Equivalent definitions of Arthur packets for real classical groups

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1 Introduction

Guided by the theory of the trace formula, Arthur conjectured a classification of automorphic representations of a connected reductive algebraic group G in terms of *A-parameters* ([Art84], [Art89]). These A-parameters are global objects, which conjecturally decompose as a product of local A-parameters. The local part of Arthur's conjectures states that associated to an A-parameter ψ over a local field F is a finite set Π_ψ, called an Arthur packet. This is a set of irreducible representations of $G(F)$ satisfying conditions to be given in Problems A-E below.

Assume for the moment that G is either the split form of GL_N, or a quasisplit form of Sp_N or SO_N, $N \geq 2$ over a local field F of characteristic 0. For these groups, Arthur defines a set, which we denote Π_ψ^{Ar}, and proves that it satisfies most of the conditions [Art13]. His approach uses harmonic analysis, and both local and global methods.

On the other hand, for a general connected real reductive group Adams, Barbasch and Vogan define a set which we denote by Π_ψ^{ABV}, and prove these satisfy most of Arthur's conditions [ABV92]. Their methods are quite different, being based on the connection between representations and equivariant sheaves.

It has long been expected that the definitions agree when both are defined ([Art08]*Section 8), *i.e.* for the real quasisplit groups Sp_N and SO_N. The main result of this paper is that, aside from the case of even rank special orthogonal groups, it is indeed true that

$$\Pi_\psi^{\mathrm{Ar}} = \Pi_\psi^{\mathrm{ABV}}.$$

The case of even rank special orthogonal groups requires a slightly modified identity, which we give later in the introduction.

Imitating [ABV92], we state Arthur's original conjectures as a sequence of problems. We then describe the two approaches to these problems. We assume that the reader is somewhat familiar with the theory of endoscopy for tempered representations [She08]. We follow the notation of [ABV92] and also provide references from [ABV92] for convenience.

Let Γ be the Galois group of $\mathbb{C}/\mathbb{R}$ and let ${}^\vee G^\Gamma = {}^\vee G \rtimes \Gamma$ be the Galois form of the L-group of G. An *A-parameter for* G, is a group homomorphism

$$\psi_G : W_\mathbb{R} \times \mathrm{SL}_2 \to {}^\vee G^\Gamma \tag{1}$$

such that $\psi_G|_{W_\mathbb{R}}$ is a tempered L-parameter and $\psi_G|_{\mathrm{SL}(2,\mathbb{C})}$ is algebraic. For the definition of $W_\mathbb{R}$ and tempered L-parameters see [Bor79].

Problem A Associate to ψ_G a finite linear combination of irreducible characters η_{ψ_G} of $G(\mathbb{R})$ which is a *stable* distribution ([She79], [ABV92]*Definition 18.2).

The finite set Π_{ψ_G} of irreducible characters occurring in the stable distribution η_{ψ_G} is defined to be the *Arthur packet* (or *A-packet*) of ψ_G. Let A_{ψ_G} be the component group of the centralizer in ${}^\vee G$ of the image of ψ_G. For the groups in question this is a finite abelian group ([Art13]**p.* 32).

Problem B Associate to each $\pi \in \Pi_{\psi_G}$ a non-zero finite-dimensional representation $\tau_{\psi_G}(\pi)$ of A_{ψ_G}.

Problem C Prove that

$$\eta_{\psi_G} = \sum_{\pi \in \Pi_{\psi_G}} \varepsilon_\pi \dim(\tau_{\psi_G}(\pi))\, \pi$$

for some $\varepsilon_\pi = \pm 1$.

Problem D Prove that the stable distributions η_{ψ_G} satisfy analogues of Shelstad's theorem on endoscopic lifting for tempered representations [ABV92]*Chapter 26.

Problem E Prove that the irreducible representations of Π_{ψ_G} are all unitary.

For the remainder of this section we assume G is either the split form of GL_N, a quasisplit form of Sp_N, or SO_N. [Art13]*Theorem 2.2.1 is a solution to nearly all of these problems, the only exceptions being the signs ε_π in Problem C, and in the case that $G = \mathrm{SO}_{2N}$, a general weakening of the results due to the existence of an outer automorphism. We shall return to both of these points.

