Involutions

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These are notes on several topics related to certain representation of the Weyl group. These include the coherent continuation representation and the representation of W on involutions studied in [4] and [3].

We fix a connected complex reductive group $G = G(\mathbb{C})$, with Weyl group W. At various points we will also assume G is defined over $\mathbb{R}$, or equivalently we are given an algebraic involutive automorphism of G.

Let $\mathcal{G} = \mathrm{Lie}(G)$, and if G is defined over $\mathbb{R}$, define $G(\mathbb{R})$ and $\mathfrak{g}_0 = \mathrm{Lie}(G(\mathbb{R}))$. Let $\mathcal{N} = \mathcal{N}(G)$ be the set of nilpotent orbits of $G = G(\mathbb{C})$ acting on $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Suppose $\mathcal{O} \in \mathcal{N}(G)$. Let $\mathcal{D}(\mathcal{O})$ be the space spanned by the orbital integrals of $G(\mathbb{R})$-orbits in $\mathcal{O} \cap G(\mathbb{R})$. This is a space of tempered, invariant distributions supported on $\mathcal{O}$.

We have the following objects.

$$\begin{aligned}
G_X &= \mathrm{Stab}_G(X)\\
A(\mathcal{O}) &= G_X/(G_X)^0\\
\overline{A}(\mathcal{O}) &= \text{ Lusztig's canonical quotient of } A(\mathcal{O}) \quad (\mathcal{O} \text{ special})\\
H^1(\Gamma, A(\mathcal{O})), H^1(\Gamma, \overline{A}(\mathcal{O})) &= \text{Galois cohomology spaces}\\
\overline{A}(\mathcal{O})_2 &= \{g \in \overline{A}(\mathcal{O}) \mid g^2 = 1\}\\
[\overline{A}(\mathcal{O})_2] &= \text{conjugacy classes in } \overline{A}(\mathcal{O})_2\\
r(\mathcal{O}) &= \dim(D(\mathcal{O}))\\
&= |\mathcal{O} \cap \mathfrak{g}_0/G(\mathbb{R})|\\
D^{\mathrm{st}}(\mathcal{O}) &= \text{subspace of } D(\mathcal{O}) \text{ consting of stable distributions}\\
s(\mathcal{O}) &= \dim(D^{\mathrm{st}}(\mathcal{O}))
\end{aligned}$$

0.1 Some Group Cohomology

Suppose there is an action of $\mathbb{Z}/2\mathbb{Z}$ on a (possibly non-abelian) group G, and write τ for the action of the non-trivial element. Then

$$\begin{aligned}
H^0(\mathbb{Z}/2\mathbb{Z}, G) &= G^\tau\\
H^1(\mathbb{Z}/2\mathbb{Z}, G) &= \{g \mid g\tau(g) = 1\}/[g \sim xg\tau(x^{-1})]
\end{aligned}$$

In particular if the action is trivial then

$$H^1(\mathbb{Z}/2\mathbb{Z}, G) = \{g \mid g^2 = 1\}/\text{conjugacy} = [G_2]$$

Thus if Γ acts triviall y on $\overline{A}(\mathcal{O})$ ($\mathcal{O}$ is special) so

$$H^1(\Gamma, \overline{A}(\mathcal{O})) \leftrightarrow [\overline{A}(\mathcal{O})_2]$$

According to [5, Section 1.9] this holds if G is simple, quasi-split and classical.

By general Galois cohomology arguments

$$(\mathcal{O} \cap \mathfrak{g}_0)/G(\mathbb{R}) \longleftrightarrow \text{ ker: } H^1(\Gamma, G_X) \to H^1(\Gamma, G)$$

1 The space $\mathcal{M}(\overline{A}(\mathcal{O}))$

Now let G be a finite group. For $x \in G$ let $G_x = \text{Cent}_G(x)$, and let $\widehat{G_x}$ be the set of irreducible representations of G_x.

Define:

$$\mathcal{M}(G) = \{(x, \xi) \mid x \in G, \xi \in \widehat{G_x}\}/G$$

where the quotient is by the natural action of G on the pairs. This set has a basepoint $(1, 1)$.

