \exp(\mathfrak{a}).$$

then the Cartan decomposition (2.1g) gives a Lie group direct product decomposition

$$H(\mathbb{R}) = T(\mathbb{R}) \times A. \tag{2.2b}$$

The group A is *not* algebraic: if we define $B = H^{-\theta}$ then B is an abelian algebraic group, and

$$A = \text{Lie group identity component of } B(\mathbb{R})$$

The notation in (2.2b) (particularly for A) is very traditional and rather useful (for describing continuous characters of $H(\mathbb{R})$, for example). But the non-algebraic nature of A must always be remembered.

The roots of H in G are complex-valued algebraic (and in particular holomorphic) characters of H:

$$\alpha\colon H \to \mathbb{C}^\times \qquad (\alpha \in R(G,H)). \tag{2.2c}$$

As holomorphic characters, the roots are determined by their restrictions to $H(\mathbb{R})$, or the differentials of those restrictions:

$$\alpha_{\mathbb{R}}\colon H(\mathbb{R}) \to \mathbb{C}^\times, \qquad d\alpha_{\mathbb{R}}\colon \mathfrak{h}(\mathbb{R}) \to \mathbb{C}. \tag{2.2d}$$

We will often write just α for either $\alpha_{\mathbb{R}}$ or its differential, relying on the context to avoid ambiguity. But for the structural results we are now describing, it is helpful to maintain an explicit distinction. In accordance with tradition, we will write the group structure on roots as $+$, even though it corresponds to multiplication of characters of H.

Because the automorphism θ is assumed to preserve H, it automatically acts on the roots. A moment's thought shows that $\sigma_{\mathbb{R}}$ also permutes the root spaces, and therefore acts on the roots by the requirement

$$[\sigma_{\mathbb{R}}(\alpha)](h) =_{\text{def}} \overline{\alpha(\sigma_{\mathbb{R}}^{-1}(h))}.$$

These two actions are related by

$$\theta(\alpha) = \sigma_{\mathbb{R}}(-\alpha). \tag{2.2e}$$

The root α is called *real* if $d\alpha_{\mathbb{R}}$ is real valued (equivalently, if $\alpha_{\mathbb{R}}$ is real-valued). Because of (2.2e),

$$\alpha \text{ is real} \iff \sigma_{\mathbb{R}}(\alpha) = \alpha \iff \theta(\alpha) = -\alpha. \tag{2.2f}$$

In case α is real, the root subgroup

$$\phi_\alpha\colon SL(2) \to G$$

may be chosen to be defined over $\mathbb{R}$ with the standard real form of $SL(2)$:

$$\phi_\alpha\colon SL(2,\mathbb{R}) \to G(\mathbb{R}) \tag{2.2g}$$

The root β is called *imaginary* if $d\beta_{\mathbb{R}}$ is imaginary-valued (equivalently, if $\beta_{\mathbb{R}}$ takes values in the unit circle). Because of (2.2e),

$$\beta \text{ is imaginary} \iff \sigma_{\mathbb{R}}(\beta) = -\beta \iff \theta(\beta) = \beta. \tag{2.2h}$$

In case β is imaginary, the root subgroup ϕ_β is defined over $\mathbb{R}$, but with one of two different real forms of $SL(2)$. In case

$$\phi_\beta\colon SU(1,1) \to G(\mathbb{R}), \tag{2.2i}$$

we say that β is *noncompact imaginary*. In case

$$\phi_\beta\colon SU(2) \to G(\mathbb{R}), \tag{2.2j}$$

we say that β is *compact imaginary*.

Finally, the root γ is called *complex* if $d\gamma_{\mathbb{R}}$ is neither real nor purely imaginary valued (equivalently, if $\gamma_{\mathbb{R}}$ takes non-real values of absolute value not equal to 1). Because of (2.2e),

$$\gamma \text{ is complex} \iff \sigma_{\mathbb{R}}(\gamma) \neq \pm\gamma \iff \theta(\gamma) \neq \pm\gamma. \tag{2.2k}$$

It is equivalent to require that the root subgroup

$$\phi_\gamma \colon SL(2) \to G$$

is not defined over $\mathbb{R}$ for any real structure on $SL(2)$.

There are (up to conjugation by $K(\mathbb{R})$) two maximal tori of particular interest to us. First is the *maximally split torus*

$$H_s(\mathbb{R}) = T_s(\mathbb{R}) \cdot A_s. \tag{2.3a}$$

This torus is characterized by the three equivalent requirements

$$\begin{aligned} &\dim_{\mathbb{R}} A_s \text{ is as large as possible} \\ &\dim_{\mathbb{R}} T_s(\mathbb{R}) \text{ is as small as possible} \\ &\text{there are no noncompact imaginary roots of } H_s \text{ in } G. \end{aligned} \tag{2.3b}$$

Inside the root system $R(G, H_s)$ we can find a set of positive roots $R_s^+ = R^+(G, H_s)$ satisfying

$$\text{the nonimaginary roots in } R_s^+ \text{ are preserved by } -\theta. \tag{2.3c}$$

There is a unique Weyl group element specified by the requirement

$$w_{0,s}(R^+(G, H_s) = \theta R^+(G, H_s);$$

it commutes with θ (as an automorphism of H), and so acts on $T(\mathbb{R})$ and A.

Next, the *maximally compact torus* (sometimes called the *fundamental torus*)

$$H_c(\mathbb{R}) = T_c(\mathbb{R}) \cdot A_c. \tag{2.4a}$$

This torus is characterized by the four equivalent requirements

$$\begin{aligned} &\dim_{\mathbb{R}} A_c \text{ is as small as possible} \\ &\dim_{\mathbb{R}} T_c(\mathbb{R}) \text{ is as large as possible} \\ &T_c(\mathbb{R})_0 \text{ is a maximal torus in } K(\mathbb{R})_0 \\ &\text{there are no real roots of } H_c \text{ in } G. \end{aligned} \tag{2.4b}$$

Inside the root system $R(G, H_c)$ we can find a set of positive roots $R_c^+ = R^+(G, H_c)$ satisfying

$$R_c^+ \text{ is preserved by } \theta. \tag{2.4c}$$

Our next goal is to recall the parametrization (due to Cartan and Weyl) of finite-dimensional representations of $G(\mathbb{R})$ by highest weights. In order to do that, we need two more bits of notation. For each root α, recall that the *coroot* $\alpha^\vee$ is the restriction to the maximal torus of the root $SL(2)$:

$$\alpha^\vee : \mathbb{C}^\times \to H, \quad \alpha^\vee(z) = \phi_\alpha \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}. \tag{2.5a}$$

The homomorphism ϕ_α is unique only up to conjugation by diagonal matrices in $SL(2)$, but $\alpha^\vee$ is (therefore) absolutely unique. The homomorphism $\alpha^\vee$ is specified by its differential

$$H_\alpha =_{\text{def}} d\phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}. \tag{2.5b}$$

If α is *real*, so that ϕ_α is defined over $\mathbb{R}$, then we can define

$$m_\alpha =_{\text{def}} \phi_\alpha \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \alpha^\vee(-1) = \exp(i\pi H_\alpha) \in H(\mathbb{R}), \tag{2.5c}$$

an element of order (one or) two in the real Cartan subgroup $H(\mathbb{R})$.

A character

$$\begin{aligned} \gamma\colon H(\mathbb{R}) &\to \mathbb{C}^\times \text{continuous} \\ d\gamma(\mathbb{R})\colon \mathfrak{h}(\mathbb{R}) &\to \mathbb{C} \qquad \text{real linear} \\ d\gamma\colon \mathfrak{h} &\to \mathbb{C} \qquad \text{complex linear} \end{aligned} \tag{2.5d}$$

is called *weakly integral* if

$$d\gamma(H_\alpha) \in \mathbb{Z} \qquad (\alpha \in R(G, H)). \tag{2.5e}$$

It is called *strongly integral* if it is integral, and also

$$\gamma(m_\alpha) = (-1)^{d\gamma(H_\alpha)} \qquad (\alpha \in R(G, H) \text{ real}). \tag{2.5f}$$

Proposition 2.6. *Suppose G is a reductive algebraic group as in* (2.1)*, and H is a real θ-stable maximal torus as in* (2.2)*.*

1. *Every irreducible finite-dimensional representation of $G(\mathbb{R})$ remains irreducible on restriction to the identity component, and so defines an irreducible finite-dimensional complex representation of the complex reductive Lie algebra $\mathfrak{g}$.*
2. *The H-weights of finite-dimensional representations of $G(\mathbb{R})$ are precisely the strongly integral characters of $H(\mathbb{R})$ (see* (2.5f)*).*
3. *If γ is a strongly integral character of $H(\mathbb{R})$, then there is a finite-dimensional representation $F(\gamma)$ of $G(\mathbb{R})$ having extremal weight γ.*
4. *If $H = H_s$ is maximally split, then the representation $F(\gamma)$ is uniquely determined.*

Most of this is proven in [6, Section 0.4].

Corollary 2.7. *In the setting of Proposition 2.6, define*

$$G(\mathbb{R})^{[H]} = G(\mathbb{R})_0 \cdot H(\mathbb{R}),$$

the subgroup of $G(\mathbb{R})$ *generated by the identity component and the fixed maximal torus. Put*

$$\pi_0^H(G(\mathbb{R})) = G(\mathbb{R})/G(\mathbb{R})^{[H]},$$

the quotient of the component group by the image of the component group of $H(\mathbb{R})$*. Define*

$$G(\mathbb{R})^\sharp = \{g \in G(\mathbb{R}) \mid \mathrm{Ad}(g) \in \mathrm{Ad}(G(\mathbb{R})_0\} \supset G_0(\mathbb{R}),$$

$$\pi_0^\sharp(G(\mathbb{R})) = G(\mathbb{R})/G(\mathbb{R})^\sharp.$$

1. *Each group* $G(\mathbb{R})^{[H]}$ *contains* $G(\mathbb{R})^\sharp$*, with equality for the maximally compact Cartan* $H = H_c$ *of* (2.4a)*.*

2. *Each group* π_0^H *is a quotient of* $\pi_0^\sharp$*, which is a finite product of copies of* $\mathbb{Z}/2\mathbb{Z}$*.*

3. *Suppose* γ *is a strongly integral character of* $H(\mathbb{R})$*. Then there is a simply transitive action of the character group of* π_0^H *on the set of finite-dimensional irreducible representations of* $G(\mathbb{R})$ *having extremal weight* γ*. The action is given by tensoring with the irreducible characters of* $G(\mathbb{R})/G(\mathbb{R})^{[H]}$*.*

Proof. For (1), suppose $g \in G(\mathbb{R})^\sharp$. Choose (according to the definition of $G(\mathbb{R})^\sharp$) $g_0 \in G(\mathbb{R})_0$ so that $\mathrm{Ad}(g) = \mathrm{Ad}(g_0)$. This means in particular that $g_0^{-1}g \in Z(G(\mathbb{R})) \subset H(\mathbb{R})$, which is the first assertion of (1). The last assertion we will address in Section 4 after we have discussed restricted roots.