The main idea of Arthur's approach is to express a symplectic or special orthogonal group G as a *twisted endoscopic group* of $(\mathrm{GL}_N, \vartheta)$ [KS99]*Section 2. In this pair ϑ is the outer automorphism of GL_N of order two defined by

$$\vartheta(g) = \tilde{J}\,(g^{-1})^{\intercal}\,\tilde{J}^{-1}, \quad g \in \mathrm{GL}_N, \tag{2}$$

where $\tilde{J}$ is the anti-diagonal matrix

$$\tilde{J} = \begin{bmatrix} & 0 & & & 1 \\ & & & -1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{N-1} & & & & 0 \end{bmatrix}.$$

The semidirect product $\mathrm{GL}_N \rtimes \langle\vartheta\rangle$ is a disconnected algebraic group with non-identity component $\mathrm{GL}_N \rtimes \vartheta$. The group G is attached to the pair $(\mathrm{GL}_N, \vartheta)$ through the existence of an element $s\vartheta \in \mathrm{GL}_N \rtimes \vartheta$ for which ${}^{\vee}G$ is the identity component of the fixed-point set $({}^{\vee}\mathrm{GL}_N)^{s\vartheta}$. Furthermore, there is a natural inclusion

$$\epsilon : {}^{\vee}G^{\Gamma} \hookrightarrow {}^{\vee}\mathrm{GL}_N^{\Gamma} \tag{3}$$

which allows us to define the A-parameter

$$\psi = \epsilon \circ \psi_G \tag{4}$$

for GL_N using (1).

As in Problem D there are theorems on *twisted* endoscopic lifting for tempered representations ([She12], [Mez13], [Mez16]). One may therefore extend Problem D to

Problem D$'$ Prove that the stable distributions η_{ψ_G} satisfy analogues of both standard and twisted endoscopic lifting for tempered representations.

The solution to Problem D$'$ in the twisted setting above opens a path towards defining η_{ψ_G}. We may take for granted the existence of an irreducible character π_ψ of $\mathrm{GL}_N(\mathbb{R})$ ([Art13]**p.* 64) such that π_ψ solves Problems A-E, *i.e.*

$$\eta_\psi = \pi_\psi.$$

Now suppose $\pi_{\tilde{\psi}}$ is an extension of π_ψ to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. Let $\mathrm{Tr}_\vartheta(\pi_{\tilde{\psi}})$ be the *twisted trace* of $\pi_{\tilde{\psi}}$, which is obtained by restricting the distribution character of $\pi_{\tilde{\psi}}$ to the non-identity component $\mathrm{GL}_N \rtimes \vartheta$. The extension $\pi_{\tilde{\psi}}$ is not unique; we choose it following [Art13]**pp.* 62-63 by fixing a Whittaker datum. Towards a solution to Problem A, Arthur defines a stable virtual character $\eta^{\mathrm{Ar}}_{\psi_G}$ using a twisted endoscopic transfer identity

$$\mathrm{Tr}_\vartheta(\pi_{\tilde{\psi}}) = \mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}(\eta^{\mathrm{Ar}}_{\psi_G}). \tag{5}$$

The endoscopic transfer map $\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}$ is defined on the space of stable virtual characters of $G(\mathbb{R})$, and $\eta^{\mathrm{Ar}}_{\psi_G}$ is fixed under the action of any outer automorphisms. This defines $\eta^{\mathrm{Ar}}_{\psi_G}$ uniquely. Arthur proves the existence of $\eta^{\mathrm{Ar}}_{\psi_G}$ satisfying (5) using the solution to Problem D$'$ in the tempered setting.

Adams, Barbasch and Vogan use completely different methods to study Problems A-E. They first construct a pairing between characters and equivariant sheaves. They then apply techniques from microlocal geometry to these sheaves and use the pairing to transfer these back to the world of virtual characters. An outline of their methods is given in the introduction to [ABV92]. Here we summarize the main ideas, specialized to the case of quasisplit classical groups.