For $x \in G$ define

$$S_x = \{g \in G_x \mid g^2 = x\}/G_x$$

(the quotient is by the conjugation action of G_x). For $x \in G, y \in G_x$ set

$$G_{x,y} = \text{Cent}_{G_x}(y) = G_x \cap G_y.$$

We define a function $\hat{\phi}_0$ on $\mathcal{M}(G)$:

$$\hat{\phi}_0(x, \xi) = \sum_{s \in S_x} \dim(\xi^{G_{x,s}})$$

Let's unwind this. On the left hand side $x \in G, \xi \in \widehat{G_x}$. On the right, each s is a G_x-conjugacy class in G_x. If $t \in s$ then $G_{x,t} = \text{Cent}_{G_x}(t)$, and $\xi^{G_{x,t}}$ is the set of fixed points of this group acting on ξ. If t' is another element of s, then $t' = hth^{-1}$ for some $h \in G_x$. Then $G_{x,t'} = h(G_{x,t})h^{-1}$, and $\xi^{G_{x,t'}} = \xi(h)\xi^{G_{x,t}}$. Therefore the dimensions of these spaces are the same, which we've written as $\dim(\xi^{G_{x,s}})$.

For example consider $\widehat{\phi}_0(1, 1)$. Then $S_x = \{g \in G \mid g^2 = 1\}$, and each dimension term is 1, so

$$\widehat{\phi}_0(1, 1) = |\{g \in G \mid g^2 = 1\}|/G,$$

the number of conjugacy classes of elements of order 1 or 2.

Suppose G is an elementary two-group. Then for all $x \in G$, $S_x = G$, and for all $s \in S_x$, $G_{x,s} = G$. Therefore

$$\widehat{\phi}_0(x,\xi) = \sum_{s\in G} \dim(\xi^G) = \begin{cases} |G| & \xi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

Suppose $\mathcal{O}$ is a nilpotent orbit, and $\phi \in \widehat{A(\mathcal{O})}$. Write Springer$(\mathcal{O},\phi) \in \widehat{W}$ for the irreducible representation of $\mathcal{O}$ given by the Springer correspondence. This is a bijection from $\{(\mathcal{O},\phi) \mid \text{Springer}(\mathcal{O},\phi) \neq 0\}$ to $\widehat{W}$ (and the number of excluded pairs $(\mathcal{O},\phi)$ is small, and often 0). The special representations of W are the representations Springer$(\mathcal{O},1)$ where $\mathcal{O}$ is special.

Write d for duality of nilpotent orbits.

Definition 1.1 *We define the* special piece *of a nilpotent orbit $\mathcal{O}$:*

$$\mathcal{SP}(\mathcal{O}) = \{\mathcal{O}' \mid d(\mathcal{O}') = d(\mathcal{O})\}$$

Define the special support *of $\mathcal{O}$, denoted $\mathcal{O}_s$ to be the unique special orbit in $\mathcal{SP}(\mathcal{O})$. In other words, $\mathcal{O}_s = d^2(\mathcal{O})$.*

Thus $\mathcal{SP}(\mathcal{O})$ consists of a single special orbit $\mathcal{O}_s$, (the special support of $\mathcal{O}$) and all of the orbits $\mathcal{O}'$ (including $\mathcal{O}$), contained in the closure of $\mathcal{O}_s$ but in the closure of no smaller special orbit.

Definition 1.2 *Set*

$$\mathcal{W}(G) = \{(\mathcal{O}, m) \mid \mathcal{O} \in \mathcal{N}(G), m \in \mathcal{M}(\overline{A}(\mathcal{O}))\}$$

Thus m is a G-orbit of pairs (x,ξ) with $x \in \overline{A}(\mathcal{O})$ and $\xi \in \widehat{Cent_{\overline{A}(\mathcal{O})}}(x)$.

The notation $\mathcal{W}$ is intended to suggest something to do with the Weyl group, any suggestions for a better name here?