The first assertion in (2) is an immediate consequence of (1). The second we will prove in Proposition 3.4(6) below.

For (3), Proposition 2.6 guarantees that there is an irreducible finite-dimensional $F(\gamma)$ of extremal weight γ, and that $F(\gamma)$ remains irreducible for $G(\mathbb{R})_0$. The rest of (3) is a formal consequence. □

We turn next to the calculation of Hermitian duals.

Proposition 2.8. *Suppose again that* G *is a reductive algebraic group as in* (2.1)*, and* H *is a real* θ*-stable maximal torus as in* (2.2)*. We use the decomposition*

$$H(\mathbb{R}) = T(\mathbb{R}) \times A$$

of (2.2b)*. Write*

$$\begin{aligned} X^*(T) &= \{\textit{continuous characters } \lambda_{\mathbb{R}}\colon T(\mathbb{R}) \to S^1\} \\ &\simeq \{\textit{algebraic characters } \lambda\colon T \to \mathbb{C}^\times\} \\ &\simeq X^*(H)/(1-\theta)X^*(H) \end{aligned}$$

for the characters of the compact group $T(\mathbb{R})$. We identify characters of the vector group $A = \exp(\mathfrak{a})$ with the complex dual space

$$\mathfrak{a}_{\mathbb{C}}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{C}),$$

sending $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ to the character

$$\exp(X) \mapsto \exp(\nu(X)) \qquad (X \in \mathfrak{a})$$

or equivalently

$$a \mapsto a^{\nu} \qquad (a \in A).$$

1. *Characters of $H(\mathbb{R})$ may be indexed by pairs*

$$\gamma = (\lambda, \nu) \in X^*(T) \times \mathfrak{a}_{\mathbb{C}}^*.$$

2. *The differential of such a character γ is*

$$d\gamma = (d\lambda, \nu) \in i\mathfrak{t}(\mathbb{R})^* \times \mathfrak{a}_{\mathbb{C}}^*.$$

3. *The Hermitian dual of γ is*

$$\gamma^h = (\lambda, -\overline{\nu}).$$

4. *If γ is strongly integral, then the Hermitian dual of a finite-dimensional representation $F(\gamma)$ (of extremal weight γ) is a finite-dimensional representation $F(\gamma^h)$ (of extremal weight γ^h).*

5. *Suppose γ^h is conjugate by W to γ, so that $F(\gamma)^h$ also has extremal weight γ. If π_0^H is trivial (see Corollary 2.7) (in particular, if $G(\mathbb{R})$ is connected) then the (uniquely determined) $F(\gamma)$ must admit an invariant Hermitian form. If π_0^H is not trivial, then either all or none of the $|\pi_0^H|$ choices for $F(\gamma)$ admits an invariant Hermitian form.*

6. *Suppose H_s is maximally split as in* (2.3), *and (λ_s, ν_s) is a strongly integral R_s^+-dominant weight. Then*

$$F(\lambda_s, \nu_s)^h = F(w_{0,s} \cdot \lambda_s, -w_{0,s} \cdot (\overline{\nu_s})).$$

   In particular, there is a nonzero invariant Hermitian form if and only if

$$\nu_s = -w_{0,s} \cdot (\overline{\nu_s}), \quad w_{0,s} \cdot \lambda_s = \lambda_s.$$

7. *Suppose H_c is maximally compact as in* (2.4), *and (λ_c, ν_c) is a strongly integral R_c^+-dominant weight. Then*

$$F(\lambda_c, \nu_c)^h = F(\lambda_c, -\overline{\nu_c}).$$

   In particular, there is a nonzero invariant Hermitian form only if

$$\nu_c \text{ is purely imaginary.}$$

Describing sufficient conditions for the existence of a form using the maximally compact Cartan H_c is complicated; we will address this in Corollary 4.3.

Sketch of Proof. The identification of algebraic characters of T with continuous characters of the compact real form $T(\mathbb{R})$ is a feature of any reductive algebraic group. Parts (1)–(3) are immediate. The Hermitian dual of a direct sum is the direct sum of the Hermitian duals, so (4) follows. In case $H = H_s$, the extremal weight $(\lambda_s, -\overline{\nu_s})$ is evidently dominant for the positive system $w_{0,s} \cdot R^+(G, H_s)$, and (5) follows.

For (6), the only difficulty is that $F(\lambda_c, \nu_c)$ is not unique: we only know that $F(\lambda_c, \nu_c)^h$ is *some* representation of highest weight $(\lambda_c, -\overline{\nu_c})$. This proves the necessity of the condition in (6) (for existence of an invariant form). The hypothesis for the last assertion in (6) amounts to

$$G(\mathbb{R}) = G(\mathbb{R})_0 T_c(\mathbb{R}),$$

which implies that the representation $F(\lambda_c, \nu_c)$ is unique. □

3 Restricted Weyl group

Our goal is to study invariant Hermitian forms on extremal weight spaces with respect to a maximally compact Cartan

$$H_c = T_c(\mathbb{R}) \cdot A_c$$

as in (2.4). In order to do that, we first need to understand how the roots and Weyl group of H_c restrict to T_c; that is the subject of this section. Fix a θ-stable system of positive roots

$$R_c^+ \subset R(G, H_c).$$

Write

$$W = W(G, H_c) = W(R(G, H_c)) \subset \mathrm{Aut}(H_c) \tag{3.1a}$$

for the Weyl group of H_c in G. We are interested in several subgroups of W, including

$$\begin{aligned} W^\theta &= \text{centralizer of } \theta \text{ in } W \\ W_{\mathrm{imag}} &= W(R_{c,\mathrm{imag}}) \subset W^\theta \qquad \text{Weyl group of imaginary roots} \\ W_K &= N_K(H)/(K \cap H) \simeq N_K(T)/T \subset W^\theta \\ W_{K_0} &= N_{K_0}(T_0)/T_0 \subset W_K \qquad \text{compact Weyl group} \end{aligned} \tag{3.1b}$$

The reason we do not call W_K the "compact Weyl group" is that it need not be the Weyl group of a root system.

The first important fact about the maximally compact Cartan is that *no root is trivial on $T_{c,0}$*. The reason is that (for any θ-stable real Cartan) the roots vanishing on $\mathfrak{t}$ are exactly the real roots (see (2.2f)); and on H_c there are no real roots (see (2.4)). We can therefore introduce the *restricted roots*

$$\begin{aligned} R_{\mathrm{res}}(G, T_{c,0}) &= \{\overline{\alpha} = \alpha|_{T_{c,0}} \mid \alpha \in R(G, H_c)\} \\ &\subset X^*(T_0) = X^*(H_c)/X^*(H_c)^{-\theta}. \end{aligned} \tag{3.1c}$$

The dual lattice to $X^*(H_c)/X^*(H_c)^{-\theta}$ is

$$X_*(T_{c,0}) = X_*(H_c)^{\theta} \tag{3.1d}$$

The *restricted coroots* are by definition

$$\overline{\alpha}^\vee = \begin{cases} \alpha^\vee & \alpha = \theta\alpha \text{ imaginary} \\ \alpha^\vee + \theta\alpha^\vee & \alpha \text{ complex, } \alpha + \theta\alpha \text{ not a root} \\ 2(\alpha^\vee + \theta\alpha^\vee) & \alpha \text{ complex, } \alpha + \theta\alpha \text{ a root.} \end{cases} \tag{3.1e}$$

Proposition 3.2. *The restricted roots and coroots form a root datum*

$$\mathcal{R}_{\mathrm{res}} = (X^*(T_{c,0}), R_{\mathrm{res}}, X_*(T_{c,0}), R^\vee_{\mathrm{res}})$$

in the torus $T_{c,0}$. This root datum is not reduced when the third case for coroots arises. Restriction to $T_{c,0}$ defines an isomorphism

$$W^\theta|_{T_{c,0}} = W(\mathcal{R}_{\mathrm{res}}).$$

Inside this root datum are several smaller root data.

1. *The* reduced restricted root datum*, written $\mathcal{R}_{\mathrm{res,red}}$, consisting of the restricted roots $\overline{\alpha}$ so that $2\overline{\alpha}$ is not a restricted root; equivalently, those falling in cases (1) and (2) of* (3.1e)*. This subsystem is preserved by W^θ, and has the same Weyl group:*

$$W(\mathcal{R}_{\mathrm{res,red}}) = W(\mathcal{R}_{\mathrm{res}}).$$

2. *The* complex subsystem*, written $\mathcal{R}_{\mathrm{res,cplx}}$, consisting of the restrictions to $T_{c,0}$ of the complex roots and the corresponding coroots. This subsystem is preserved by the action of W^θ, and so defines a normal subgroup*

$$W_{\mathrm{cplx}} \lhd W^\theta.$$

3. *The* imaginary subsystem*, written $\mathcal{R}_{\mathrm{res,imag}}$, consisting of the restrictions to $T_{c,0}$ of the imaginary roots and the corresponding coroots. This subsystem is preserved by the action of W^θ, and so defines a normal subgroup*

$$W_{\mathrm{imag}} \lhd W^\theta.$$

The imaginary roots have a $\mathbb{Z}/2\mathbb{Z}$ grading in which the compact imaginary roots are even and the noncompact imaginary roots are odd (cf. (2.2h)*). This grading is respected by W_{cplx}, but not usually by W_{imag}.*

4. *The* root datum for K, *written* $\mathcal{R}_K$. *Its roots are the disjoint union of the complex roots and the compact imaginary roots:*

$$\mathcal{R}_K = \mathcal{R}_{\rm cplx} \amalg \mathcal{R}_{\rm imag,cpt}.$$

Sketch of Proof. That the restricted roots are a root system is classical. The term "restricted roots" most often refers to restriction to the split part of a maximally split torus. The fact those restrictions constitute a root system is proved in [2, Section VII.2]. Helgason's arguments can be applied (with substantial simplifications) to show that $\mathcal{R}_{\rm res}$ is a root datum.

Another classical fact is that $\mathfrak{t}_c$ contains regular elements, so that no element of W can fix all elements of $\mathfrak{t}_c$. This proves that restriction to $T_{c,0}$ is an injective group homomorphism on W^θ. That the image contains $W(\mathcal{R}_{\rm res})$ follows from

$$s_{\overline{\alpha}} = \begin{cases} s_\alpha|_{T_{c,0}} & \alpha \text{ imaginary} \\ (s_\alpha s_{\theta\alpha})|_{T_{c,0}} & \alpha \text{ complex, } \alpha+\theta\alpha \text{ not a root} \\ s_{\alpha+\theta\alpha}|_{T_{c,0}} & \alpha+\theta\alpha \text{ a root.} \end{cases}$$

(Only the second assertion requires thought, and it is very easy.)

That W^θ is generated by elements of these three kinds is due perhaps to Knapp; a proof may be found in [7, Proposition 3.12].

For (1), the complex roots are precisely those having a non-trivial restriction to the -1 eigenspace $\mathfrak{a}$ of θ. That W^θ preserves these roots is obvious. In particular, the reflections in complex restricted roots preserve complex roots. This last fact is the main part of the proof that the complex roots are a root datum.

