Unitary representations and bottom layer K-types

David Vogan

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the Unitary Dual Problem
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Outline

Introduction

$SO(4,1)$

All kinds of induction

What's the unitary dual look like?

G real reductive Lie $\supset$ K maximal compact.

Assume $G =$ real pts of conn reductive cplx algebraic group.

Want to describe $\widehat{G}_u =$ unitary dual: equiv classes of irreducible unitary representations. This is hard.

This is hard.

Harish-Chandra: larger set $\widehat{G}_a =$ adm dual easier.

HC, Langlands: $\widehat{G}_a$ parametrized by countable union of (ratl vec space)$\otimes_{\mathbb{Q}}\mathbb{C}$/(finite group)

Describing $\widehat{G}_u$ means describing subset of each $E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}/F$.

Admissible rep X is unitary if

1. admits non-zero G-invt Hermitian form $\langle , \rangle_X$, and
2. form $\langle , \rangle_X$ is definite.

What's the unitary dual look like II?

$$\widehat{G}_a \leftrightsquigarrow \bigcup_\delta [E(\delta)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}]/F(\delta).$$

Admissible repn $X(\delta,\nu)$ is unitary if and only if (1) Hermitian, and (2) form is definite.

Knapp-Zuckerman: cond (1) $\leftrightsquigarrow$ "real points" of $\widehat{G}_a$:

$$X(\delta,\nu) \text{ Herm } \iff \exists f \in F(\delta), \quad -\overline{\nu} = f \cdot \nu.$$

Easy unitary: $f = 1$, ν pure imag, $X(\delta,\nu)$ tempered.

Fairly easy unitary: if ν Herm, nonzero imag part, then $X(\delta,\nu)$ unitarily induced from proper $P = LN$.

Imag ν: all unitary. Nonreal ν: unitarity settled on smaller L.

Hard unitary: $\nu \in E(\delta)_{\mathbb{R}}$ real, $f \cdot \nu = -\nu$.

Theorem. For each δ, hard unitary ν are compact rational polyhedron $C_u(\delta) \subset E(\delta)_{\mathbb{R}}$.

What `atlas` does: $\nu \in E(\delta)_{\mathbb{Q}} \rightsquigarrow$ is $X(\delta,\nu)$ unitary?

This oracle determines any one $C_u(\delta)$ by a finite calculation.

Subject of this talk

Unitary dual determined by knowledge of countably many compact rational polyhedra $C_u(\delta) \subset E(\delta)_{\mathbb{R}}$

Each $C_u(\delta)$ computable in finite time $\rightsquigarrow$ unitary dual of one G computable in countably infinite time.

This is good but not good enough.

Subject today: find conditions for

$$C_u(\delta) = C_u(\delta'), \qquad C_u(G,\delta) = C_u(L,\delta_L)$$

with L proper reductive subgroup.

Know since 1980s: there are enough such equalities to make $\widehat{G}_u$ computation finite.

Goal: sharpen to make computation feasible.

What's that look like?

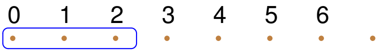
$G = SO(4,1)$; $\widehat{G}_u \subset \widehat{G}_a$ found 1962 by Takeshi Hirai.

δ	classical	atlas	$E_{\mathbb{Q}}$	$C_u(\delta)$
$(n+3/2, m+1/2)$ $n \geq m \geq 0$	disc ser	$(0,[n+3/2, m+1/2])$	0	0
dim $2n+1$ $O(3)$ rep ($n \geq 1$)	princ ser	$(1,[n+1/2, \pm 1/2])$	$\mathbb{Q}$	$[-\frac{1}{2}, \frac{1}{2}]$
dim 1 $O(3)$ rep	spherical princ ser	$(1,[1/2, \pm 1/2])$	$\mathbb{Q}$	$[-\frac{3}{2}, \frac{3}{2}]$

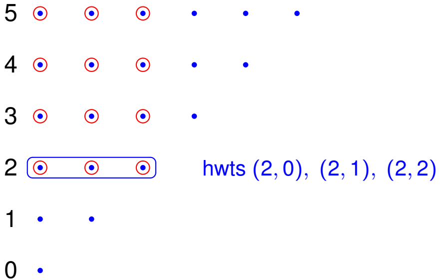
$C_u(\delta) = [-\frac{1}{2}, \frac{1}{2}] \rightsquigarrow$ Bargmann comp ser of $SO(2) \times SO(2,1)$.

```
atlas> set G=SO(4,1)
atlas> set q=parameter(KGB(G)[1],[2+1/2,1/2],[0,1/2])
   {n=2,  nu=1/2}
atlas> is_unitary(q)
Value:  true
atlas> set r=parameter(KGB(G)[1],[2+1/2,1/2],[0,1])
   {n=2,  nu=1}
atlas> is_unitary(r)
Value:  false
```

Where's the bottom layer?