Lusztig defines a map

$$\Psi : \widehat{W} \to \mathcal{W}(G). \tag{1.3}$$

It satisfies the following properties. Recall if $\mathcal{O} \in \mathcal{N}$ then $\mathcal{O}_s = d^2(\mathcal{O})$ is the special support of $\mathcal{O}$.

(1) Ψ is injective.

(2) $\Psi(\text{Springer}(\mathcal{O},\phi)) = (\mathcal{O}_s, m)$ where $\mathcal{O}_s = d^2(\mathcal{O})$ is the special support of $\mathcal{O}$, for some $m \in \mathcal{M}(\overline{A}(\mathcal{O}))$. In particular the orbits occuring in $\mathcal{W}(G)$ are all special.

(3) $\Psi(\text{Springer}(\mathcal{O},1)) = (\mathcal{O}_s, (x,1))$ for some $x \in [\overline{A}(\mathcal{O}_s)]$.

(4) If $\mathcal{O}$ is special $\Psi(\text{Springer}(\mathcal{O},1)) = (\mathcal{O},(1,1))$, and Ψ restricts to a bijection from the special representations in $\widehat{W}$ to $\{(\mathcal{O},(1,1)\}$ where $\mathcal{O}$ is special. The inverse map is $(\mathcal{O},(1,1)\} \mapsto \text{Springer}(\mathcal{O})$.

(5) Fix a special orbit $\mathcal{O}$, and set

$$\mathcal{C}(\mathcal{O}) = \{\Psi^{-1}(\mathcal{O},m) \mid m \in \mathcal{M}(\overline{A}(\mathcal{O}))\}$$

Then $\mathcal{C}(\mathcal{O})$ is a two-sided cell, and all two-sided cells arise this way. Thus the two-sided cells are parametrized by special orbits, and $\mathcal{C}(\mathcal{O}) \hookrightarrow \mathcal{M}(\overline{A}(\mathcal{O}))$. Each two-sided cell contains a unique special representation σ, which satisfies $\Psi(\sigma) = (\mathcal{O},(1,1)$.

(6) Suppose $\mathcal{O}$ is special. By (3) if $\mathcal{O}' \in \mathcal{SP}(\mathcal{O})$ then $\Psi(\text{Springer}(\mathcal{O}',1)) = (\mathcal{O},(x,1))$ for some $x \in [\overline{A}(\mathcal{O})]$. The map $\mathcal{O}' \mapsto x$ injective:

$$\mathcal{SP}(\mathcal{O}) \hookrightarrow [\overline{A}(\mathcal{O})]. \tag{1.4}$$

(7) Suppose $\mathcal{O}$ is special and $d(\mathcal{O})$ is even. Then the map (1.4) is a bijection

$$\mathcal{SP}(\mathcal{O}) \longleftrightarrow [\overline{A}(\mathcal{O})]$$

(8) If $\mathcal{O}$ is special then every element $(\mathcal{O},(x,1)) \in \mathcal{W}(G)$ is in the image of Ψ. This gives a set $\{\sigma \in \widehat{W} \mid \Psi(\sigma) = (\mathcal{O},(x,1))$, parametrized by $[\overline{A}(\mathcal{O})]$. This is a Lusztig cell (see below).

This result is assembled from various sources. Here are some references and explanations.

The map Ψ is defined in [7], Sections 4.4-4.13, and Properties (1)-(5) are part of the definitions. Assertion (6) is part of [6, Theorem 0.4].

If $G = E_8$ and $\mathcal{O} = E_8(a_4)$ then $\overline{A}(\mathcal{O}) = \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{SP}(\mathcal{O}) = \mathcal{O}$. This is an example where $\mathcal{SP}(\mathcal{O})$ embeds in $\overline{A}(\mathcal{O})$, but not surjectively. Note that $d(\mathcal{O}) = A_2 + A_1$ which is not even.

On the other hand if $\mathcal{O} = A_2 + A_1$ then $d(\mathcal{O}) = E_8(a_4)$ is even, $|\mathcal{SP}(\mathcal{O})| = 2$, $\overline{A}(\mathcal{O}) = \mathbb{Z}_2$, and (7) holds.