Part (2) is exactly parallel, except that this time the condition is trivial restriction to $\mathfrak{a}$. The grading was already explained after (2.2h); that it is preserved by compact imaginary reflections is clear. We postpone for a moment the assertion that it is preserved by complex reflections.

Because K_0 is a reductive algebraic group with maximal torus $T_{c,0}$, we have the root datum of K_0 in $X^*(T_{c,0})$ and $X_*(T_{c,0})$. Evidently this includes the compact imaginary roots. From each complex root β with root vector X_β we get a root vector

$$X_\beta + \theta X_\beta \in \mathfrak{k}$$

for $\overline{\beta}$; so the complex roots are automatically roots for K_0. This proves (3).

Because the complex root reflections have representatives in K, they must preserve the compact/noncompact grading on the imaginary roots. This completes the proof of (2). □

Recall that we have fixed a θ-stable set of positive roots R_c^+; this defines automatically a positive root system $R_{\rm res}^+$ for the restricted roots, and also for the complex, imaginary, and compact roots. Write Γ_c for the Dynkin diagram of R_c^+ (a graph with a vertex for each simple α and an edge labelled r from α to β whenever $\alpha + r\beta$ is a root). Then θ defines an automorphism of Γ_c. The

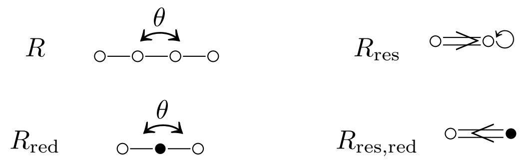

Table 1: Restricted and reduced roots for $SL(5, R)$

Dynkin diagram Γ_{res} for the restricted roots has as vertex set the orbits of θ on Γ_c. A fixed point on Γ_c corresponds to an imaginary simple root in Γ_{res}; an orbit consisting of two non-adjacent simple roots α and $\theta(\alpha)$ corresponds to a complex simple root $\overline{\alpha}$ in the second case of (3.1e); and an orbit consisting of two adjacent simple roots α and $\theta(\alpha)$ corresponds to a non-reduced complex simple root $\overline{\alpha}$ in the third case of (3.1e). (Such a vertex $\overline{\alpha}$ is joined to itself in the restricted Dynkin diagram Γ_{res} since $2\overline{\alpha}$ is the (imaginary) root $\alpha + \theta(\alpha)$. See for example the top right diagram in Table 1.)

The reduced restricted roots are the restrictions of roots which involve either both or neither of a pair $(\alpha, \theta\alpha)$ of adjacent simple roots. Such roots in R are themselves a θ-stable subsystem R_{red}. The simple roots of R_{red} are those of R, except that each adjacent complex pair $(\alpha, \theta\alpha)$ is replaced by the single imaginary simple root $\alpha + \theta\alpha$.

This process is illustrated for $SL(5, \mathbb{R})$ in Table 1. The Dynkin diagram in the upper left is for R, showing the action of θ reversing the line. The diagram on the upper right is for the restricted roots, obtained by folding the diagram on the left in half. The diagram on the lower left eliminates the complex roots for which $2\overline{\alpha}$ is a root, by replacing the two middle roots by their sum. On the lower right are the restricted reduced roots: the complex restricted root $\overline{\alpha}$ has been replaced by an imaginary (restricted) root $\alpha + \theta\alpha$. In each diagram imaginary vertices are indicated with a filled circle, and complex vertices with an empty circle.

We offer one more example, the restricted root system for the split real form of E_6. In this case the Cartan involution θ interchanges the long legs of the Dynkin diagram. There are no adjacent pairs $(\alpha, \theta\alpha)$, so the restricted root system is already reduced. Its Dynkin diagram is obtained by folding together the long legs of the E_6 diagram, obtaining a diagram of type F_4. Again imaginary vertices are illustrated with a filled circle, and complex vertices with an empty circle.

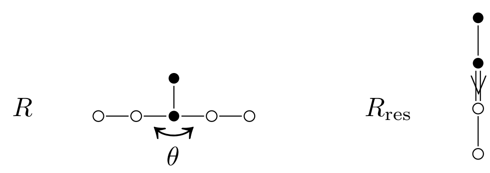

Table 2: Restricted roots for the split real form of E_6

Returning to general G, define

$$\begin{aligned} 2\rho_{\text{cplx}} &= \sum_{\alpha \in R^+_{\text{cplx}}} \alpha \in X^*(T_{c,0}) \\ 2\rho_{\text{imag}} &= \sum_{\beta \in R^+_{\text{imag}}} \beta \in X^*(T_{c,0}) \\ 2\rho_K &= \sum_{\gamma \in R^+_K} \gamma \in X^*(T_{c,0}). \end{aligned} \tag{3.3a}$$

Define the *singular imaginary roots* by

$$R^{\text{sing}}_{\text{imag}} = \{\delta \in R_{\text{res}} \mid \langle 2\rho_{\text{cplx}}, \delta^\vee \rangle = 0\}, \tag{3.3b}$$

the *singular complex roots* by

$$R^{\text{sing}}_{\text{cplx}} = \{\delta \in R_{\text{res}} \mid \langle 2\rho_{\text{imag}}, \delta^\vee \rangle = 0\}, \tag{3.3c}$$

and the *singular noncompact roots* by

$$R^{\text{sing}}_{\text{ncpt}} = \{\delta \in R_{\text{res}} \mid \langle 2\rho_K, \delta^\vee \rangle = 0\}. \tag{3.3d}$$

Using these root systems, we can begin to understand the restricted Weyl group $W^\theta = W_{\text{res}}$.

Proposition 3.4. *We use the notation of Proposition 3.2.*

1. *The weight $2\rho_{\text{cplx}}$ is dominant for R^+_{res} and regular for the complex roots. Therefore the singular imaginary roots form a Levi subsystem in R_{res}, consisting entirely of imaginary roots. We get a semidirect product decomposition*
$$W^\theta = W_{\text{cplx}} \rtimes W^{\text{sing}}_{\text{imag}}.$$

2. *The weight $2\rho_{\text{imag}}$ is dominant for R^+_{res} and regular for the imaginary roots. Therefore the singular complex roots form a Levi subsystem in R_{res}, consisting entirely of complex roots. We get a semidirect product decomposition*
$$W^\theta = W^{\text{sing}}_{\text{cplx}} \ltimes W_{\text{imag}}.$$

For the last items, we modify our θ-stable choice of positive roots R_c^+ to a new choice

$$R_c^{+,K} \quad \textit{making } 2\rho_K \textit{ dominant.}$$

3. *The weight $2\rho_K$ is dominant for $R_{\mathrm{res}}^{+,K}$ and regular for the roots of K. Therefore the singular noncompact roots form a Levi subsystem with respect to $R_{\mathrm{res}}^{+,K}$ consisting entirely of noncompact imaginary roots*

$$\{\pm\beta_1, \dots, \pm\beta_r\}.$$

This root system is of type A_1^r, so has Weyl group

$$W_{\mathrm{ncpt}}^{\mathrm{sing}} = (\mathbb{Z}/2\mathbb{Z})^r.$$

For $1 \leqslant j \leqslant r$, choose a root $SL(2)$

$$\phi_{\beta_j} \colon SU(1,1) \to G(\mathbb{R}), \qquad \phi_{\beta_j}\left(\mathrm{Ad}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(g)\right) = \theta(\phi_{\beta_j}(g))$$

as in (2.2h). *Define*

$$\sigma_j = \phi_{\beta_j}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

a representative in $N_G(H_c)$ for the simple reflection s_{β_j}, and

$$m_j = \phi_{\beta_j}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in T_{c,0}(\mathbb{R}) \subset K(\mathbb{R}).$$

For $B \subset \{1, \dots, r\}$, define

$$H_B = \sum_{i \in B} \beta_j^\vee \in X_*(H_c), \qquad \sigma_B = \prod_{j \in B} \sigma_j$$

$$s_B = \prod_{j \in B} s_j \in W_{\mathrm{ncpt}}^{\mathrm{sing}} \qquad m_B = \prod_{j \in B} m_j = \exp(2\pi i H_B/2) = \sigma_B^2.$$

Then

$$\theta(\sigma_B) = \sigma_B^{-1} = m_B \sigma_B.$$

4. *The Weyl group element s_B admits a representative in K if and only if there is a coweight $\ell_B \in X_*(H_c)$ satisfying*

$$\ell_B + \theta(\ell_B) = H_B.$$

In this case the representative may be taken to be

$$\widetilde{s}_B = \exp(\pi i \ell_B)\sigma_B.$$

5. *Define*

$$W^{\text{sing}}_{\text{ncpt}}(K) = \{s_B \in (\mathbb{Z}/2\mathbb{Z})^r \mid H_B \in (1+\theta)X_*(H_c)\} \qquad (B \subset \{1,\ldots,r\}).$$

Then there is a semidirect product decomposition

$$W_K = W_{K_0} \rtimes W^{\text{sing}}_{\text{ncpt}}(K),$$

the second factor being an abelian group with every element of order one or two.

6. *Define*

$$K^\sharp = \{k \in K \mid \mathrm{Ad}_G(k) \in \mathrm{Ad}_G(K_0)\} = K_0 T_c$$

(cf. Corollary 2.7; $K^\sharp(\mathbb{R}) = G(\mathbb{R})^\sharp \cap K$). Then

$$G(\mathbb{R})/G(\mathbb{R})^\sharp \simeq K(\mathbb{R})/K^\sharp(\mathbb{R}) \simeq K/K^\sharp \simeq W^{\text{sing}}_{\text{ncpt}}(K).$$

Sketch of Proof. We recommend examining Table 3 to get a more concrete picture of the constructions in the proposition.

Part (2) is [7, Proposition 3.12(c)]; part (1) can be proven in exactly the same way. For (3), the dominance of $2\rho_K$ comes from the choice of positive roots, and the regularity for K is a general fact about positive root sums in a root datum. This implies that the singular noncompact roots are roots in a Levi factor for the restricted root datum, and are all noncompact imaginary. In particular, the sum of two distinct singular noncompact roots cannot be a root; for if it were root, the grading would necessarily make it even, and so compact, and therefore not singular.

The absence of root sums shows that the noncompact singular system consists of orthogonal simple roots, and is therefore of type A_1^r. The assertions before (4) all take place in $SU(1,1)^r$, where they are easy computations.

