LAST TIME:

If $A \subseteq B$ is an inclusion of groups with compatible P-action, we have an exact sequence

$$0 \to H^0(P, A) \to H^0(P, B) \to (B/A)^P \to H^1(P, A) \to H^1(P, B)$$

B^P orbits on $(B/A)^P \overset{1-1}{\longleftrightarrow} \ker\{H^1(P,A) \to H^1(P,B)\}$.

$G = GL(n, \mathbb{R})$, count max'l tori

Borovoi: $\underset{\text{fundamental Cartan}}{H_f} = Z(\underset{\text{maxl compact torus}}{T})$

$H^1(P, G) \simeq H^1(P, H_f)/W_f$

$|H^1(P, H_f)| = 2^{\# \text{ of } SO(2) \text{ factors}} = 2^c$

$H_f(\mathbb{R}) = (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b \times \underline{\underline{(SO(2))^c}}$

$GL(n, \mathbb{R}) \leadsto H_f \simeq (\mathbb{C}^\times)^{\lfloor \frac{n}{2} \rfloor} \times (\mathbb{R}^\times)^{n - 2\lfloor \frac{n}{2} \rfloor}$

$|H^1(P, GL_n)| = 1$

To compute # of maxl tori (under $GL_n(\mathbb{R})$ conjugacy),

fix a real maxl torus H

$$0 \to H^0(P, N(H)) \to H^0(P, GL_n) \to \underbrace{(GL_n/N(H))^P}_{GL_n(\mathbb{R})\text{-orbits} \overset{1-1}{\longleftrightarrow} H^1(P, N(H))} \to H^1(P, N(H)) \to \underline{\underline{H^1(P, GL_n)}}.$$

$H = H_s$ split max'l torus

$N(H_s) = S_n \ltimes (\mathbb{C}^\times)^n$

σ acts on $N(H_s)$ by conjugating the $\mathbb{C}^\times$ factors and acts trivially on S_n.

Computation $\Rightarrow$ $H^1(P, S_n \ltimes (\mathbb{C}^\times)^n)$

$\updownarrow$

involutions in S_n up to S_n-conjugacy

1 involution for each integer

$0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$

$\swarrow \quad \downarrow \quad \searrow$

$\mathbb{R}^n \quad \mathbb{R}^{n-2} \times \mathbb{C} \quad (\mathbb{C}^\times)^{\lfloor \frac{n}{2} \rfloor} \times (\mathbb{R}^\times)^{0 \text{ or } 1}$

$\underset{G}{\overset{\|}{U(p,q)}}$ and nilpotent orbits $(n = p+q)$

H_f = diagonal matrices $= (U(1))^n$

Borovoi: $H^1(P, H_f)$ = elts of order 2 in $(U(1))^n$

$H^1(P, G) \simeq H^1(P, H_f)/W_f$

$W_f = \dfrac{N(T)}{H_f} = S_n$

If $x \in H^1(P, H_f)$, $\alpha \in S_n = W_f$,

$\alpha \bullet x = g^{-1} x \bar{g}$ where $g \in N(T)$, $g \mapsto \alpha$, $N(T) \to W_f$: $\alpha \in W_f$

g = permutation matrix

$\bar{g} = \mathrm{Ad}\left(\begin{bmatrix} I_p & \\ & -I_q \end{bmatrix}\right)(g)$

$e_1, e_2, \ldots, e_n$ standard basis

$g(e_i) \to e_{\rho(i)}$

$\bar{g}(e_i) = e_{\rho(i)} \cdot (-1)^{i \in [p+1, q]} \cdot (-1)^{\rho(i) \in [p+1, q]}$

$g^{-1} x \bar{g}(e_i) = e_i \cdot x_{\rho(i)} (-1)^{i \in [p+1, q]} (-1)^{\rho(i) \in [p+1, q]}$