Property (7) is implicit in [2, Proposition 3.2], although the references given in the proof there do not address this fact. Probably it follows from a close reading of [6].

Suppose $\mathcal{O}$ is special and let ${}^\vee\mathcal{O} = d(\mathcal{O})$. It is well known that $\overline{A}(\mathcal{O}) \simeq \overline{A}({}^\vee\mathcal{O})$. If $\overline{A}(\mathcal{O}) = 1$ then (5) implies

$$|\mathcal{SP}(\mathcal{O})| = |\mathcal{SP}({}^\vee\mathcal{O})| = 1$$

so (7) is true in this case. Thus (7) only has content if $\overline{A}(\mathcal{O}) \neq 1$.

We use (7) quite a bit so we state it separately.

Proposition 1.5 *Suppose $\mathcal{O}$ is special, and the dual of $\mathcal{O}$ is even. Then*

$$|[\overline{A}(\mathcal{O})]| = |\mathcal{SP}(\mathcal{O})|.$$

An alternative statement is: suppose $\mathcal{O}$ is even. Then

$$|[\overline{A}(\mathcal{O})]| = |[\overline{A}(d(\mathcal{O}))]| = |\mathcal{SP}(d(\mathcal{O}))|.$$

and keep in mind this is a formula for the size of the special piece not of the even orbit $\mathcal{O}$, but of its (possiby not even) dual.

Here is confirmation that the Proposition holds for E_8. It is enough to consider orbits with $\overline{A}(\mathcal{O}) \neq 1$.

Here is a list of special nilpotent orbits $\mathcal{O}$ for E_8 for which $\overline{A}(\mathcal{O}) \neq 1$.

$\mathcal{O}$	diagram	${}^\vee\mathcal{O}$	$\overline{A}(\mathcal{O})$	$\lvert\mathcal{SP}(\mathcal{O})\rvert$	$\lvert\mathcal{SP}({}^\vee\mathcal{O})\rvert$
A2	00000002	E8(a3)	**S2**	2	**2**
A2+A1	10000001	E8(a4)	S2	2	1
2A2	20000000	E8(a5)	**S2**	2	**2**
D4(a1)	00000020	E8(b5)	**S3**	3	**3**
D4(a1)+A1	01000010	E8(a6)	S3	3	1
A4	20000002	E7(a3)	**S2**	1	**2**
D4(a1)+A2	02000000	E8(b6)	**S2**	2	**2**
A4+A1	10000101	E6(a1)+A1	S2	1	1
D5(a1)	10000102	E6(a1)	S2	2	1
A4+2A1	00010001	D7(a2)	S2	2	1
E6(a3)	20000020	D6(a1)	**S2**	2	**2**
E8(a7)	00002000	E8(a7)	**S5**	7	7
D6(a1)	01100012	E6(a3)	S2	2	**2**
E6(a1)	20000202	D5(a1)	**S2**	1	**2**
D7(a2)	10010101	A4+2A1	S2	1	2
E6(a1)+A1	10010102	A4+A1	S2	1	1
E8(b6)	00020002	D4(a1)+A2	**S2**	2	**2**
E7(a3)	20010102	A4	S2	2	1
E8(a6)	00020020	D4(a1)+A1	**S3**	1	**3**
E8(b5)	00020022	D4(a1)	**S3**	3	**3**
E8(a5)	20020020	2A2	**S2**	2	**2**
E8(a4)	20020202	A2+A1	**S2**	1	**2**
E8(a3)	20020222	A2	**S2**	2	**2**

The Proposition says: if $\mathcal{O}$ is *even*, the number of conjugacy classes of $\overline{A}(\mathcal{O})$ and the cardinality of the *special piece of* ${}^\vee\mathcal{O}$ agree, as indicated by the bold face entries.

1.1 Example

We interrupt or program for an example.