For (4), any representative of s_B is of the form

$$\widetilde{s}_B = h\sigma_B, \qquad \text{some } h = \exp(i\pi\ell) \in H_c. \tag{3.5a}$$

Therefore

$$\theta\widetilde{s}_B = \theta(h)m_B\sigma_B,$$

and $\widetilde{s}_B$ belongs to K if and only if

$$h\sigma_B = \theta(h)m_B\sigma_B, \qquad h\theta(h)^{-1} = m_B,$$

or equivalently

$$\exp(i\pi(\ell - \theta\ell)) = \exp(-i\pi H_B). \tag{3.5b}$$

The kernel of $\exp(2\pi i)$ on $\mathfrak{h}_c$ is $X_*(H_c)$, so the conclusion is that there must be an element $\ell_B \in X_*(H_c)$ satisfying

$$(\ell - \theta\ell)/2 + H_B/2 = \ell_B.$$

R	R_{res}	R_{cplx}	R_{imag}	$R^{\text{sing}}_{\text{cplx}}$	$R^{\text{sing}}_{\text{imag}}$	$W_{\text{cplx}} \rtimes W^{\text{sing}}_{\text{imag}}$	$W^{\text{sing}}_{\text{cplx}} \ltimes W_{\text{imag}}$
A_{2n-1}	C_n	D_n	A_1^n	A_{n-1}	A_1	$W(D_n) \rtimes \{\pm 1\}$	$S_n \ltimes \{\pm 1\}^n$
A_{2n}	BC_n	B_n	A_1^n	A_{n-1}	$\varnothing$	$W(B_n) \rtimes 1$	$S_n \ltimes \{\pm 1\}^n$
D_{n+1}	B_n	A_1^n	D_n	A_1	A_{n-1}	$\{\pm 1\}^n \rtimes S_n$	$\{\pm 1\} \ltimes W(D_n)$
E_6	F_4	D_4	D_4	A_2	A_2	$W(D_4) \rtimes S_3$	$S_3 \ltimes W(D_4)$

Table 3: **Restricted root systems**

Because H_B is in the $+1$ eigenspace of θ and $\ell - \theta(\ell)$ in the -1 eigenspace, this equation is equivalent to two equations

$$H_B = \ell_B + \theta(\ell_B), \quad (\ell - \theta\ell)/2 = (\ell_B - \theta(\ell_B))/2. \tag{3.5c}$$

So the existence of $\widetilde{s}_B$ guarantees the existence of ℓ_B as the Proposition requires. Conversely, given ℓ_B as in the proposition, choosing $\ell = \ell_B$ makes (3.5c) true, proving that

$$\widetilde{s}_B = \exp(i\pi\ell_B)\sigma_B \tag{3.5d}$$

is a representative for s_B in K.

For (5), suppose $w \in W_K$. Clearly $w(R_K^+)$ is another positive root system for R_K, so there is a unique $w_1 \in W(R_K) = W_{K_0}$ satisfying

$$w(R_K^+) = w_1(R_K^+), \qquad w(2\rho_K) = w_1(2\rho_K). \tag{3.5e}$$

Therefore $w_2 = w_1^{-1}w$ fixes $2\rho_K$. By Chevalley's theorem, w_2 is a product of reflections fixing $2\rho_K$; that is, $w_2 \in W^{\text{sing}}_{\text{ncpt}}$. Now (5) follows.

Part (6) is elementary. □

If θ acts trivially on the roots in H_c, then all roots are imaginary, and there is not much content to Proposition 3.4(1)–(2). If θ interchanges two simple factors $R_L \simeq R_0$ and $R_R \simeq R_0$ of the root system, then all the roots are complex, and W^θ is the diagonal copy of $W(R_0)$. The remaining and most interesting (indecomposable) possibility is that θ acts as a nontrivial automorphism of order 2 of a simple root system R. There is up to isomorphism exactly one such automorphism for the simple root systems of types A_n ($n \geqslant 2$), D_n ($n \geqslant 4$), and E_6, and none for the other simple systems. Table 3 lists the restricted root systems in each case, and some of the other root systems described in Proposition 3.2. In each case the last two columns give two semidirect product decompositions of $W_{\text{res}} = W^\theta$ from Proposition 3.4.

One can give a similarly exhaustive enumeration of the results of Proposition 3.4(3–6), but the details are substantially more complicated; so we will content ourselves with a few examples. If the complex group G is simply connected, then X_* is the coroot lattice, which has as a basis the simple coroots. Because the roots β_i are simple, the equation in Proposition 3.4(4) can have no solution unless B is empty. That is (still for G simply connected)

$$W^{\text{sing}}_{\text{ncpt}}(K) = 1, \qquad K = K^\sharp = K_0T_c.$$

(In fact K must be connected in this case.)

We get interesting departure from this behavior only when X_* includes more than the coroots. Enlarging X_* means passing to central quotients of G; the most interesting case is for the adjoint group. Here are some examples.

Suppose first that $G = PSp(2n,\mathbb{R})$, the projective symplectic group. In this case

$$\begin{aligned} X^* &= \{\lambda \in \mathbb{Z}^n \mid \sum_i \lambda_i \in 2\mathbb{Z}\} \quad R(G,H_c) = \{\pm 2e_i, (\pm e_i \pm e_j) \mid i \neq j\} \\ X_* &= \langle \mathbb{Z}^n, (1/2,\ldots,1/2)\rangle \qquad R^\vee(G,H_c) = \{\pm e_i, (\pm e_i \pm e_j) \mid i \neq j\} \end{aligned} \tag{3.6a}$$

The action of θ on H_c is trivial, so all the roots are imaginary. The compact ones are

$$R_{\text{cpt}} = \{(e_i - e_j) \mid i \neq j\}, \quad 2\rho_K = (n-1, n-3, \ldots, -n+1) \in X^*. \tag{3.6b}$$

We therefore calculate

$$R^{\text{sing}}_{\text{ncpt}} = \{\pm(e_1+e_n), \ldots, (e_{[n/2]} + e_{n-[n/2]+1})\} \cup \{2e_{(n+1)/2}\}; \tag{3.6c}$$

the last root is present only if n is odd. The corresponding simple coroots are

$$\{(e_1+e_n),\ (e_2+e_{n-1}),\ \ldots,\ (e_{[n/2]}+e_{n-[n/2]+1})\} \cup \{e_{(n+1)/2}\},$$

again with the last term present only if n is odd. The elements H_B have all coordinates 1 or 0, symmetrically distributed. Since θ acts by the identity, B contributes to $W^{\text{sing}}_{\text{ncpt}}$ if and only if H_B is divisible by two in X_*; that is, if and only if

$$B = \varnothing \quad \text{or} \quad B = \{1,\ldots,r\}.$$

The nontrivial Weyl group element is

$$w_B(t_1,\ldots,t_n) = (t_n^{-1},\ldots,t_1^{-1}) \tag{3.6d}$$

(reverse order and invert all entries). (More precisely, that is the Weyl group element in $Sp(2n)$, acting on the maximal torus $(\mathbb{C}^\times)^n$. In our case that torus is divided by ± 1.)

Suppose next that $G = PSO(2n,2n)$, the projective special orthogonal group (the split form of D_{2n}). In this case

$$\begin{aligned} X^* &= \{\lambda \in \mathbb{Z}^{2n} \mid \sum_i \lambda_i \in 2\mathbb{Z}\} \quad R(G,H_c) = \{(\pm e_i \pm e_j) \mid i \neq j\} \\ X_* &= \langle \mathbb{Z}^{2n}, (1/2,\ldots,1/2)\rangle \quad R^\vee(G,H_c) = \{(\pm e_i \pm e_j) \mid i \neq j\} \end{aligned} \tag{3.7a}$$

We will sometimes write a semicolon between the first n and the last n coordinates of X^* for clarity. The action of θ on H_c is trivial, so all the roots are imaginary. The compact ones are

$$\begin{aligned} &R_{\text{cpt}} = \{(\pm e_p \pm e_q),\ (\pm e_{n+p} \pm e_{n+q}) \mid 1 \leqslant p \neq q \leqslant n\}, \\ &2\rho_K = (n-1, n-2, \ldots, 1, 0; n-1, n-2, \ldots, 1, 0) \in X^*. \end{aligned} \tag{3.7b}$$

We therefore calculate

$$R^{\rm sing}_{\rm ncpt} = \{\pm(e_p - e_{n+p}) \mid 1 \leqslant p \leqslant n-1\} \cup \{(e_n \pm e_{2n})\}; \tag{3.7c}$$

The corresponding simple coroots are the same. The elements H_B have coordinates $1 \leqslant p \leqslant n-1$ equal to 1 or 0, with the same value on coordinate $p+n$. The coordinates n and $2n$ are either $(0,0)$ or $(1,\pm 1)$ or $(2,0)$. Since θ acts by the identity, B contributes to $W^{\rm sing}_{\rm ncpt}(K)$ if and only if H_B is divisible by two in X_*; that is, if and only if

$$H_{B_0} = 0, \qquad B_0 = \varnothing;$$

$$H_{B_\pm} = (1,\ldots,1;-1,\ldots,\mp 1), \quad B_\pm = \{(e_p - e_{n+p}) | p \leqslant n-1\} \cup \{(e_n \mp e_{2n})\}$$

$$H_{B_2} = (0,\ldots,2;0,\ldots,0), \qquad B_2 = \{(e_n - e_{2n})\ ,\ (e_n + e_{2n})\}.$$

The three nontrivial Weyl group elements are

$$\begin{aligned} w_{B_\pm}(s_1,\ldots,s_n;t_1,\cdots,t_n) &= (t_1,\ldots,t_n^{\pm 1};s_1,\ldots,s_n^{\pm 1})\\ w_{B_2}(s_1,\ldots,s_n;t_1,\ldots,t_n) &= (s_1,\ldots,s_{n-1},s_n^{-1};t_1,\ldots,t_{n-1},t_n^{-1}). \end{aligned} \tag{3.7d}$$

(More precisely, those are the Weyl group elements in $SO(4n)$, acting on the maximal torus $(\mathbb{C}^\times)^{2n}$. In our case that torus is divided by ± 1.) Because $T_c = H_c$ is connected, the group $K^\sharp = K_0 T_c$ is connected. Therefore the group of connected components of K is

$$K/K_0 = W^{\rm sing}_{\rm ncpt}(K) = (\mathbb{Z}/2\mathbb{Z})^2,$$

the Klein four-group.

We are going to need to understand the cosets of W_{K_0} in W^θ. We conclude this section with that. Define

$$W^1 = \{w \in W^\theta \mid wR^+_{\rm res} \supset R^+_K\}; \tag{3.8a}$$

equivalently, these are the restricted Weyl group elements making only noncompact imaginary roots change sign. The reason these elements are of interest is that they are natural coset representatives for W_{K_0} in W^θ:

$$W^\theta = W_{K_0}\cdot W^1, \qquad W^1 \simeq W_{K_0}\backslash W^\theta. \tag{3.8b}$$

Corollary 3.9. *In the setting of* (3.8)*,*

$$W^1 \subset W^{\rm sing}_{\rm imag}.$$

More precisely,

$$W^1 = \left\{ w \in W^{\rm sing}_{\rm imag} \mid wR^{+,\rm sing}_{\rm imag} \supset R^{+,\rm sing}_{\rm imag,cpt} \right\},$$

$$W_{K_0}\backslash W^\theta \simeq W_{K_{\rm imag,0}}\backslash W_{\rm imag}.$$

The groups on the right in the last formula come from the (maximal cuspidal Levi) subgroup

$$L_{\rm imag} = G^{A_c}$$

corresponding to the imaginary roots of H_c*.*

This is immediate from Proposition 3.4(1).