Adams, Barbasch and Vogan introduce a complex variety $X({}^{\vee}G^{\Gamma})$ equipped with a ${}^{\vee}G$-action [ABV92]*Section 6, so that the ${}^{\vee}G$-orbits are in bijection with the equivalence classes of L-parameters. The advantage to working with orbits of $X({}^{\vee}G^{\Gamma})$ lies in the additional topological structure. The orbits provide a stratification of $X({}^{\vee}G^{\Gamma})$ which naturally leads to the notions of local systems and constructible sheaves. We define a *complete geometric parameter* to be a pair

$$\xi = (S, \mathcal{V})$$

consisting of an orbit $S \subset X({}^\vee G^\Gamma)$, together with a ${}^\vee G$-equivariant local system $\mathcal{V}$ on S ([ABV92]*Definition 7.6). The set of complete geometric parameters is denoted by $\Xi({}^\vee G^\Gamma)$. This definition ignores more general local systems [ABV92], which are equivariant for an algebraic cover of ${}^\vee G$. These aren't needed here, and this simplifies the discussion. By [ABV92]*Theorem 10.11 there is a canonical bijection

$$\Xi({}^\vee G^\Gamma) \longleftrightarrow \Pi(G/\mathbb{R}) \tag{6}$$

The set on the right is the set of (equivalence classes of) irreducible representations of certain *forms* of G, including a fixed quasisplit form. We write bijection (6) as

$$\xi \mapsto \pi(\xi).$$

Each irreducible representation $\pi(\xi)$ is the unique irreducible quotient of a standard representation $M(\xi)$, so we also have a bijection

$$\xi \mapsto M(\xi)$$

between $\Xi({}^\vee G^\Gamma)$ and a set of standard representations. Let $K\Pi(G/\mathbb{R})$ be the Grothendieck group of the admissible representations of the strong involutions of G appearing on the right of (6). This Grothendieck group has two bases, namely $\{\pi(\xi)\}$ and $\{M(\xi)\}$ for $\xi \in \Xi({}^\vee G^\Gamma)$.

There is a parallel construction in terms of sheaves for the dual group ${}^\vee G$. Suppose $\xi \in \Xi({}^\vee G^\Gamma)$. The local system of this complete geometric parameter is a ${}^\vee G$-equivariant sheaf on S. Applying the functors of extension by zero to the closure of S, and then taking the direct image gives an irreducible ${}^\vee G$-equivariant constructible sheaf $\mu(\xi)$ on $X({}^\vee G^\Gamma)$. This defines a bijection

$$\xi \mapsto \mu(\xi)$$

between complete geometric parameters and irreducible ${}^\vee G$-equivariant constructible sheaves. Alternatively, one may apply the functors of intermediate extension and direct image. This defines an irreducible ${}^\vee G$-equivariant perverse sheaf $P(\xi)$, and a bijection

$$\xi \mapsto P(\xi)$$

between complete geometric parameters and irreducible ${}^\vee G$-equivariant perverse sheaves. The Grothendieck groups of the categories of ${}^\vee G$-equivariant constructible and perverse sheaves are isomorphic ([ABV92]*Lemma 7.8, [BBD82]). We identify the two Grothendieck groups and write them as $KX({}^\vee G^\Gamma)$. The sets $\{\mu(\xi)\}$ and $\{P(\xi)\}$ for $\xi \in \Xi({}^\vee G^\Gamma)$ each form a basis of $KX({}^\vee G^\Gamma)$.

We now define a pairing

$$\langle\cdot,\cdot\rangle_G : K\Pi(G/\mathbb{R}) \times KX({}^\vee G^\Gamma) \to \mathbb{Z} \tag{7}$$

using the bases of standard representations and constructible sheaves:

$$\langle M(\xi), \mu(\xi')\rangle_G = e(\xi)\,\delta_{\xi,\xi'}, \quad \xi,\xi' \in \Xi({}^\vee G^\Gamma).$$

Here $e(\xi)$ is the Kottwitz sign ([ABV92]*Definition 15.8), and $\delta_{\xi,\xi'}$ is the Kronecker delta. It is natural to ask what the formula for this pairing is in terms of the bases of irreducible representations and perverse sheaves. It is a deep fact that in these alternative bases the pairing is also, up to signs, given by the Kronecker delta function. More precisely

$$\langle \pi(\xi), P(\xi')\rangle_G = e(\xi)\,(-1)^{d(\xi)}\,\delta_{\xi,\xi'}, \quad \xi \in \Xi({}^\vee G^\Gamma).$$

where $d(\xi)$ is the dimension of the orbit S in $\xi = (S, \mathcal{V})$ ([ABV92]*Theorem 1.24).