Consider the special orbits $E_8(b_4) \subset E_8(a_4)$. Both special pieces are empty. The duals are: $E_8(b_4) \to A2+2A_1$, $E_8(a_4) \to A_2+A_1$, and we have the picture The duals are $4A_1 \subset A_2+A_1 \subset A_2+2A_1$; the special piece of A_2+A_1 is (itself and) $4A_1$.

Here is the Springer correspondence.

orbit $\mathcal{O}$	$A(\mathcal{O})$	$\overline{A}(\mathcal{O})$	ϕ	$\sigma(\mathcal{O},\phi)$	$a(\sigma)$	$b(\sigma)$
$E_8(a_4)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\phi_{210,4}$	4	4
$E_8(a_4)$			ϵ	$\phi_{160,7}$	4	7
$E_8(b_4)$	$\mathbb{Z}/2\mathbb{Z}$	1	1	$\phi_{560,5}$	6	6
$E_8(b_4)$			ϵ	$\phi_{50,8}$	6	8
A_2+A_1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\phi_{210,52}$	52	52
A_2+A_1			ϵ	$\phi_{160,55}$	52	55
A_2+2A_1	1	1	1	$\phi_{560,47}$	47	47
$4A_1$	1	1	1	$\phi_{50,56}$	56	56

Here are the special orbits, families/double cells, and the corresponding elements (x,ξ) of $\mathcal{M}(\overline{A}(\mathcal{O}))$.

orbit	σ	x	ξ
$E_8(a_4)$	$\phi_{210,4}$	1	1
	$\phi_{50,8}$	g_2	1
	$\phi_{160,7}$	1	ϵ
$E_8(b_4)$	$\phi_{560,5}$	1	1
A_2+A_1	$\phi_{210,52}$	1	1
	$\phi_{50,56}$	g_2	1
	$\phi_{160,55}$	1	ϵ
A_2+2A_1	$\phi_{560,47}$	1	1

Here are the special orbits and Lusztig cells.

orbit $\mathcal{O}$	$\mathcal{L}(\mathcal{O})$
$E_8(a_4)$	$\phi_{210,4},\phi_{50,8}$
$E_8(b_4)$	$\phi_{60,6}$
A_2+A_1	$\phi_{210,52},\phi_{50,56}$
A_2+2A_1	$\phi_{560,47}$

Definition 1.1.1 *Let $\mathcal{O}$ be a special orbit. Let*

$$\Sigma(\mathcal{O})=\{\sigma\in\widehat{W}\mid\Psi(\sigma)=(\mathcal{O},m)\text{ for some }m\in\overline{A}(\mathcal{O})\}$$

This is the double cell defined by $\mathcal{O}$.

The Lusztig (left) cell defined by $\mathcal{O}$ is:

$$\mathcal{L}(\mathcal{O})=\{\sigma\in\widehat{W}\mid\Psi(\sigma)=(\mathcal{O},(x,1))\ \textit{for some}\ x\in[\overline{A}(\mathcal{O})]\}\subset\Sigma(\mathcal{O})$$

The Springer (left) cell defined by $\mathcal{O}$ is:

$$\mathcal{S}(\mathcal{O})=\{\sigma\in\widehat{W}\mid\Psi(\sigma)=(\mathcal{O},(1,\xi))\ \textit{for some}\ \xi\in[\widehat{\overline{A}(\mathcal{O})}]\}\subset\Sigma(\mathcal{O})$$

Note that

$$\mathcal{L}(\mathcal{O})\cap\mathcal{S}(\mathcal{O})=\{\text{Springer}(\mathcal{O})\}$$

and this is the unique special representation in $\Sigma(\mathcal{O})$.

Lemma 1.1.2 *Suppose $\mathcal{O}$ is special. Then $|\mathcal{L}(\mathcal{O})|=|[\overline{A}(\mathcal{O})]|$.*

Recall

$$\mathcal{L}(\mathcal{O})=\{\Psi^{-1}(\mathcal{O},(x,1))\mid x\in[\overline{A}(\mathcal{O})]\}.$$

Thus

$$\{\text{Springer}(\mathcal{O}',1)\mid\mathcal{O}'\in\mathcal{SP}(\mathcal{O})\}\subset\mathcal{L}(\mathcal{O})$$

with equality if $d(\mathcal{O})$ is even.

