Langlands parameters and finite-dimensional representations

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Outline

What Langlands can do for you

Representations of compact Lie groups

Representations of finite Chevalley groups

Representations of *p*-adic maximal compacts

Old reasons for listening to Langlands

GL_n everybody's favorite reductive group/local F.

Want to understand $\widehat{GL_n(F)} =$ set of irr repns (hard).

Classical approach (Harish-Chandra *et alia* 1950s):

1. find big compact subgp $K \subset GL_n(F)$;
2. understand $\widehat{K}$ (supposed to be easy?)
3. understand reps of $GL_n(F)$ restricted to K.

Langlands (1960s) studies $\widehat{GL_n(F)}$ (global reasons).

Global suggests: $\widehat{GL_n(F)} \overset{\approx}{\leftrightsquigarrow}$ n-diml reps of $\mathrm{Gal}(\overline{F}/F)$.

Harris/Taylor prove: $\widehat{GL_n(F)}$. $\overset{\text{bij}}{\leftrightsquigarrow}$ n-diml of Weil-Deligne(F).

Meanwhile (Howe *et alia* 1970s...) continue $GL_n(F)|_K$.

One difficulty (of many): $\widehat{K}$ not so easy after all.

First question: what's Langlands tell us about $\widehat{K}$?

Representations of compact Lie groups

This is introduction number two.

Suppose K is a compact Lie group.

Famous false fact[1] : we understand $\widehat{K}$.

Proof we don't: $O(n) =$ maximal compact in $GL_n(\mathbb{R})$.

Fix irreducible $\tau \in \widehat{O(n)}$.

How do you write down τ? ("Highest weight??")

How do you calculate mult of τ in principal series?

I'll explain: $\widehat{O(n)} \leftrightsquigarrow$ temp irr of $GL_n(\mathbb{R})$/unram twist

$\leftrightsquigarrow$ certain Langlands parameters. . .

Second question: what's Langlands tell us about $\widehat{K}$?
(the big compact subgroup of $GL_n(F)$).

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[1] This alliteration is homage to American politics. Fifty-four forty or fight. Tippecanoe and Tyler too. Totally trust Trump. It's just something we're good at.

Representations of finite Chevalley groups

This is introduction number three.

Suppose G is a reductive group defined over $\mathbb{F}_q$.

Deligne-Lusztig and Lusztig described irr reps of $G(\mathbb{F}_q)$.

Can their results be formulated in spirit of Langlands?

Deligne-Lusztig use ratl max torus $T \subset G$, character

$$\theta\colon T(\mathbb{F}_q) \to \mathbb{C}^\times.$$

Lusztig: $(T, \theta) \rightsquigarrow$ semisimple conj class $x \in {}^\vee G(\mathbb{F}_q)$.

This is a step in the right direction, but not quite a Langlands classification.

Third question: what's Langlands tell us about $\widehat{G(\mathbb{F}_q)}$?

Structure of compact conn Lie grps

K compact connected Lie $\supset$ T maximal torus.

$X^*(T) =_{\text{def}}$ lattice of chars $\lambda\colon T \to S^1 \subset \mathbb{C}^\times$

$X_*(T) =_{\text{def}}$ lattice of cochars $\xi\colon S^1 \to T$.

Adjoint rep of T on cplx Lie algebra decomposes

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \in X^*(T)\backslash\{0\}} \mathfrak{k}_{\mathbb{C},\alpha};$$

$$\rightsquigarrow R = R(K,T) = \text{roots of } T \text{ in } K \subset X^*(T).$$

Each root α gives rise to root TDS

$$\phi_\alpha\colon SU(2) \to K, \quad \operatorname{im} d\phi_\alpha \subset \mathfrak{t} + \mathfrak{k}_{\mathbb{C},\alpha} + \mathfrak{k}_{\mathbb{C},-\alpha}$$

defined up to conjugation by T.

$\phi_\alpha|_{\text{diagonal}} \rightsquigarrow \alpha^\vee\colon S^1 \to K$ coroot for α.

$$\rightsquigarrow R^\vee = R^\vee(K,T) \subset X_*(T),$$

(finite set in bijection with R) coroots of T in K.

We do understand compact conn Lie grps

cpt conn Lie $K \supset T$ max torus $\rightsquigarrow (X^*, R, X_*, R^\vee)$: dual lattices (X^*, X_*), finite subsets $(R, R^\vee)$ in bijection.