$H^1(P, H_f)/S_p \times S_q \sim (a, b)$

a: # of $-1 \in U(1)$ in first p

b: # of $-1 \in U(1)$ last q

$H^1(P, H_f)$ = elts of order 2 in $(SO(2))^c = U(1)^n$

$(\underbrace{\pm 1, \pm 1, \ldots}_{p} \mid \underbrace{\ldots, \pm 1}_{q})$

$S_p \qquad S_q$

$\downarrow$

$(+1, +1, \ldots +1, -1, -1, \ldots -1 \mid +1, +1, \ldots +1, -1, -1)$

$[n] = [1, \ldots, p] \sqcup [p+1, \ldots, p+q]$

$\downarrow \qquad \downarrow$

$A_p \sqcup B_p \qquad A_q \sqcup B_q \qquad |B_p| = |B_q|$

$\downarrow \qquad \downarrow$

$A_p \sqcup B_q \qquad A_q \sqcup B_p$

$\downarrow$ permute $\qquad \downarrow$ permute

$[\rho(1), \rho(2), \ldots \qquad \rho(n)]$

Counts # of -1's

- Doesn't change # of (-1)'s
- Adds k -1 to both first p or the last q
- Removes k -1 to both the first p and the last q

$$H^1(P, G) \sim \left\{ \begin{array}{l} \# \text{ of } -1\text{'s in first } q \\ - \# \text{ of } -1\text{'s in last } p \end{array} \right\}$$

$\|$

$[-p, -p+1, \ldots, q-1, q]$

$n+1$ elements in $H^1(P, H_f)$

$U(n, 0) \qquad H^1(P, U(n)) \sim [0, 1, \ldots n]$

$0 \mapsto U(n)$, $1 \mapsto U(n-1, 1)$, $n \mapsto U(0, n)$

Nilpotent orbits $\longleftrightarrow$ $\mathfrak{sl}_2$-triple in $\mathfrak{u}(p,q)$

(*) Compute the centralizer of $\mathfrak{sl}_2(\mathbb{R})$-triple

$\mathbb{C}^n \cong \bigoplus V_i$ where V_i is a direct sum of copies of the $(i+1)$-dim irred of $\mathfrak{sl}_2(\mathbb{R})$

Collingwood-McGovern: On each h.w space $V_i[i]$, get (skew)-Hermitian form by

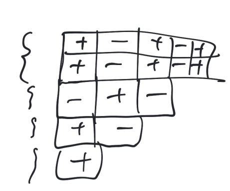
$(x, y)_i = \langle x, Y^i y \rangle$ for $x, y \in V_i[i]$

and $\mathfrak{sl}_2(\mathbb{R}) = \langle X, H, Y \rangle$

$\Rightarrow$ Any intertwining operator for $\mathfrak{sl}_2(\mathbb{R})$ in $\mathfrak{u}(p,q)$ preserves these $(\,,\,)_i$ forms

Centralizer $= \prod U(a_i, b_i)$ $\qquad a_i + b_i$ = # of irred in V_i

$H^1(P, \text{Centralizer}) = \prod_i \underline{\underline{(\# \text{ of irred in } V_i + 1)}}$



$V_4 = L_4 \oplus L_4$

$V_2 = L_2$

$V_1 = L_1$

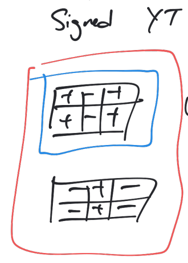
$V_0 = L_0$

$H^1(P, \text{Centralizer}) \longleftrightarrow$ Signed Young Tableaux with fixed shape

$$0 \to H^0(P, \text{Centralizer}) \to H^0(P, G) \to \boxed{H^0(P, G/\text{centralizer})} \to \boxed{H^1(P, \text{Centralizer})} \to \underbrace{H^1(P, G)}_{\text{(pure) real forms } \mathfrak{u}(p,q)}$$

$H^1(P, \text{Centralizer})$ = Signed YT

Sends Signed YT to $\mathfrak{u}(p,q)$ with p = # of $+$'s, q = # of -1's



$\underline{U(4,2)}$

$U(2,4)$