Restriction to $O(2)$ of $SO(2,1)$ princ series



Restriction to $K = O(4)$ of $SO(4,1)$ princ series

Bottom layer for this princ series is three $O(4)$ reps.

Mults and sigs match $O(2)$ reps in $SO(2,1)$ princ series.

I interchanged x and y axes in the diagram above. Fixing that in `picture` is probably beyond my skills, certainly beyond my patience.

Signatures on the bottom layer

0	1	2	3	4	5	6
+	−	+	−	−	−	−

Signature of $SO(2,1)$ sph princ series, $\nu = 3$

5	−	+	−	•	•	•
4	−	+	−	•	•	
3	−	+	−	•		
2	+	−	+			
1	•	•				
0	•					

sig on higher layers
unrelated to bottom layer

hwts $(2,0)$, $(2,1)$, $(2,2)$

Signature of $SO(4,1)$ princ series $\delta = 5$-diml, $\nu = 3$

sig in sph series for $SO(2,1)$ neg on 1, $\nu > 1/2$.

Means 1 is nonunitarity certificate.

$\rightsquigarrow$ sig in $\delta = 5$ series for $SO(4,1)$ neg on $(2,1)$, $\nu > 1/2$.

Means $(2,1)$ is nonunitarity certificate.

Unitary dual of $SO(4,1)$

Know in advance about $SO(2,1)$ spherical series $J(\nu)$:

1. C_u(spherical) $= [-1/2, 1/2]$.
2. Lowest $O(2)$-type of any $J(\nu)$ is $\mu(0)$, form pos there.
3. If $|\nu| > 1/2$, form neg on $\mu(1)$; nonunitarity certif.

Deduce about $SO(4,1)$ princ series $J(n,\nu)$:

1. $C_u(n) \supset [-1/2, 1/2]$.
2. Lowest $O(4)$-type of $J(n,\nu)$ is $\mu(n,0)$, form pos there.
3. If $|\nu| > 1/2$, $J(n,\nu)$ form neg on $\mu(n,1)$
   if $(n,1)$ is highest wt for $O(4)$;
   that is, if $n \geq 1$: nonunitarity certif.

Remains to calculate spherical comp series $C_u(0)$:

```
atlas> set G=SO(4,1)       (value of ν)
atlas> set p=parameter(KGB(G)[1],[1/2,1/2],[0,2])
atlas> is_unitary(p)
Value:  false      (so ν = 2 excluded from C_u(0))
atlas> is_unitary(p*(3/4))
Value:  true      (so ν = 3/2 included in C_u(0))
atlas> is_unitary(p*(1/2))
Value:  true      (so C_u(0) = [−3/2,3/2]).
```

Calculation gives nonunitarity certif $\mu(1,0)$ for $|\nu| > 3/2$.

Induction, schminduction

Gelfand, Mackey and a host of glamorous costars invented parabolic induction.

$\mathbb{R}$-alg $P \subset G$ called parabolic if $G(\mathbb{C})/P(\mathbb{C})$ projective.

Then $P = LU$ with U conn unip, L reductive; any $\pi_L \in \widehat{L}_a$ extends (triv on U) to P, defines finite length

$$\pi_G = \mathrm{Ind}_P^G(\pi_L).$$

Ind: unitary $\to$ unitary, depends on L, not P.

Relates nicely to maximal compact K:

$$\left(\mathrm{Ind}_P^G(\pi_L)\right)\Big|_K = \mathrm{Ind}_{P\cap K}^K(\pi_L|_{P\cap K}).$$

But this is not general enough.

Zuckerman and a host of glamorous costars invented cohomological parabolic induction.

Write $\mathfrak{g}$ for cplx Lie alg of G, $\theta =$ Cartan involution.