Pair $(\alpha, \alpha^\vee) \rightsquigarrow$

$$s_\alpha \colon X^* \to X^*, \quad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \quad s_{\alpha^\vee} = {}^t s_\alpha \colon X_* \to X_*.$$

PROPERTIES: for all $\alpha \in R$

1. RD1: $\langle \alpha, \alpha^\vee \rangle = 2$ (so $s_\alpha^2 = \mathrm{Id}$)
2. RD2: $s_\alpha R = R$, $s_\alpha^\vee R^\vee = R^\vee$, $(s_\alpha \beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$
3. RDreduced: $2\alpha \notin R$, $2\alpha^\vee \notin R^\vee$.

Axioms $\leftrightsquigarrow$ root datum; $W = \langle s_\alpha \mid \alpha \in R \rangle =$ Weyl group.

Root datum is based if we fix $(R^+, R^{\vee,+})$ (pos roots).

Axioms symm in X^*, X_*: $(X_*, R^\vee, X^*, R) =$ dual root datum.

Theorem (Grothendieck)

1. Each root datum $\leftrightsquigarrow$ unique cpt conn Lie grp.
2. $k = \overline{k}$: root datum $\leftrightsquigarrow$ unique conn reductive alg grp $/k$.
3. $k \neq \overline{k}$: red alg grp $/k \rightsquigarrow \mathrm{Gal}(\overline{k}/k) \curvearrowright$ based root datum.

Representations of compact conn Lie grps

Recall cpt conn Lie $K \supset T$ max torus $\rightsquigarrow (X^*, R, X_*, R^\vee)$.

$$(X^*, R, X_*, R^\vee) \rightsquigarrow K_{\mathbb{C}} \quad \text{complex conn reductive alg}$$
$$= \mathrm{Spec}(K\text{-finite functions on } K)$$

$K =$ max compact subgp of $K_{\mathbb{C}}$.

irr reps of K = irr alg reps of $K_{\mathbb{C}} = X^*/W$.

${}^\vee K =_{\text{def}}$ cplx alg group $\leftrightsquigarrow (X_*, R^\vee, X^*, R)$ cplx dual gp.

Theorem (Cartan-Weyl)

1. $\widehat{K} \leftrightarrow \left(\text{homs } \phi_c \colon S^1 \to {}^\vee K\right) / ({}^\vee K\text{-conj}), \quad E(\phi_c) \leftrightarrow \phi_c$.
2. Each side is X^*/W.

Theorem (Zhelobenko) Put $\widehat{K_{\mathbb{C}}}$ = cont ∞-diml irr reps of $K_{\mathbb{C}}$.

1. $\widehat{K_{\mathbb{C}}} \leftrightarrow (\text{homs } \phi \colon \mathbb{C}^\times \to {}^\vee K) / ({}^\vee K\text{-conj}), \quad X(\phi) \leftrightarrow \phi$.
2. $X(\phi)|_K \approx \mathrm{Ind}_T^K(\mathbb{C}_{\phi|_{S^1}})$.
3. $E(\phi|_{S^1}) =$ lowest K-type of $X(\phi)$.

Langlands classification for real groups

G complex reductive alg group, $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$. Fix

inner class of real forms $\sigma =$ action $\Gamma \curvearrowright$ (based root datum).

Definition Cartan involution for σ is inv alg aut θ of G such that $\sigma\theta = \theta\sigma$ is compact real form of G.

inner class of real forms $\sigma =$ inner class of alg invs θ.

Definition L-group for $(G, \{\sigma\})$ is ${}^L G =_{\text{def}} {}^\vee G \rtimes \Gamma$.

Definition Weil grp $W_{\mathbb{R}} = \langle \mathbb{C}^\times, j \rangle$, $1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma \to 1$.

Definition Langlands param $= \Big(\underbrace{\phi \colon W_{\mathbb{R}} \to {}^L G}_{\text{ss image, respect } \Gamma} \Big) /$ conj by ${}^\vee G$.

Theorem (Langlands, Knapp-Zuckerman)

1. Param $\phi \rightsquigarrow$ L-packet $\Pi(\phi)$ of reps $\pi_j \in \widehat{G(\mathbb{R}, \sigma_j)}$.
2. L-packets disjoint; cover all reps of all real forms.
3. $\Pi(\phi)$ indexed by $\widehat{\left({}^\vee G^\phi / {}^\vee G_0^\phi\right)}$.
4. (3) is (slightly& correctably) false: Adams-Barbasch-Vogan.

Langlands classification for real max cpts

G cplx reductive endowed with inner class of real forms $\sigma \leftrightsquigarrow$ inner class of alg invs θ; ${}^L G = L$-group.

$K = G^\theta =$ cplxified max cpt of $G(\mathbb{R}, \sigma)$.