4 Restricted weights

There are many useful classical facts about the set of weights of a finite-dimensional representation, like the fact that all weights are in the convex hull of the extremal weights. In this section we first formulate those facts for restricted weights with respect to a maximally compact Cartan. Then we consider the behavior of invariant Hermitian forms on the restricted extremal weight spaces.

Fix therefore a θ-stable system of positive roots

$$R_c^+ \subset R(G, H_c),$$

and a strongly integral R_c^+-dominant weight

$$\gamma_c = (\lambda_c, \nu_c). \tag{4.1a}$$

Write

$$F(\gamma_c) = \text{(some) finite-dimensional irreducible, highest weight } \gamma_c \tag{4.1b}$$

as in Proposition 2.6(3). Eventually we will impose also the requirement

$$\nu_c \text{ is purely imaginary;} \tag{4.1c}$$

the requirement that ν_c be imaginary is the condition from Proposition 2.8 for the existence of a $G(\mathbb{R})^\sharp$ invariant Hermitian form on $F(\gamma_c)$.

Every continuous character of $H_c(\mathbb{R})$ restricts to a continuous character of $T_c(\mathbb{R})$, which is in turn the restriction of a unique algebraic character in $X^*(T_c)$. The *restricted weights* of the finite-dimensional representation $F(\gamma_c)$ are the characters

$$\{\overline{\phi} \in X^*(T_c) \mid \overline{\phi} = \text{restriction of character } \phi \text{ of } H_c(\mathbb{R}) \text{ in } F(\gamma_c).\} \tag{4.1d}$$

It was more convenient to discuss the general theory of restricted roots on the connected torus $T_{c,0}$, but it is more convenient to discuss restricted weights on all of T_c. Passage back and forth is facilitated by the fact

$$X^*(T_c) \xrightarrow{\text{res}} X^*(T_{c,0}) \quad \text{is injective on restricted root lattice } \mathbb{Z}R_{\text{res}}; \tag{4.1e}$$

the lattice means the lattice of T_c-weights of $S(\mathfrak{g})$. Using this fact, we will freely replace any restricted root $\overline{\alpha} \in X^*(T_{c,0})$ by its unique extension to T_c as a weight of $\mathfrak{g}$. Define

$$2\rho_{\text{res}}^\vee = \sum_{\overline{\alpha} \in R^+_{\text{res,red}}} \overline{\alpha}^\vee, \tag{4.1f}$$

the sum of the coroots for the positive reduced restricted roots. If $\overline{\phi} \in X^*(T_c)$ is any character, then there is a unique character $w\phi$ (for $w \in W^\theta$) with the property that $w\overline{\phi}$ is weakly dominant for R^+_{res}. We define the *restricted height of* $\overline{\phi}$ by

$$\text{ht}_{\text{res}}(\overline{\phi}) = \langle w\overline{\phi}, 2\rho^\vee_{\text{res}}\rangle = \langle w\phi, 2\rho^\vee_{\text{res}}\rangle \tag{4.1g}$$

a nonnegative integer. (The last pairing is independent of the choice of $\phi \in X^*(H_c)$ restricting to $\overline{\phi}$, because the restricted coroots are θ-fixed.) Clearly

$$\mathrm{ht}_{\mathrm{res}}(\overline{\phi}) = \mathrm{ht}_{\mathrm{res}}(x\overline{\phi}) \qquad (x \in W^\theta). \tag{4.1h}$$

Here is the description we want of restricted weights.

Proposition 4.2. *Suppose we are in the setting of* (4.1) *so that in particular $F(\gamma_c)$ is an irreducible finite-dimensional representation of $G(\mathbb{R})$ of highest weight*

$$\gamma_c = (\lambda_c, \nu_c).$$

1. *The set of restricted weights (and their multiplicities) is invariant under the restricted Weyl group W^θ.*
2. *An R^+_{res}-dominant restricted weight $\overline{\phi}$ is a restricted weight of $F(\gamma_c)$ if and only if*

   $$\lambda_c = \overline{\phi} + \sum_{\overline{\alpha} \in R^+_{\mathrm{res}}} n_{\overline{\alpha}} \overline{\alpha}, \qquad (n_{\overline{\alpha}} \in \mathbb{N}).$$

   In this case

   $$\mathrm{ht}_{\mathrm{res}}(\overline{\phi}) \leqslant \mathrm{ht}_{\mathrm{res}}(\lambda_c),$$

   with equality if and only if $\overline{\phi} = \lambda_c$.
3. *Suppose a restricted weight $\overline{\phi}$ is a weight of $F(\gamma_c)$. Then*

   $$\lambda_c = \overline{\phi} + \sum_{\overline{\alpha} \in R^+_{\mathrm{res}}} n_{\overline{\alpha}} \overline{\alpha}, \qquad (n_{\overline{\alpha}} \in \mathbb{N})$$

   and

   $$\mathrm{ht}_{\mathrm{res}}(\overline{\phi}) \leqslant \mathrm{ht}_{\mathrm{res}}(\lambda_c),$$

   with equality if and only if

   $$\overline{\phi} = w\lambda_c, \qquad \textit{some } w \in W^\theta.$$

   We call $\{w\lambda_c \mid w \in W^\theta\}$ the restricted extremal weights *of $F(\gamma_c)$.*
4. *The R^+_K-dominant restricted extremal weights are*

   $$W^1\lambda_c,$$

   with W^1 as in Corollary 3.9. Each such extremal weight is therefore uniquely of the form

   $$w\lambda_c = \lambda_c - \sum_{\beta \in R^{+,\mathrm{sing}}_{\mathrm{imag}} \ \textit{simple}} n_\beta \beta,$$

   with notation as in (3.3).

Part (1) is elementary. Part (2) is exactly parallel to a standard fact about weights of finite-dimensional representations, and can be proved in the same way. Then (3) follows from (1) and (2). Part (4) follows from Corollary 3.9. We omit the details.

Corollary 4.3. *Suppose we are in the setting of* (4.1)*, and that* (4.1c) *also holds, so that $F(\gamma_c)$ admits a $G(\mathbb{R})_0$-invariant Hermitian form*

$$\langle \cdot, \cdot \rangle_{F(\gamma_c)}.$$

We normalize this form to be positive on the λ_c restricted weight space.

1. *The form $\langle \cdot, \cdot \rangle_{F(\gamma_c)}$ is nondegenerate on each (one-dimensional) restricted extremal weight space $w\lambda_c$, and so either positive or negative there. Write*

$$\epsilon_{F(\gamma_c)}(w) = \pm 1$$

for this sign.

2. *The sign $\epsilon_{F(\gamma_c)}(w)$ is invariant under left multiplication by $W(K_0)$, and so is determined by its restriction to the coset representatives W^1 of Corollary 3.9.*

3. *Write the simple roots for the Levi subsystem $R^{\text{sing}}_{\text{imag}}$ as the disjoint union of compact and noncompact imaginary roots:*

$$\Pi^{+,\text{sing}}_{\text{imag}} = \Pi^{+,\text{sing}}_{\text{imag,cpt}} \amalg \Pi^{+,\text{sing}}_{\text{imag,ncpt}}$$

(notation as in Proposition 4.2(4)). For $w \in W^1$, we have

$$\epsilon_{F(\gamma_c)}(w) = \prod_{\beta \in \Pi^{\text{sing}}_{\text{imag,ncpt}}} (-1)^{n_\beta}.$$

4. *The form $\langle \cdot, \cdot \rangle_{F(\gamma_c)}$ is invariant by $K(\mathbb{R})$ (and therefore by $G(\mathbb{R})$) if and only if*

$$\epsilon_{F(\gamma_c)}(xw) = 1, \qquad \text{all } x \in W^{\text{sing}}_{\text{ncpt}}(K)$$

(see Proposition 3.4(5)).

Proof. se:extrsig Because all characters of the compact group $T_c(\mathbb{R})$ are Hermitian, the Hermitian pairing necessarily makes the distinct restricted weight spaces orthogonal, and so (by nondegeneracy) defines a nondegenerate form on each restricted weight space. Now (1) is immediate. The form is preserved by $G(\mathbb{R})_0 \supset K(\mathbb{R})_0$, and the Weyl group elements in $W(K_0)$ have representatives in $K(\mathbb{R})_0$. So (2) follows. Part (3) can be proven by induction on the length of w. It is obvious if $w = 1$; so suppose $w \neq 1$, and choose a simple reflection s_β so that

$$\ell(s_\beta w) = \ell(w) - 1. \tag{4.4a}$$

Because w is in the Levi subgroup $W^{\text{sing}}_{\text{imag}}$, the root β must be imaginary. Define

$$m = \langle s_\beta w\lambda, \beta^\vee\rangle. \tag{4.4b}$$

Then

$$w\lambda = s_\beta w\lambda - m\beta. \tag{4.4c}$$

The proposed formula for $\epsilon(w)$ therefore satisfies

$$\epsilon(w) = \epsilon(s_\beta w) \cdot \begin{cases} 1 & \text{if } \beta \text{ is compact} \\ (-1)^m & \text{if } \beta \text{ is noncompact.} \end{cases} \tag{4.4d}$$

To complete the induction argument, we must show that ϵ actually satisfies (4.4d). The $m+1$-dimensional space

$$E(w,\beta) = \text{span of the weight spaces } \{w\lambda - j\beta \mid 0 \leqslant j \leqslant m\} \tag{4.4e}$$

is an irreducible representation of $SL(2)$, by means of the root $SL(2)$ ϕ_β (see (2.2)). If β is compact, $E(w,\beta)$ a Hermitian representation of $SU(2)$, so the form is definite, and $\epsilon(s_\beta w) = \epsilon(w)$, as required by (4.4d).

If β is noncompact, then $E(w,\beta)$ is an irreducible Hermitian representation of $SU(1,1)$. For such a representation, calculation in $SU(1,1)$ shows that the signature of the form alternates in j on the weights $w\lambda - j\beta$. Consequently

$$\epsilon(w) = (-1)^m \epsilon(s_\beta w), \tag{4.4f}$$

again as required by (4.4d). This completes the induction, and the proof of (3).

For (4), if the form is $K(\mathbb{R})$-invariant, then it must be definite on each of the irreducible representations of $K(\mathbb{R})$ generated by an extremal weight. Because the elements of $W^{\text{sing}}_{\text{ncpt}}(K)$ have representatives in $K(\mathbb{R})$ (Proposition 3.4(5)), the invariance property in (4) follows.

We omit the proof of the converse, which we will not use. □

5 Dirac operator and signature calculation

We have so far avoided introducing invariant bilinear forms on $\mathfrak{g}$, because the idea of root data teaches us to do that. But now it is time to talk about Dirac operators, and there the choice of forms appears to be critical and unavoidable. We begin by introducing the forms and the corresponding Casimir operators. (The Casimir operators will play the role of Laplacians, of which the Dirac operator is a kind of square root.)