Using the pairing (7) we may regard virtual characters as $\mathbb{Z}$-valued linear functionals on $KX({}^\vee G^\Gamma)$. Of particular importance are the stable virtual characters. The theory of microlocal geometry provides a family of linear functionals

$$\chi_S^{\mathrm{mic}} : KX({}^\vee G^\Gamma) \to \mathbb{Z} \tag{8}$$

parameterized by the ${}^\vee G$-orbits $S \subset X({}^\vee G^\Gamma)$. These microlocal multiplicity maps appear in the theory of *characteristic cycles* ([ABV92]*Chapter 19, [BGK+87]), and are associated with ${}^\vee G$-equivariant local systems on a conormal bundle over $X({}^\vee G^\Gamma)$ ([ABV92]*Section 24, [GM88]). The virtual characters associated by the pairing to these linear functionals are stable ([ABV92]*Theorems 1.29 and 1.31).

Now we return to the Arthur parameter ψ_G given in (1). Associated to ψ_G is a Langlands parameter ϕ_{ψ_G} [Art89]*Section 4 defined by

$$\phi_{\psi_G}(w) = \psi_G\left(w, \begin{bmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{bmatrix}\right), \quad w \in W_{\mathbb{R}}. \tag{9}$$

Let $S_{\psi_G} \subset X({}^\vee G^\Gamma)$ be the ${}^\vee G$-orbit of ϕ_{ψ_G}. We define $\eta^{\mathrm{mic}}_{\psi_G}$ to be the virtual character associated to $\chi^{\mathrm{mic}}_{S_{\psi_G}}$ by the pairing. That is, $\eta^{\mathrm{mic}}_{\psi_G}$ is the unique virtual character satisfying

$$\langle \eta^{\mathrm{mic}}_{\psi_G}, \mu\rangle_G = \chi^{\mathrm{mic}}_{S_{\psi_G}}(\mu), \quad \mu \in KX({}^\vee G^\Gamma).$$

As a distribution, the stable virtual character $\eta^{\mathrm{mic}}_{\psi_G}$ is supported on real forms of G which include the quasisplit form $G(\mathbb{R})$. In this more general context, Adams, Barbasch and Vogan show that $\eta^{\mathrm{mic}}_{\psi_G}$ satisfies the conditions of Problems B-D. For our purposes however, it suffices to consider the restriction

$$\eta^{\mathrm{ABV}}_{\psi_G} = \eta^{\mathrm{mic}}_{\psi_G}|_{G(\mathbb{R})} \tag{10}$$

of $\eta^{\mathrm{mic}}_{\psi_G}$ to the quasisplit form $G(\mathbb{R})$.

Having sketched the construction of $\eta^{\mathrm{Ar}}_{\psi_G}$ and $\eta^{\mathrm{ABV}}_{\psi_G}$, we come to the main result.

Theorem 1.1. *Let G be a real quasisplit form of Sp_N or SO_{2N+1} and suppose ψ_G is an Arthur parameter for G. Then*

$$\eta^{\mathrm{Ar}}_{\psi_G} = \eta^{\mathrm{ABV}}_{\psi_G}.$$

For the precise statement, including the case of SO_{2N}, see Theorem 9.3. We continue by giving an outline of the proof under the assumption that G is not a special orthogonal group of even rank.

Arthur's definition of $\eta^{\mathrm{Ar}}_{\psi_G}$ is given in terms of the twisted endoscopic transfer map $\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}$ appearing in (5). The first step in the proof of Theorem 1.1 is to compare $\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}$ with the analogous twisted endoscopic lifting map Lift_0 defined in [CM18]*Section 5. We wish to prove

$$\mathrm{Lift}_0 = \mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}. \tag{11}$$

The construction of the map Lift_0 follows the construction in [ABV92]*Section 26 and is given in terms of a pairing analogous to (7) in the setting of twisted characters and sheaves. Associated to the involution ϑ is a $\mathbb{Z}$-module of *twisted characters* $K\Pi(\mathrm{GL}_N(\mathbb{R}), \vartheta)$ [AV15] . On the dual side we have a $\mathbb{Z}$-module of *twisted sheaves* $KX({}^\vee \mathrm{GL}_N^\Gamma, \vartheta)$ [LV14]. We wish to define a pairing

$$\langle \cdot, \cdot \rangle : K\Pi(\mathrm{GL}_N(\mathbb{R}), \vartheta) \times KX({}^\vee \mathrm{GL}_N^\Gamma, \vartheta) \to \mathbb{Z}. \tag{12}$$