Defn Compact Weil grp $W_{\mathbb{R},c} = \langle S^1, j \rangle$, $1 \to S^1 \to W_{\mathbb{R},c} \to \Gamma \to 1$.

$W_{\mathbb{R},c}$ is not $O(2)$.

Defn Compact param $= \big(\underbrace{\phi_c \colon W_{\mathbb{R},c} \to {}^L G}_{\text{respect } \Gamma}\big)/$ conj by ${}^\vee G$.

Theorem.

1. Param $\phi_c \rightsquigarrow L_c$-pkt $\Pi_c(\phi_c)$ of irr reps μ_j of $K_j = G^{\theta_j}$.
2. L_c-packets disjoint; cover all reps of all $K = G^\theta$.
3. $\Pi_c(\phi_c)$ indexed by $\left({}^\vee G^{\phi_c}/{}^\vee G_0^{\phi_c}\right)\widehat{}$.
4. $\{$lowest K-types of all $\pi \in \Pi(\phi)\} = \Pi_c(\phi|_{W_{\mathbb{R},c}})$.
5. (3) is (correctably) false...

Example of O_{2n}

$G = GL_{2n}(\mathbb{R})$, ${}^L G = GL_{2n}(\mathbb{C}) \times \Gamma$.

Cartan involution is $\theta g = {}^t g^{-1}$, $K = O_{2n}(\mathbb{C})$.

Recall $W_{\mathbb{R},c} = \langle S^1, j\rangle$, $je^{i\theta}j^{-1} = e^{-i\theta}$, $j^2 = -1 \in S^1$.

Theorem says $\widehat{O_{2n}} \leftrightsquigarrow$ $2n$-diml reps of $W_{\mathbb{R},c}$.

Irr reps of $W_{\mathbb{R},c}$ are

1. 1-diml trivial rep $\delta_+(e^{i\theta}) = 1, \quad \delta_+(j) = 1$.
2. 1-diml sign rep $\delta_-(e^{i\theta}) = 1, \quad \delta_-(j) = -1$.
3. For $m > 0$ integer, 2-dimensional representation

$$\tau_m(e^{i\theta}) = \begin{pmatrix} e^{im\theta} & 0 \\ 0 & e^{-im\theta} \end{pmatrix}, \quad \tau_m(j) = \begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix}.$$

$2n$-diml rep $\leftrightsquigarrow$ pos ints $m_1 > \cdots > m_r > 0$, non-neg ints $(a_1, \ldots, a_r, p, q)$ so $2n = 2a_1 + \cdots + 2a_r + p + q$.

Rep is $a_1\tau_{m_1} + \cdots + a_r\tau_{m_r} + p\delta_+ + q\delta_-$.

Highest weight for O_{2n} rep is

$$(\underbrace{m_1+1, \ldots, m_1+1}_{a_1 \text{ times}}, \ldots, \underbrace{m_r+1, \ldots, m_r+1}_{a_r \text{ times}}, \underbrace{1, \ldots, 1}_{\min(p,q)}, \underbrace{0, \ldots, 0}_{|q-p|/2}).$$

Example of U_2

$G = Sp_4(\mathbb{R})$, $K(\mathbb{R}) = U_2$, ${}^L G = SO_5(\mathbb{C}) \times \Gamma$.

Thm says $\widehat{U_n} \leftrightsquigarrow$ 5-diml orth reps of $W_{\mathbb{R},c}$. Irrs:

1. 1-diml trivial rep $\delta_+(e^{i\theta}) = 1, \quad \delta_+(j) = 1$ (orth).
2. 1-diml sign rep $\delta_-(e^{i\theta}) = 1, \quad \delta_-(j) = -1$ (orth).
3. For $m > 0$ integer, 2-dimensional representation

$$\tau_m(e^{i\theta}) = \begin{pmatrix} e^{im\theta} & 0 \\ 0 & e^{-im\theta} \end{pmatrix}, \quad \tau_m(j) = \begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix} \quad \begin{matrix} \text{orth } m \text{ even} \\ \text{sympl } m \text{ odd} \end{matrix}.$$

Here are 5-diml orth reps of $W_{\mathbb{R}} \rightsquigarrow U_2$ highest wts.