We continue to work with our complex connected reductive algebraic group G which is defined over $\mathbb{R}$, and with a chosen Cartan involution θ as in (2.1), so that we have

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{k}(\mathbb{R}) + \mathfrak{s}(\mathbb{R})$$

as in (2.1f). Fix a non-degenerate $\text{Ad}(G)$-invariant symmetric bilinear form

$$B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \tag{5.1a}$$

We require also that B is preserved by θ, and that

$$B \text{ is real negative definite on } \mathfrak{k}(\mathbb{R}), \text{ real positive definite on } \mathfrak{s}(\mathbb{R}). \tag{5.1b}$$

If G is semisimple, the Killing form meets these requirements; in general they are easy to achieve. The properties are inherited by many real and θ-stable reductive subalgebras. For example, if $H = T(\mathbb{R})A$ is a maximal torus as in (2.2b), then

$$B \text{ is real negative definite on } \mathfrak{t}(\mathbb{R}), \text{ real positive definite on } \mathfrak{s}(\mathbb{R}). \tag{5.1c}$$

In particular B is nondegenerate on $\mathfrak{h}$, and so dualizes to a Weyl group invariant symmetric bilinear form B^* on $\mathfrak{h}^*$. Because the roots take imaginary values on $\mathfrak{t}(\mathbb{R})$ and real values on $\mathfrak{a}$, we get

$$B^* \text{ is positive definite on the root lattice.} \tag{5.1d}$$

The decomposition

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}(\mathfrak{g}) = \mathfrak{g}_{\mathrm{ss}} + \mathfrak{z}(\mathfrak{g}) \tag{5.1e}$$

(the second summand being the center) is orthogonal for B. On each maximal torus this gives

$$\mathfrak{h} = \mathfrak{h}_{\mathrm{ss}} + \mathfrak{z}(\mathfrak{g}), \qquad \mathfrak{h}_{\mathrm{ss}} = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]; \tag{5.1f}$$

the first summand is the span of the coroots. Dualizing gives an orthogonal decomposition

$$\mathfrak{h}^* = \mathfrak{h}^*_{\mathrm{ss}} + \mathfrak{z}(\mathfrak{g})^*, \tag{5.1g}$$

and the first summand is the span of the roots.

If $\{X_i\}$ is any basis of $\mathfrak{k}$, there is a unique *dual basis* $\{X^j\}$ defined by the requirements

$$B(X_i, X^j) = \delta_{ij}. \tag{5.1h}$$

The *Casimir operator for K* (with respect to B) is

$$\Omega_K = \sum_i X_i X^i \in U(\mathfrak{k}). \tag{5.1i}$$

It is independent of the choice of basis, and is fixed by $\mathrm{Ad}(K)$; in particular, it belongs to the center of the enveloping algebra $U(\mathfrak{k})$. Consequently Ω_K acts by a complex scalar operator

$$\mu(\Omega_K) \in \mathbb{C} \tag{5.1j}$$

on any irreducible representation μ of $\mathfrak{k}$. In the same way, if $\{Z_p\}$ is any basis of $\mathfrak{g}$ and $\{Z^q\}$ the dual basis, we get the *Casimir operator for G*

$$\Omega_G = \sum_p Z_p Z^p \in U(\mathfrak{g}), \tag{5.1k}$$

which acts by a complex scalar

$$\pi(\Omega_G) \in \mathbb{C} \tag{5.1l}$$

on any irreducible representation π of $\mathfrak{g}$.

If μ is an irreducible representation of K of highest weight $\xi \in X^*(T_c)$ with respect to R_K^+ (see (3.3a)), then

$$\begin{aligned}\mu(\Omega_K) = B(d\xi + d(2\rho_K)/2, d\xi + d(2\rho_K)/2 \\ - B(d(2\rho_K)/2, d(2\rho_K)/2) \geqslant 0;\end{aligned} \tag{5.1m}$$

equality holds if and only if $d\xi$ vanishes on all coroots of K. In accordance with our policy of ignoring the difference between characters and their differentials when it is harmless, we will usually write this result as

$$\mu(\Omega_K) = B(\xi + \rho_K, \xi + \rho_K) - B(\rho_K, \rho_K).$$

In the same way, if $(\pi, F(\gamma_c))$ is an irreducible representation of G as in (4.1), then

$$\begin{aligned}\pi(\Omega_G) &= B(\gamma_c + \rho, \gamma_c + \rho) - B(\rho, \rho) \\ &= B(\lambda_c + \rho, \lambda_c + \rho) - B(\rho, \rho) + B(\nu_c, \nu_c).\end{aligned} \tag{5.1n}$$

We turn now to the Dirac operator. The key to its definition is the (positive definite) real quadratic space

$$(\mathfrak{s}(\mathbb{R}), B), \qquad \mathrm{Ad}\colon K \to O(\mathfrak{s}(\mathbb{R}), B). \tag{5.2a}$$

The *Clifford algebra* $C(\mathfrak{s}(\mathbb{R}))$ is the real associative algebra with 1 generated by $\mathfrak{s}(\mathbb{R})$ subject to the relations

$$X^2 + B(X, X) = 0 \qquad (X \in \mathfrak{s}(\mathbb{R})), \tag{5.2b}$$

or equivalently

$$XY + YX + 2B(X, Y) = 0 \qquad (X, Y \in \mathfrak{s}(\mathbb{R})). \tag{5.2c}$$

By definition $C(\mathfrak{s}(\mathbb{R}))$ is a quotient of the tensor algebra of $\mathfrak{s}(\mathbb{R})$, from which it inherits a filtration indicated by lower subscripts:

$$C(\mathfrak{s}(\mathbb{R}))_m = \text{span of products of at most } m \text{ elements of } \mathfrak{s}(\mathbb{R}).$$

We have

$$\mathrm{gr}\, C(\mathfrak{s}(\mathbb{R})) \simeq \textstyle\bigwedge \mathfrak{s}(\mathbb{R}). \tag{5.2d}$$

Here are the basic facts about the spin cover of a compact orthogonal group.

Proposition 5.3. *Define*

$$C(\mathfrak{s}(\mathbb{R}))^\times = \textit{invertible elements of the Clifford algebra,}$$

an open subgroup of the algebra. The conjugation action of this group on the Clifford algebra is by algebra automorphisms. Regard $C(\mathfrak{s}(\mathbb{R}))$ *as a Lie algebra under the commutator of the associative algebra structure; this is the Lie algebra of the group* $C(\mathfrak{s}(\mathbb{R}))^\times$*. Then there is a natural inclusion of Lie algebras*

$$\mathfrak{so}(\mathfrak{s}(\mathbb{R})) \simeq \bigwedge^2 \mathfrak{s}(\mathbb{R}) \overset{j}{\hookrightarrow} C(\mathfrak{s}(\mathbb{R}))_2,$$

The spin group *is by definition the corresponding Lie subgroup of* $C(\mathfrak{s}(\mathbb{R}))^\times$*:*

$$\mathrm{Spin}(\mathfrak{s}(\mathbb{R})) = \exp\big(j(\mathfrak{so}(\mathfrak{s}(\mathbb{R})))\big) \subset C(\mathfrak{s}(\mathbb{R}))^\times.$$

The spin group action on $C(\mathbb{R})$ *by conjugation preserves the filtration, and so descends to an action on*

$$\mathrm{gr}\, C(\mathfrak{s}(\mathbb{R})) \simeq \bigwedge \mathfrak{s}(\mathbb{R}),$$

The action on $\mathfrak{s}(\mathbb{R})$ *preserves the quadratic form (because it comes from Clifford algebra automorphisms), so defines*

$$\mathrm{Spin}(\mathfrak{s}(\mathbb{R})) \xrightarrow{\pi} SO(\mathfrak{s}(\mathbb{R})).$$

The differential of π *is the inverse of the Lie algebra isomorphism* j*; so* π *is a covering map. As long as* $\dim \mathfrak{s}(\mathbb{R}) \geqslant 2$*, we have*

$$\ker \pi = \{\pm 1\} \subset C(\mathfrak{s}(\mathbb{R}))^\times,$$

so the covering is two to one.

Here is the representation theory of the Clifford algebra.

Proposition 5.4. *Write*

$$\dim \mathfrak{s}(\mathbb{R}) =_{\mathrm{def}} n = 2m + \epsilon, \qquad m = [n/2].$$

The complexified Clifford algebra has dimension $2^n = 2^\epsilon \cdot (2^m)^2$*. It is the direct sum of* $2^\epsilon = 1$ *or* 2 *copies of a matrix algebra of rank* 2^m*. In particular, the center of the Clifford algebra has dimension* 2^ϵ*; it is spanned by* 1 *and (if* n *is odd)*

$$z = e_1 \cdots e_{2m+1},$$

with $\{e_i\}$ *an orthonormal basis of* $\mathfrak{s}(\mathbb{R})$*. This central element depends only on the orientation defined by the chosen orthonormal basis, and satisfies*

$$z^2 = (-1)^{m-1}.$$

The Clifford algebra has 2^ϵ *irreducible representations, called* spin representations*, each of dimension* 2^m*. In case* n *is odd, these two representations are distinguished by the scalar by which* z *acts: we write* $(\sigma_{[\pm]}, S_{[\pm]})$ *for an irreducible representation on which* z *acts by* $\pm i^{m-1}$*.*

If n is even, the spin representation (σ, S) has a $\mathbb{Z}/2\mathbb{Z}$ grading

$$S = S_+ \oplus S_-,$$

with each summand of dimension 2^{m-1}. The generators $X \in \mathfrak{s}(\mathbb{R})$ carry S_+ to S_-. The action of the spin group

$$\mathrm{Spin}(\mathfrak{s}(\mathbb{R})) \subset C(\mathfrak{s}(\mathbb{R}))^\times$$

preserves $S_\pm$, and acts irreducibly on each; these are the half-spin representations *$\sigma_\pm$ of (the double cover of) an even special orthogonal group.*

If n is odd, the two spin representations $\big(\sigma_{[\pm]}, S_{[\pm]}\big)$ are isomorphic as representations of the spin group. The action is irreducible; this is the spin representation *σ of (the double cover of) an odd special orthogonal group.*

Suppose that the weights for $SO(\mathfrak{s}(\mathbb{R}))$ acting $\mathfrak{s}(\mathbb{C})$ are

$$\{\pm\mu_1, \ldots, \pm\mu_m\} \amalg \{0\};$$

the last zero is present only if $\epsilon = 1$. Then the weights of (either) spin representation S are

$$(1/2)\sum_{j=1}^{m} \epsilon_j \mu_j,$$

with $\epsilon_j = \pm 1$. Each such weight has multiplicity one.

It is possible to enlarge $\mathrm{Spin}(\mathfrak{s}(\mathbb{R})) \subset C(\mathfrak{s}(\mathbb{R}))^\times$ to a double cover of the full orthogonal group $O(\mathfrak{s}(\mathbb{R}))$. This is interesting for us because

$$\mathrm{Ad}\colon K \to O(\mathfrak{s}(\mathbb{R}))$$

need not have image inside SO. All of the discussion starting in (5.7) below can accordingly be extended to some double cover $\widetilde{K}$ of K. But this is a bit complicated, and plays no essential role in this paper; so we omit it.