One of the technical difficulties in defining this pairing lies in making canonical choices of extensions. Suppose $\xi \in \Xi({}^\vee \mathrm{GL}_N^\Gamma)$ (see (6)), with associated standard representation $M(\xi)$. If $M(\xi)$ is fixed by ϑ then it extends in two ways to a representation of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. Each of the two resulting characters restricts to $\mathrm{GL}_N(\mathbb{R}) \rtimes \vartheta$ to give a twisted character, and the $\mathbb{Z}$-module $K\Pi(\mathrm{GL}_N(\mathbb{R}), \vartheta)$ is defined so that if $M(\xi)^\pm$ are the two extensions, then $M(\xi)^- = -M(\xi)^+$ in $K\Pi(\mathrm{GL}_N(\mathbb{R}), \vartheta)$.

Similarly, the ${}^\vee\mathrm{GL}_N$-equivariant constructible sheaf $\mu(\xi)$ extends in two ways to a $({}^\vee\mathrm{GL}_N \rtimes \langle\vartheta\rangle)$-equivariant constructible sheaf on $X({}^\vee\mathrm{GL}^\Gamma)$. The two extensions $\mu(\xi)^\pm$ again differ by sign in $KX({}^\vee \mathrm{GL}_N^\Gamma, \vartheta)$.

A standard problem in the twisted theory is how to choose the extensions. On the sheaf-theoretic side, we use a special property of irreducible ${}^\vee\mathrm{GL}_N$-equivariant sheaves, namely that they are all constant sheaves. We define $\mu(\xi)^+$ to be the irreducible $({}^\vee\mathrm{GL}_N \rtimes \langle\vartheta\rangle)$-equivariant constant sheaf on $X({}^\vee \mathrm{GL}_N^\Gamma)$.

On the representation-theoretic side, the literature offers two ways to choose an extension of $M(\xi)$. As mentioned earlier, Arthur uses Whittaker data to fix a preferred extension which we denote $M(\xi)^{\sim}$ and call the *Whittaker extension.* On the other hand [AV15] gives an extension which we label $M(\xi)^{+}$ and call the *Atlas extension.*

Having chosen the extensions we define pairing (12) by

$$\langle M(\xi)^{\sim}, \mu(\xi')^{+}\rangle = \delta_{\xi,\xi'}. \tag{13}$$

The endoscopic lifting map Lift_0 is defined through pairings (7), (12) and the map ϵ (see (3)) as follows. The map ϵ naturally induces a map

$$X({}^{\vee}G) \to X({}^{\vee}\mathrm{GL}_N^{\Gamma}).$$

The usual inverse image functor on constructible sheaves then induces a homomorphism

$$\epsilon^{*} : K_{\mathbb{C}}X({}^{\vee}\mathrm{GL}_N^{\Gamma}, \vartheta) \to K_{\mathbb{C}}X({}^{\vee}G^{\Gamma})$$

on the complexifications of the $\mathbb{Z}$-modules. The adjoint of ϵ^{*} with respect to the pairings is the homomorphism

$$\epsilon_{*} : K_{\mathbb{C}}\Pi(G/\mathbb{R}) \to K_{\mathbb{C}}\Pi(\mathrm{GL}_N(\mathbb{R}), \vartheta)$$

defined by

$$\langle \epsilon_{*}(\eta), \mu\rangle = \langle \eta, \epsilon^{*}(\mu)\rangle_G, \quad \eta \in K_{\mathbb{C}}\Pi(G/\mathbb{R}), \quad \mu \in K_{\mathbb{C}}X({}^{\vee}\mathrm{GL}_N^{\Gamma}, \vartheta).$$

Here, the pairings on the left and right are (12) and (7), respectively. Finally, the endoscopic lifting map

$$\mathrm{Lift}_0 : K_{\mathbb{C}}\Pi(G(\mathbb{R}))^{\mathrm{st}} \to K_{\mathbb{C}}\Pi(GL_N(\mathbb{R}), \vartheta)$$

is defined to be the restriction of ϵ_{*} to the stable subspace of $K_{\mathbb{C}}\Pi(G(\mathbb{R}))$, the complex virtual characters of $G(\mathbb{R})$. That is, if $\eta \in K_{\mathbb{C}}\Pi(G/\mathbb{R})$ is stable then $\mathrm{Lift}_0(\eta)$ is defined by

$$\langle \mathrm{Lift}_0(\eta), \mu\rangle = \langle \eta, \epsilon^{*}(\mu)\rangle_G \tag{14}$$

for all $\mu \in KX({}^{\vee}\mathrm{GL}_N^{\Gamma}, \vartheta)$.