$\tau_{2m_1} + \tau_{2m_2} + \delta_+$	$(m_1 > m_2 \geq 1)$	$(m_1+1, m_2+2), (m_1+1, -m_2)$ $(m_2, -m_1-1), (-m_2-2, -m_1-1)$
$\tau_{2m} + \tau_{2m} + \delta_+$	$(m \geq 1)$	$(m+1, -m), (m, -m-1)$
$\tau_{2\ell-1} + \tau_{2\ell-1} + \delta_+$	$(\ell \geq 1)$	$(\ell+1, -\ell-1)$
$4\delta_- + \delta_+$		$(1,1), (-1,-1)$
$2\delta_- + 3\delta_+$		$(1,0), (0,-1)$
$5\delta_+$		$(0,0)$

Finite Chevalley groups

$k = \mathbb{F}_q$ finite field; $\Gamma = \mathrm{Gal}(\overline{k}/k) = \varprojlim_m \mathbb{Z}/m\mathbb{Z}$.

Generator is arith Frobenius Frob $=$ qth power map.

k-ratl form of conn reductive alg G $=$ action of Γ on based root datum $=$ fin order aut.

Definition L-group for G/k is ${}^L G =_{\mathrm{def}} {}^\vee G \rtimes \Gamma$.

Here ${}^\vee G$ taken over $\mathbb{C}$, or $\overline{\mathbb{Q}}_\ell$, or. . . : field for repns.

Def (MacDonald) Weil grp $W_k = \varprojlim_m \mathbb{F}_{q^m}^\times$; $W_k \to \Gamma$ trivial.

Def (MacDonald) Langlands param $= \Big(\underbrace{\rho\colon W_k \to {}^\vee G}_{\text{respect }\Gamma}\Big)/{}^\vee G$ conj.

$\rho(W_k) \subset {}^\vee G$ (not ${}^L G$) since $W_k \to 1 \in \Gamma$.

Respect Γ $=$ exists $f \in {}^L G$ mapping to Frob, $\mathrm{Ad}(f)\phi(\gamma) = \rho(\mathrm{Frob}\,\gamma)$.
KEEP COSET $f\,{}^\vee G_0^{\rho_c}$ as part of ρ_c.

Deligne-Langlands param $\phi = \Big((\rho_\phi, N_\phi)\ \ (N \in {}^\vee\mathfrak{g}^{\rho_\phi}, \mathrm{Ad}(f)N = qN)\Big)$.

Langlands parameters for $\mathbb{F}_q$

$G \supset B \supset T$ conn red alg $/\mathbb{F}_q$, Frob: $G \to G$ Frobenius.

Get Γ action on W permuting gens $\rightsquigarrow {}^\Gamma W = W \rtimes \Gamma$

$\tilde{w} = w\,\mathrm{Frob}$ (another) Frobenius morphism $T \to T$.

Deligne-Lusztig built chars of $G(\mathbb{F}_q)$ from virt chars $R_{\theta'}^{T'}$:
T' ratl maxl torus, θ' char of $T'(\mathbb{F}_q)$.

Proposition. For any rational = Frob-stable max torus $T' \subset G$, $\exists!$ W-conj class of $\tilde{w}$ so $(T', \mathrm{Frob}) \simeq (T, \tilde{w})$.

Prop (Macdonald) $\widehat{T^{\tilde{w}}} \simeq \{\rho\colon W_k \to {}^\vee T \mid w\,\mathrm{Frob}\,\phi(\gamma) = \phi(\mathrm{Frob}\,\gamma)\}$.

Conclusion: L-params ρ' for G = DL-pairs (T', θ').

$R_{\theta'}^{T'}$ and $R_{\theta''}^{T''}$ overlap $\iff$ ρ', ρ'' ${}^\vee G$-conjugate.

$\widehat{G(\mathbb{F}_q)}$ partitioned by Langlands parameters.

So far this is Deligne-Lusztig 1976: (relatively) easy.

Using Deligne-Langlands params to shrink L-pkts harder. . .

Lusztig's big orange book

F finite $\rightsquigarrow \mathcal{S}(F) =_{\text{def}} \{(f,\sigma) | f \in F, \sigma \in \widehat{F^f}\}/(\text{conj by } F)$.

$$\mathcal{S}((\mathbb{Z}/2\mathbb{Z})^n) = (\mathbb{Z}/2\mathbb{Z})^n \times (\widehat{\mathbb{Z}/2\mathbb{Z}})^n.$$

$\mathcal{S}(S_3) = \left\{(1,\mathbb{C}), (1,\text{refl}), (1,\text{sgn}), (s_2,\mathbb{C}), (s_2,\text{sgn}), (s_3,\mathbb{C}), (s_3,\omega), (s_3,\omega^2)\right\}$.

$G \supset B \supset T$ conn red alg $/\mathbb{F}_q$, ${}^L G$ L-group.

Def $\phi = (\rho, N)$ special if $N \in {}^\vee\mathfrak{g}^\rho$ is special nilp.

Recall that ϕ remembers coset $f\,{}^\vee G_0^{\rho,N}$.