The real form $C(\mathfrak{s}(\mathbb{R}))$ of the complexified Clifford algebra corresponds to a conjugate-linear automorphism

$$\sigma_{\mathbb{R}}\colon C(\mathfrak{s}(\mathbb{R}))_{\mathbb{C}} \to C(\mathfrak{s}(\mathbb{R}))_{\mathbb{C}}, \qquad \sigma_{\mathbb{R}}(X) = X \qquad (X \in \mathfrak{s}(\mathbb{R})). \tag{5.5a}$$

There is also a (complex-linear) algebra antiautomorphism τ characterized by

$$\tau(X) = -X \qquad (X \in \mathfrak{s}(\mathbb{R})). \tag{5.5b}$$

(The reason for the existence of τ is that the requirement (5.5b) respects the defining relations of the Clifford algebra.) If (π, M) is any $C(\mathfrak{s}(\mathbb{R}))_{\mathbb{C}}$-module, the Hermitian dual vector space M^h (consisting of conjugate-linear functionals on M; see for example [1, Section 8]) becomes a $C(\mathfrak{s}(\mathbb{R}))_{\mathbb{C}}$-module by the requirement

$$\pi^h(c) = \pi\big(\tau(\sigma_{\mathbb{R}}(c))\big)^h \qquad (c \in C(\mathfrak{s}(\mathbb{R}))_{\mathbb{C}}) \tag{5.5c}$$

or equivalently

$$\langle m, X\cdot\mu\rangle = \langle -X\cdot m, \mu\rangle \qquad (m\in M, \mu\in M^h, X\in\mathfrak{s}(\mathbb{R})). \tag{5.5d}$$

Here we write $\langle\cdot,\cdot\rangle$ for the Hermitian pairing between M and its Hermitian dual M^h. Passage to the Hermitian dual obviously fixes the unique simple $C(\mathfrak{s}(\mathbb{R}))_{\mathbb{C}}$-module S in the even-dimensional case, so S admits an invariant Hermitian form

$$\langle\cdot,\cdot\rangle_S\colon S\times S\to\mathbb{C}. \tag{5.5e}$$

In the odd-dimensional case, we find for the central element z described in Proposition 5.4 that

$$\sigma_{\mathbb{R}}(z) = z, \qquad \tau(z) = (-1)^{m-1}z$$

Since z acts on $S_\pm$ by the scalar $(\pm i)^{m-1}$, it follows that z acts on the Hermitian dual $S_\pm^h$ by the scalar

$$(-1)^{m-1}\overline{(\pm i)^{m-1}} = (\pm i)^{m-1}.$$

Therefore $S_\pm^h \simeq S_\pm$, and $S_\pm$ admits an invariant Hermitian form

$$\langle\cdot,\cdot\rangle_{S_\pm}\colon S_\pm\times S_\pm\to\mathbb{C}. \tag{5.5f}$$

Proposition 5.6. *In the setting of* (5.5)*, the invariant Hermitian forms* $\langle,\rangle_S$ *and* $\langle,\rangle_{S_\pm}$ *are all definite. We normalize them henceforth to be* positive*. The characteristic invariance property is*

$$\langle X\cdot s, s'\rangle + \langle s, X\cdot s'\rangle = 0 \qquad (X\in\mathfrak{s}(\mathbb{R}));$$

that is, the action of Clifford multiplication is by skew-adjoint operators.

These Hermitian forms are also invariant under the action of the spin group $\mathrm{Spin}(\mathfrak{s}(\mathbb{R}))$.

Suppose

$$(\xi, V) \text{ is a } (\mathfrak{g}, K_0)\text{-module}; \tag{5.7a}$$

that is, that V is at the same time a complex representation of the Lie algebra $\mathfrak{g}$, and a locally finite continuous representation of the Lie group K, and that

$$\text{the differential of } \xi|_K \text{ is equal to the restriction to } \mathfrak{k}_0 \text{ of } \xi|_{\mathfrak{g}}. \tag{5.7b}$$

Let (σ, S) be a spin representation of the complexified Clifford algebra $C(\mathfrak{s}_{\mathbb{R}})_{\mathbb{C}}$. In the odd-dimensional case, we simply choose one of the two representations S_+ or S_-. Finally, fix any basis

$$\{X_1,\cdots,X_n\}\subset\mathfrak{s}(\mathbb{R}) \tag{5.7c}$$

for the -1 eigenspace of the Cartan involution on the real Lie algebra, and let

$$\{X^1,\cdots,X^n\}\subset\mathfrak{s}(\mathbb{R}), \qquad B(X_i, X^j) = \delta_{ij} \tag{5.7d}$$

be the dual basis with respect to the symmetric invariant form B of (5.1). The *Dirac operator for* (ξ, V) is the linear operator on $V \otimes S$ defined by

$$D = \sum_{j=1}^{n} \xi(X_j) \otimes \sigma(X^j) \in \mathrm{End}(V \otimes S). \tag{5.7e}$$

It will be convenient as in the discussion of the Clifford algebra to write

$$n = 2m + \epsilon, \qquad \dim S = 2^m = 2^{[n/2]}. \tag{5.7f}$$

The adjoint action defines a group homomorphism

$$\mathrm{Ad}\colon K_0 \to SO(\mathfrak{s}(\mathbb{R})). \tag{5.7g}$$

Using the covering

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(\mathfrak{s}(\mathbb{R})) \xrightarrow{\pi} SO(\mathfrak{s}(\mathbb{R})) \longrightarrow 1, \tag{5.7h}$$

from Proposition 5.3, we can define a pushout

$$\widetilde{K}_0 = \{(s,k) \in \mathrm{Spin}(\mathfrak{s}(\mathbb{R})) \times K_0 \mid \mathrm{Ad}(k) = \pi(s)\}. \tag{5.7i}$$

There is a short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{K}_0 \xrightarrow{\pi} K_0 \longrightarrow 1. \tag{5.7j}$$

Projection on the first factor defines a homomorphism $\widetilde{\mathrm{Ad}}$,

$$\widetilde{\mathrm{Ad}}\colon \widetilde{K}_0 \longrightarrow \mathrm{Spin}(\mathfrak{s}(\mathbb{R})). \tag{5.7k}$$

In this way S becomes a representation of $\widetilde{K}_0$ by

$$\sigma_{\widetilde{K}_0} = \sigma \circ \widetilde{\mathrm{Ad}}. \tag{5.7l}$$

The nonzero weights of T_c on $\mathfrak{s}$ are

$$\{\pm\gamma_j \mid 1 \leqslant j \leqslant r\} \ \amalg \ \{0\}. \tag{5.7m}$$

Here the γ_j are the complex positive roots and the noncompact imaginary positive roots; and the multiplicity of the weight zero is $\dim A_c$. In light of Proposition 5.4, it follows that the weights of $\widetilde{K}_0$ on S are

$$(1/2)\sum_{j=1}^{r} \epsilon_j \gamma_j, \tag{5.7n}$$

with $\epsilon_j = \pm 1$. The multiplicity of such a weight is the number of expressions for it of this form, times $2^{[\dim A_c]/2}$. (The multiplicity arises because the weights μ_j appearing in Proposition 5.4 are the r pairs $\pm\gamma_j$, together with $[\dim A_c/2]$ pairs of zeros.)

Of course V is a representation of $\widetilde{K}_0$ by $\xi \circ \pi$, which we will just call ξ. Therefore

$$(\sigma_{\widetilde{K}_0} \otimes \xi, S \otimes V) \tag{5.7o}$$

is a representation of $\widetilde{K}_0$.

Here are the basic facts about Parthasarthy's Dirac operator.

Proposition 5.8 (Parthasarathy [5]). *In the setting of* (5.7), *the Dirac operator D is independent of the choice of basis (of $\mathfrak{s}(\mathbb{R})$), and commutes with the representation $\sigma_{\widetilde{K}_0} \otimes \xi$ of $\widetilde{K}_0$. Consequently*

$$\ker D \subset S \otimes V$$

is a representation of $\widetilde{K}_0$ (as indeed is every eigenspace of D).

The square of the Dirac operator is

$$D^2 = -1_S \otimes \xi(\Omega_G) + (\sigma_{\widetilde{K}_0} \otimes \xi)(\Omega_K) - [B(\rho_G, \rho_G) - B(\rho_K, \rho_K)] \cdot 1_S \otimes 1_V.$$

Suppose next that Ω_G acts on V by a complex scalar $\xi(\Omega_G)$ (as is automatic if ξ is irreducible). Then D^2 is diagonalized by the decomposition of $\sigma_{\widetilde{K}_0} \otimes \xi$ into irreducible representations of $\widetilde{K}_0$. All of the eigenvalues differ from $\xi(\Omega_G)$ by real scalars.

Suppose finally that V admits a nondegenerate invariant Hermitian form $\langle , \rangle_V$. Then D is self-adjoint for the Hermitian form

$$\langle , \rangle_S \otimes \langle , \rangle_V.$$

If V has signature (p, q), then $S \otimes V$ has signature $(2^m p, 2^m q)$ (notation as in Proposition 5.4).

Here is Kostant's result about the spectrum of the Dirac operator.

Proposition 5.9 ([4]). *Suppose that $F(\gamma_c)$ is an irreducible finite-dimensional representation of $G(\mathbb{R})$ of highest weight*

$$\gamma_c = (\lambda_c, \nu_c)$$

as in Proposition 4.2, and S is a spin representation of $\mathrm{Spin}(\mathfrak{s}(\mathbb{R}))$ *as in Proposition 5.4. Regard $S \otimes F(\gamma_c)$ as a representation of $\widetilde{K}_0$ as in Proposition 5.8.*

1. *Every irreducible representation $\widetilde{\tau}$ of $\widetilde{K}_0$ on $S \otimes F(\gamma_c)$ has highest weight of the form*
$$\overline{\phi} + w\rho_G - \rho_K - 2\rho(B),$$
for some $w \in W^1$ (see Corollary 3.9), $\overline{\phi}$ a wR^+_{res}-dominant restricted weight of $F(\gamma_c)$, and B a set of noncompact imaginary roots in wR^+.

2. *The scalar $\widetilde{\tau}(\Omega_K)$ satisfies*
$$\widetilde{\tau}(\Omega_K) \leqslant \langle \lambda_c + \rho_G, \lambda_c + \rho_G \rangle - \langle \rho_K, \rho_K \rangle.$$
Equality holds if and only if
$$\overline{\phi} = w\lambda_c = w'\lambda_c, w\rho_G - 2\rho(B) = w'\rho_G$$
for some $w' \in W^1$. In particular, this largest possible eigenvalue of $\widetilde{\tau}(\Omega_K)$ is equal to
$$\langle \lambda_c + \rho_G, \lambda_c + \rho_G \rangle - \langle \rho_K, \rho_K \rangle.$$

The proposition has been formulated in such a way as to outline its proof in [4] and [3]. The highest weight of any representation of K_0 in $F(\gamma_c)$ must be a K_0-dominant restricted weight of $F(\gamma_c)$, and therefore a wR^+_{res}-dominant restricted weight $\overline{\phi}$. The highest weight of $\widetilde{\tau}$ must therefore be equal to such a weight, plus a weight of S. A weight of S is of the form $w\rho_G - \rho_K - 2\rho(B)$. This is how (1) is proved. Now the formula in (5.1) for the eigenvalue of Ω_K, together with Proposition 4.2(2), leads easily to (2).