Now that Lift_0 is defined, we may proceed to check the equality (11). Fix a ${}^{\vee}G$-orbit $S_G \subset X$. The local multiplicity function taking a constructible sheaf to the dimension of a stalk at a point in S_G is a linear functional on $KX({}^{\vee}G^{\Gamma})$. By the pairing (7) this defines an element of $K\Pi(G/\mathbb{R})$. This is a stable virtual character denoted by $\eta^{\mathrm{loc}}_{S_G}$. It is the sum of the standard representations in what is often called a *pseudopacket.*

Let $S \subset X({}^{\vee}\mathrm{GL}^{\Gamma})$ be the ${}^{\vee}\mathrm{GL}_N$-orbit containing $\epsilon(S_G)$, and let $M(S,1)$ be the standard representation defined by the trivial local system on S. By Proposition 5.3

$$\mathrm{Lift}_0(\eta^{\mathrm{loc}}_{S_G}) = (-1)^{\ell^{I}(S,1)-\ell^{I}_{\vartheta}(S,1)}\, M(S,1)^{+} \tag{15)(a}$$

(the terms in the exponent are defined in Section 4). On the other hand Arthur defines a stable character η'_{S_G} by

$$\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}(\eta'_{S_G}) = M(S,1)^{\sim}. \tag{15)(b}$$

According to [AMR17] $\eta'_{S_G} = \eta^{\mathrm{loc}}_{S_G}$. The two extensions of $M(S,1)$ are related by

$$M(S,1)^{\sim} = (-1)^{\ell^{I}(S,1)-\ell^{I}_{\vartheta}(S,1)}\, M(S,1)^{+} \tag{15)(c}$$

(see Proposition 7.8). Taken together, (15(a-c)) give

$$\mathrm{Lift}_0\left(\eta^{\mathrm{loc}}_{S_G}\right) = M(S,1)^{\sim} = \mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}(\eta^{\mathrm{loc}}_{S_G}). \tag{16}$$

Identity (11) follows from the fact that the $\eta^{\mathrm{loc}}_{S_G}$ form a basis of the stable virtual characters.

Going back to (5), and using (11) we see $\eta^{\mathrm{Ar}}_{\psi_G}$ is determined by

$$\mathrm{Lift}_0(\eta^{\mathrm{Ar}}_{\psi_G}) = \pi_{\psi}^{\sim}.$$

Therefore to prove Theorem 1.1 it is enough to show

$$\mathrm{Lift}_0(\eta^{\mathrm{ABV}}_{\psi_G}) = \pi_{\psi}^{\sim}.$$

According to (14), this identity is equivalent to

$$\langle \pi_{\psi}^{\sim}, P(\xi')^{+}\rangle = \langle \eta_{\psi_G}^{\mathrm{ABV}}, \epsilon^{*}(P(\xi')^{+})\rangle_{G}. \tag{17}$$

Recall from (13) that the pairing on the left-hand side is defined in terms of standard representations and constructible sheaves. However, on the left of (17) we require a corresponding formula in terms of irreducible representations and perverse sheaves. It turns out that the exact formula required is

$$\langle \pi(\xi)^{+}, P(\xi')^{+}\rangle = (-1)^{\ell^{I}(\xi)-\ell^{I}_{\vartheta}(\xi)}\delta_{\xi,\xi'}. \tag{18}$$

We prove this formula from an identity involving the twisted Kazhdan-Lusztig-Vogan polynomials. We are in the setting of [LV14] and [AV15], so we have all of the tools of the Hecke algebra available. The proof of (18) is carried out in Section 4, and Theorem 1.1 then follows.