Back to our regularly scheduled programming.

Here is a formula for the size of a weak Arthur packet. See unipotentExceptional.pdf. For $\gamma\in\mathfrak{h}^*$ let $\mathcal{M}_\gamma(G(\mathbb{R}))$ be Grothendieck group of representations of $G(\mathbb{R})$ with infinitesimal character γ, equipped with the coherent continuation representation of W.

Proposition 1.1.3 ([2, Proposition 3.1]) *Assume ${}^\vee\mathcal{O}$ is even. Let $\gamma=\gamma({}^\vee\mathcal{O})$. Then*

$$|\Pi({}^\vee\mathcal{O})|=\dim\text{Hom}(\mathcal{L}({}^\vee\mathcal{O})\otimes\text{sgn},\mathcal{M}_\gamma(G(\mathbb{R}))$$

Corollary 1.1.4 *Suppose ${}^\vee\mathcal{O}$ is even, and set $\gamma=\gamma({}^\vee\mathcal{O})$. Let $\mathcal{O}$ be the dual (special) orbit for G. Then*

$$|\Pi({}^\vee\mathcal{O})|=\sum_{\mathcal{O}'\in\mathcal{SP}(\mathcal{O})}\dim\text{Hom}(\text{Springer}(\mathcal{O}',1),\Pi_\gamma(G(\mathbb{R})))$$

The special orbits A_4+A_1 for E_7 and $A_4+A_1, E_6(a_1)+A_1$ for E_8 are said to be *exceptional.* We also say the the special Springer representations $\text{Springer}(\mathcal{O},1)$ for these orbits are exceptional; these have dimension $512, 4096, 4096$ respectively.

Note: I'm not sure if the statements above, especially the Corollary, need to be modified for these orbits.

2 Three Representations of the Weyl group

We will work (somewhat) in the `atlas` setting, with a fixed Cartan subgroup H and choice of positive roots Δ^+. Suppose θ is an involutive automorphism of W. Define the imaginary roots $\Delta_i(\theta)$, with positive roots $\Delta_i^+(\theta) = \Delta_i(\theta) \cap \Delta^+$, $\rho_i(\theta)$, and Weyl group $W_i(\theta)$ as usual, and similarly $\Delta_r(\theta)$ etc. If θ is understood we drop it from the notation, although frequently θ is varying.

Define

$$\epsilon_i(\theta, w) = |\{\alpha \in \Delta_i^+ \mid w^{-1}(\alpha) \in -\Delta^+\}$$

The restriction of $\epsilon_i(\theta, *)$ to W^θ is a character, although $\epsilon_i(\theta, *)$ is typically not a character of W.

Suppose x is a KGB element. Then θ_x is an involutive automorphism of H and W, and define $\rho_i(x)$ accordingly. Note that for α a simple root

$$\rho_i(s_\alpha x s_\alpha) = \epsilon_i(x, s_\alpha) s_\alpha \rho_i(x)$$

and therefore

$$\rho_i(wxw^{-1}) = \epsilon_i(x, w)(w\rho_i(x)) \quad (w \in W).$$

2.1 Representation on the space of involutions

First of all let $\mathcal{I} = \{w \in W \mid w^2 = 1\}$, the involutions in W. If $w \in \mathcal{I}$ define $\theta_w(u) = wuw^{-1}$, and $\epsilon_i(w, u) = \epsilon_i(\theta_w, u)$ $(u \in W)$. Define a representation of W on the space $V_\mathcal{I}$ with basis $\{a_w \mid w \in \mathcal{I}\}$ by:

$$\pi_\mathcal{I}(w)(a_y) = \epsilon_i(\theta_y, w) a_{wyw^{-1}} \quad (w \in W, y \in \mathcal{I}).$$

This is a representation of W of dimension $|\mathcal{I}|$.