Theorem (Lusztig). Irreducible reps of $G(\mathbb{F}_q)$ are partitioned into packets $\Pi(\phi)$ by special DL parameters ϕ. The packet $\Pi(\phi)$ is indexed by $\mathcal{S}({}^\vee G^\phi/{}^\vee G_0^\phi)$ using Lusztig quotient of ${}^\vee G^\phi/{}^\vee G_0^\phi$.

To make this look like other Langlands classifications, prefer to drop requirement N special, replace Lusztig quotient by ${}^\vee G^\phi/{}^\vee G_0^\phi$, replace $\mathcal{S}(F)$ by subset $\widehat{F}$.

1st two prefs $\rightsquigarrow$ more params, 3rd $\rightsquigarrow$ fewer params.

Rewriting Lusztig's orange book by Langlands

Deligne-Langlands param for $G(\mathbb{F}_q)$ is

$$\phi_L = (\rho, N, \bar{f}),$$

1. $\rho\colon W_k \to {}^\vee G$ semisimple, $N \in {}^\vee\mathfrak{g}^\phi$ nilpotent,
2. $\bar{f} = f\left({}^\vee G^{\phi,N}\right)$, $f \in {}^L G \to \text{Frob}$
3. $\mathrm{Ad}(f)(\rho(w)) = \rho(w^q)$, $\mathrm{Ad}(f)(N) = qN$

complete geom Deligne-Langlands param has also

4. $\xi \in \widehat{{}^\vee G^{\phi,N}/{}^\vee G_0^{\rho,N}}$, $\xi|_{Z({}^\vee G)} = 1$.

Conjecture Irreducible reps of $G(\mathbb{F}_q)$ partitioned into packets $\Pi_L(\phi_L)$ by all Deligne-Langlands parameters ϕ_L. Packet $\Pi_L(\phi_L)$ indexed by irr reps ξ of ${}^\vee G^{\phi,N}/{}^\vee G_0^{\rho,N} Z({}^\vee G)$.

Lifting finite to p-adic

$G \supset B \supset T$ conn red alg $/k = \mathbb{F}_q$.

Fix p-adic $F \supset \mathcal{O} \supset \mathcal{P}$, $\quad \mathcal{O}/\mathcal{P} \simeq k$.

$\Gamma_F = \mathrm{Gal}\,\overline{F}/F; \quad 1 \to I_F \to \Gamma_F \to \Gamma_k \to 1$.

Weil group of F is preimage of $\mathbb{Z} = \langle \mathrm{Frob} \rangle$, so

$$1 \to I_F \to W_F \to \langle \mathrm{Frob} \rangle \to 1.$$

Set $P_F =$ wild ramif grp $\subset I_F$; then $I_F/P_F \simeq W_k$.

Fix p-adic $\mathbb{G} \leftrightsquigarrow$ based root datum of G/k, Γ_F acts via Γ_k.

G/k and $\mathbb{G}/F$ have same L-group ${}^L G$.

Prop L-params for $G/k =$ (tamely ramif params for $\mathbb{G}/F)|_{I_F}$.

Def cpt Weil grp $W_{F,c} =$ inertia subgroup I_F.

Def cpt param is $\rho_c \colon I_F \to {}^L G$ s.t. $\exists$ extn to L-param.

Extension to cpt Deligne-Langlands params $\phi_c = (\rho_c, N)$ easy.

Wild conjectures

$G/\overline{F}$ conn reduc alg, inner class of F-forms σ.

$\{K_j(\sigma)\}$ maxl cpt subgps of $G(F,\sigma)$.

${}^LG = {}^\vee G \rtimes \Gamma_F$ L-group for $(G, \{\sigma\})$.

Conjecture

1. Cpt DL param $\phi_c \rightsquigarrow L_c$-pkt of irr reps $\mu_j(\sigma)$ of $K_j(\sigma)$.
2. L_c packets are disjoint.
3. ϕ any ext of $\phi_c \rightsquigarrow \Pi_c(\phi_c) = \{\text{LKTs of all } \pi \in \Pi(\phi)\}$.
4. $\Pi_c(\phi_c)$ indexed by $\left({}^\vee G^{\phi_c}/{}^\vee G_0^{\phi_c}\right)\widehat{}$.
5. $\bigcup \Pi_c(\phi_c) =$ all irrs $\supset$ Bushnell-Kutzko type.

NOTE: some $K_j \twoheadrightarrow G_j(\mathbb{F}_q)$, G_j smaller than G.

Corr reps should correspond to non-special N, etc.

Chance that this is formulated properly is near zero.

I know this because I once taught Bayesian inference.

Hope that it's wrong in interesting ways.