Corollary 5.10. *Suppose we are in the setting of Proposition 5.9.*

1. *The eigenvalues of D^2 on $S \otimes F(\gamma_c)$ are less than or equal to the positive number*
$$-\langle \nu_c, \nu_c \rangle.$$
Equality occurs exactly on the representations of $\widetilde{K}_0$ of highest weights
$$w(\lambda_c + \rho_G) - \rho_K \qquad (w \in W^1),$$
with W^1 as in Corollary 3.9.
2. *Each such representation of $\widetilde{K}_0$ has multiplicity*
$$2^{[\ell/2]}, \qquad \ell = \dim A_c.$$
3. *The Hermitian form on $S \otimes F(\gamma_c)$ is definite on each such representation, of sign $\epsilon(w)$ computed in Corollary 4.3(4).*
4. *Define*
$$p_0 = \sum_{w \in W^1, \epsilon(w)=+1} \dim E(w(\lambda_c + \rho_G) - \rho_K),$$
$$q_0 = \sum_{w \in W^1, \epsilon(w)=-1} \dim E(w(\lambda_c + \rho_G) - \rho_K),$$
Then the the signature of the form on the largest eigenspace of D^2 is
$$2^{[\ell/2]}(p_0, q_0).$$

Sketch of Proof. Part (1) is precisely Proposition 5.9, together with Parthasarathy's formula in Proposition 5.8 for D^2. For (2), the proof of Proposition 5.9 shows that the multiplicity of such a representation of $\widetilde{K}_0$ is equal to the multiplicity of a highest weight space of S of weight $\rho_G - \rho_K$. (That was the reason for recalling the proof above.) This last weight multiplicity is computed after (5.7n); it is $2^{[\ell/2]}$. For (3), this same proof shows that the highest weight vector of such a representation is equal to a weight vector in $F(\gamma_c)$ of weight $w\lambda_c$ (which by definition has length a positive multiple of $\epsilon(w)$) tensored with a vector in S (which has positive length). Part (4) just writes (3) explicitly. □

Lemma 5.11. *Suppose that T is a linear operator on a finite-dimensional Hermitian vector space V, self-adjoint with respect to a Hermitian form of signature (P, Q); and suppose that T has purely imaginary eigenvalues.*

1. *For $x \neq 0$, the Hermitian form defines an isomorphism*

$$V_{ix}^h \simeq V_{-ix}.$$

   In particular, the eigenspaces V_{ix} and V_{-ix} have the same dimension $m(x)$, and contribute $(m(x), m(x))$ to the signature.

2. *The Hermitian form has a nondegenerate restriction to the kernel*

$$V_0 = \ker T,$$

   where it has signature (p_1, q_1).

3. *The signatures on V and V_0 satisfy*

$$P - Q = p_1 - q_1.$$

   In particular, the Signature invariant for V is equal to that for V_0:

$$\mathrm{Sig}(V) = |P - Q| = |p_1 - q_1| = \mathrm{Sig}(V_0).$$

Once stated, this result is immediate; what is true is

$$P = p_1 + \sum_{x>0} m(x), \qquad Q = q_1 + \sum_{x>0} m(x).$$

Here at last is the main theorem.

Theorem 5.12. *Suppose in the setting of* (4.1) *and* (4.1c) *that $F(\gamma_c)$ is a finite-dimensional representation of $G(\mathbb{R})$ admitting an invariant Hermitian form $\langle\cdot,\cdot\rangle_{F(\gamma_c)}$; we normalize the form to be positive on the γ_c weight space. Write $2r$ for the number of noncompact imaginary and complex restricted roots of T_c in G:*

$$2r = \dim G/H_c - \dim K/T_c.$$

Then (with notation as in Corollary 5.10)

$$\mathrm{Sig}(F(\gamma_c)) = |p_0 - q_0|/2^r.$$

Proof. The approximate idea is to apply Lemma 5.11 to the Dirac operator D. This is indeed a self-adjoint linear operator on the finite-dimensional Hermitian vector space $F(\gamma_c) \otimes S$. Write (p, q) for the signature of the form on $F(\gamma_c)$; then the form on $F(\gamma_c) \otimes S$ has signature

$$(P, Q) = 2^m(p, q) \qquad (m = [\dim \mathfrak{s}/2]) \tag{5.13a}$$

(see Proposition 5.4). Corollary 5.10 says that the eigenvalues of D^2 are less than or equal to $-\langle \nu_c, \nu_c \rangle$, and that the signature of the form on the largest eigenspace is

$$2^{[\ell/2]}(p_0, q_0). \tag{5.13b}$$

Suppose for a moment that

$$\nu_c = 0. \tag{5.13c}$$

Then the Corollary says that the eigenvalues of D^2 are less than or equal to zero. From this it follows that the eigenvalues of D (as square roots of non-positive real numbers) are purely imaginary. Therefore Lemma 5.11 applies, and tells us that

$$P - Q = 2^{[\ell/2]}(p_0 - q_0). \tag{5.13d}$$

Combining this with (5.13a) gives

$$2^{[\dim \mathfrak{s}/2]}(p - q) = 2^{[\ell/2]}(p_0 - q_0). \tag{5.13e}$$

Because of (5.7m),

$$\dim \mathfrak{s} = 2r + \ell, \tag{5.13f}$$

with r the number of complex and noncompact imaginary positive restricted roots, and ℓ the dimension of A_c. Therefore

$$(p - q) = 2^{-r}(p_0 - q_0), \tag{5.13g}$$

which is precisely the conclusion of the theorem.

So what if $\nu_c \neq 0$? In this case D^2 has at least some strictly positive eigenvalues, meaning that D has some real eigenvalues. The proof of Lemma 5.11 would tell us that we could compute Sig by restricting the form to these real eigenspaces. The largest of these real eigenvalues we understand, but the smaller ones are not easily accessible. So the proof strategy appears to fail.

There are at least two ways out. The simplest is to work not with G but with its commutator subgroup, a semisimple group. We already know that an integral weight (like (λ_c, ν_c)) must take real values on the real span of the coroots. If G is semisimple, this real span of the coroots is

$$i\mathfrak{t}_c(\mathbb{R}) + \mathfrak{a}_c(\mathbb{R}).$$

Therefore the purely imaginary linear functional ν_c on $\mathfrak{a}_c(\mathbb{R})$ must be zero, and we are back in the case (5.13c).

A second (equivalent) method is to use the strongly integral weight

$$\chi = (0, -\nu_c).$$

This weight χ is the differential of a one-dimensional unitary character $\mathbb{C}_\chi$ of $G(\mathbb{R})$, so the signature of $F(\gamma_c)$ is the same as the signature of $F(\gamma_c) \otimes \mathbb{C}_\chi$. This latter representation has highest weight $(\gamma_c, 0)$, so we are again in the case (5.13c).

A third (still equivalent!) method would be to use not the Dirac operator of (5.7e), but one built from $\mathfrak{s}(\mathbb{R}) \cap [\mathfrak{g}, \mathfrak{g}]$. The reason we did not do that is that there is a long history and literature attached to Parthasarathy's Dirac operator; we preferred to use it and to make this extra argument at the end. □

Proof of Theorem 1.2. Now $G = GL(n, \mathbb{R})$, and

$$\mathfrak{s}(\mathbb{R}) = \text{real symmetric matrices.} \tag{5.14a}$$

We will treat the case $n = 2m$ is *even*; the case of odd n is similar but slightly simpler, and we leave it to the reader. The maximal compact torus is

$$T_c(\mathbb{R}) = SO(2)^m, \qquad X^*(T_c) \simeq \mathbb{Z}^m. \tag{5.14b}$$

The restricted root system is (see Table 3)

$$R_{\text{res}} = C_m, \quad R_{\text{cplx}} = D_m, \quad R_{\text{imag}} = A_1^m; \tag{5.14c}$$

all of the imaginary roots are noncompact. We use the standard positive root system

$$R_{\text{res}}^+ = \{e_j \pm e_k \mid 1 \leqslant j < k \leqslant m\} \cup \{2e_j\}. \tag{5.14d}$$

Then we calculate

$$\rho_G = (2m-1, 2m-3, \ldots, 1), \qquad \rho_K = (m-1, m-2, \ldots, 0), \tag{5.14e}$$

$$W^1 = W(A_1) = \{1, s_m\}, \tag{5.14f}$$

with s_m the reflection in the simple root $2e_m$.

The theorem concerns a restricted highest weight

$$\lambda_c = (2\mu_1, \ldots, 2\mu_m) = ((\lambda_1 - \lambda_n), \cdots, (\lambda_m - \lambda_{m+1}))\,. \tag{5.14g}$$

In the theorem we took all the μ_j to be integers, but Proposition 2.6 says that we can allow all the μ_j to be half-integers as well.

By definition $\epsilon(1) = 1$; Corollary 4.3 says that

$$\epsilon(s_m) = (-1)^{2\mu_m} = \begin{cases} 1 & \lambda_j \in \mathbb{Z} \\ -1 & \lambda_j \in \mathbb{Z} + 1/2. \end{cases} \tag{5.14h}$$

The two highest weights of $\widetilde{K}_0$ on the largest eigenspace of D^2 are

$$(2\mu_1, \ldots, 2\mu_{m-1}, \pm 2\mu_m) + (m, \ldots, 2, \pm 1). \tag{5.14i}$$

These two representations of $\mathrm{Spin}(2m)$ differ by the outer automorphism coming from $O(2m)$, so they have the same dimension. The computation of the signature from Theorem 5.12 is therefore

$$\mathrm{Sig}(\pi(\lambda)) = \begin{cases} 2 \cdot \dim E(2\mu_1 + m, \ldots, 2\mu_m + 1)/2^{m^2} & \lambda_j \in \mathbb{Z} \\ 0 & \lambda_j \in \mathbb{Z} + 1/2. \end{cases} \tag{5.14j}$$

The dimension in this formula is calculated by the Weyl dimension formula for D_m; the weight that must be inserted in the formula is the highest weight plus ρ_K, which is

$$(2\mu_1 + 2m - 1, \dots, 2\mu_m + 1). \tag{5.14k}$$

Now the Weyl dimension formula is a *homogeneous* polynomial of degree $m^2 - m$ (the number of positive roots for D_m; so

$$\dim E(k\psi + (k-1)\rho_K) = k^{m^2-m} \dim E(\psi). \tag{5.14l}$$

If we apply this formula with $k = 2$, we get

$$\mathrm{Sig}(\pi(\lambda)) = \begin{cases} 2 \cdot \dim E(\mu_1 + 1/2, \dots, \mu_m + 1/2)/2^m & \lambda_j \in \mathbb{Z} \\ 0 & \lambda_j \in \mathbb{Z} + 1/2. \end{cases} \tag{5.14m}$$

The first formula here is precisely Theorem 1.2 (in case n is even). □

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