If $w \in \mathcal{I}$ then $\epsilon_{i,w}(u) := \epsilon_i(\theta_w, u)$ is a character when restricted to W^{θ_w}, and

$$\pi_I \simeq \sum_{w \in \mathcal{I}/W} \mathrm{Ind}_{W^{\theta_w}}^W (\epsilon_{i,w})$$

Let

$$\mathcal{T} = \mathcal{I}/W; \tag{2.1.1}$$

this is in bijection with K-conjugacy classes of Cartan subgroups of G, or equivalently $G(\mathbb{R})$-conjugacy classes of Cartan subgroups of $G(\mathbb{R})$.

2.2 Representation on the KGB space

Now suppose we are given a real form of G, and let X be the corresponding KGB space. Then w acts on X by the cross action $w : x \to w \times x$. In particular $s_\alpha \times x = x$ if and only if x is real or imaginary. Define a representation of W on the space V_X with basis $\{a_x \mid x \in X\}$ by

$$\pi_X(w)(a_x) = \epsilon_i(\theta_x, w) a_{w \times x} \quad (w \in W, x \in X)$$

This is a representation of W of dimension $|X|$.

Recall that $W_x := \mathrm{Stab}_W(x)$ is equal to $W(G^{\theta_x}, H^{\theta_x})$, and this is isomorphic to the "real" or "rational" Weyl group $W(G(\mathbb{R}), H(\mathbb{R}))$.

If $x \in X$ then $\epsilon_{i,x}(u) := \epsilon_i(\theta_x, u)$ is a character when restricted to W^{θ_x}, and

$$\pi_X \simeq \sum_{w \in X/W} \mathrm{Ind}_{W_x}^W(\epsilon_{i,x})$$

Note that W acts transitively on the fibers of the map $X \to \mathcal{I}$, and therefore

$$X/W \simeq \mathcal{I}/W = \mathcal{T}.$$

2.3 Representation on a block

Now suppose we are given a real form of G, and a block $\mathcal{B}$ for this real form. Then W acts on the space $\mathcal{B}$ by the cross action. Define a representation of W on the space $V_{\mathcal{B}}$ with basis $\{a_\gamma \mid \gamma \in \mathcal{B}\}$

$$\pi_{\mathcal{B}}(w)(a_\gamma) = \epsilon_i(\theta_{x(\gamma)}, w) a_{w \times \gamma} \quad (w \in W, \gamma \in \mathcal{B})$$

This is precisely the coherent continutation representation of W on the block; the formulas for the action of W have some Cayley transform terms, but these do not contribute to the trace (i.e. we can filter the representation by Cartan subgroups, and replace it by the associated graded).

If $\gamma \in \mathcal{B}$ let $W_\gamma = \mathrm{Stab}_W(\gamma)$, and let $\theta_\gamma = \theta_x$ where $\gamma = (x, \lambda, \nu)$. Then $\epsilon_{i,\gamma}(w) := \epsilon(\theta_\gamma, w)$ is a character of W_γ, and

$$\pi_{\mathcal{B}} \simeq \sum_{w \in \mathcal{B}/W} \mathrm{Ind}_{W_\gamma}^W(\epsilon_{i,\gamma}).$$

Note that $\mathcal{B}/W \hookrightarrow \mathcal{T}$, and this is a bijection of G is quasisplit. In particular if G is quasisplit all three formulas are sums over the same set $\mathcal{T}$.

Remark 2.3.1 The three subgroups of W just discussed are related as follows. If $\gamma = (x, \lambda, \nu)$ is a parameter then

$$\mathrm{Stab}_W(\gamma) \subset \mathrm{Stab}_W(x) \subset W^{\theta_x}$$

Here is a result of Rossmann [8]. Recall for $\mathcal{O}$ a complex nilpotent orbit:

$$r(\mathcal{O}) = \dim(D(\mathcal{O})) = |\mathcal{O} \cap \mathfrak{g}_0 / G(\mathbb{R})|,$$

the number of "real forms" of $\mathcal{O}$.

Theorem 2.3.2 *For any complex nilpotent orbit $\mathcal{O}$:*

$$r(\mathcal{O}) = \mathrm{mult}(\mathrm{Springer}(\mathcal{O}), \pi_X)$$

Here is a generalization due to Kottwitz [5, Theorem 1.8]. Recall

$$s(\mathcal{O}) = \dim(D^{\mathrm{st}}(\mathcal{O}))$$

Theorem 2.3.3 *For any complex nilpotent orbit $\mathcal{O}$:*

$$s(\mathcal{O}) = \mathrm{mult}(\mathrm{Springer}(\mathcal{O}), \pi_{\mathcal{I}})$$

Note: If G is split and classical, and $\mathcal{O}$ is not special, then by [4, Theorem 1] $s(\mathcal{O}) = 0$. This is known to be false for G split and exceptional.

3 Kottwitz's result

Kottwitz has a conjecture generalizing Theorem 2.3.3 to replace Springer($\mathcal{O}$) with any irreducible representation of W. This was proved by Kottwitz [4] (classical groups) and Casselman (exceptional groups) [3].

Theorem 3.1 *Assume G is simple.*

$$\mathrm{mult}(\sigma, \pi_{\mathcal{I}}) = \begin{cases} \widehat{\phi}(x_\sigma) & \sigma \text{ not exceptional} \\ 1 & \sigma \text{ exceptional} \end{cases}$$

Here are some cases spelled out.

(1) $\sigma = \mathrm{Springer}(\mathcal{O}, 1)$: $s(\mathcal{O})$ (Theorem 2.3.3)

(2) σ special, not exceptional: $|[\overline{A}(\mathcal{O})_2]|$

(3) σ special, G classical: $|[\overline{A}(\mathcal{O})_2]| = |[\overline{A}(\mathcal{O})]|$.

(4) G classical, σ not special: 0

(5) σ special, exceptional: 1

There is substantial overlap among these cases. For example suppose G is classical and $\sigma = \mathrm{Springer}(\mathcal{O}, 1)$. Then by cases 1, 3 and 4:

$$\mathrm{mult}(\sigma, \pi_{\mathcal{I}}) = s(\mathcal{O}) = \begin{cases} |[\overline{A}(\mathcal{O})]| & \mathcal{O} \text{ is special} \\ 0 & \text{otherwise} \end{cases}$$

Also if σ is special, so $\sigma = \mathrm{Springer}(\mathcal{O}, 1)$ with $\mathcal{O}$ special and not exceptional, then by 1 and 2:

$$\mathrm{mult}(\sigma, \pi_{\mathcal{I}}) = s(\mathcal{O}) = |[\overline{A}(\mathcal{O})_2]|$$

References

[1] Dan Barbasch and David A. Vogan Jr., *Unipotent representations of complex semisimple groups*, Ann. of Math. (2) **121** (1985), no. 1, 41–110. MR782556 (86i:22031)

[2] Dan M. Barbasch and Peter E. Trapa, *Stable combinations of special unipotent representations*, Representation theory and mathematical physics, 2011, pp. 113–127. MR2848922

[3] Bill Casselman, *Verifying Kottwitz' conjecture by computer*, Represent. Theory **4** (2000), 32–45. MR1740179

[4] Robert E. Kottwitz, *Involutions in Weyl groups*, Represent. Theory **4** (2000), 1–15. MR1740177

[5] ———, *Stable nilpotent orbital integrals on real reductive Lie algebras*, Represent. Theory **4** (2000), 16–31. MR1740178

[6] G. Lusztig, *Notes on unipotent classes*, Asian J. Math. **1** (1997), no. 1, 194–207. MR1480994 (98k:20078)

[7] George Lusztig, *Characters of reductive groups over a finite field*, Annals of Mathematics Studies, vol. 107, Princeton University Press, Princeton, NJ, 1984. MR742472 (86j:20038)

[8] W. Rossmann, *Nilpotent orbital integrals in a real semisimple Lie algebra and representations of Weyl groups*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), 1990, pp. 263–287. MR1103593