We now provide further details by running through the remaining sections in sequence. Section 2 begins with an outline of the local Langlands correspondence appearing in [ABV92]. One of the features in this correspondence is the parameterization of inner forms of $G(\mathbb{R})$ using *strong involutions*, and the subsequent inclusion of *representations of strong involutions* in the correspondence. Unlike the overview above, we shall be keeping track of the infinitesimal characters of these representations. As a result, the variety $X({}^{\vee}G^{\Gamma})$ in the overview is replaced by $X({}^{\vee}\mathcal{O}, {}^{\vee}G^{\Gamma})$, where ${}^{\vee}\mathcal{O}$ is an infinitesimal character. We assume all infinitesimal characters to be regular until Section 9. Another important theme of Section 2 is the equivalence of complete geometric parameters with Atlas parameters for $\mathrm{GL}_N(\mathbb{R})$. The equivalence between complete geometric parameters and Atlas parameters forges a connection between [ABV92] and [AV15]. The Atlas parameters are indispensable in defining the Atlas extensions, and in the ensuing Hecke operator computations of Section 4. The section closes with a discussion on twisted characters, and the $\mathbb{Z}$-module $K\Pi({}^{\vee}\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ which contains them.

Section 3 is devoted to ${}^{\vee}G$-equivariant sheaves, and their relationship with $\mathcal{D}$-modules and characteristic cycles. We recall a category of sheaves extended by an automorphism σ of ${}^{\vee}\mathrm{GL}_N$ ([ABV92]*(25.7)). The automorphism is of the form

$$\sigma = \mathrm{Int}(s) \circ \vartheta$$

where $s \in {}^{\vee}\mathrm{GL}_N$. The element s plays no meaningful role in this section, but becomes important in the theory of endoscopy (Section 5). The category of extended sheaves is the counterpart to the category of representations on $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. We define the canonical extended sheaves $\mu(\xi)^{+}$ and $P(\xi)^{+}$ in Lemma 3.4. The twisted characters that one obtains from representations of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ find their counterpart as microlocal traces ([ABV92]*(25.1)) which are supported on extensions of irreducible sheaves. The $\mathbb{Z}$-module counterpart to $K\Pi({}^{\vee}\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ is defined in (58) and is denoted by $K(X({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}^{\Gamma}), \sigma)$. The pairings (7) and (12) are also defined in this section.

We provide a terse summary of $\mathcal{D}$-modules and their relationship to equivariant sheaves, characteristic cycles, the microlocal multiplicity maps (8), and the definition of $\eta_{\psi_G}^{\mathrm{ABV}}$. The set of irreducible characters in the support of $\eta_{\psi_G}^{\mathrm{ABV}}$ is denoted by $\Pi_{\psi_G}^{\mathrm{ABV}}$ and is called the *ABV-packet* of ψ_G.

The main objective of Section 4 is to prove the equivalence of the twisted pairings (12) and (18). Our proof is an adaptation of the proof of the equivalence for ordinary pairings ([ABV92]*Sections 16-17) using the tools of [AV15]. As noted earlier, Hecke operators are among these tools. A conspicuous difference between [ABV92] and [AV15] is in the objects upon which Hecke operators act. In [ABV92] Hecke operators are defined on both characters and sheaves. By contrast, the Hecke operators of [AV15]*Section 7 are defined only on (twisted) characters. The links between characters and sheaves in the Hecke actions are the Riemann-Hilbert and Beilinson-Bernstein correspondences ([ABV92]*Theorems 7.9 and 8.3). In Sections 4.1 and 4.2 we describe these correspondences as a bijection

$$P(\xi) \longleftrightarrow \pi({}^{\vee}\xi), \quad \xi \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}G^{\Gamma}),$$

where $\pi({}^{\vee}\xi)$ is the *Vogan dual* of $\pi(\xi)$ (as the equivalence class of a Harish-Chandra module) (6.1 [AV15]). For $G = \mathrm{GL}_N$ the correspondence is extended to

$$P(\xi)^{+} \longleftrightarrow \pi({}^{\vee}\xi)^{+}$$

for ϑ-fixed complete geometric parameters ξ. Once sheaves are aligned with characters in this manner, the rest of the proof of the equivalence of the twisted pairings follows [ABV92] without incident.

Subsection 4.8 is included in Section 4 only because it uses the same machinery. This subsection presents an argument from the twisted Kazhdan-Lusztig-Vogan algorithm ([LV14], [Ada17]) which is crucial to the comparison of Whittaker and Atlas extensions in Section 7.

In Section 5 we describe the theory of endoscopy, both standard and twisted, for GL_N using the framework of [ABV92]. The standard theory of endoscopy in Subsection 5.1 is simply a specialization of [ABV92]*Section 26 to $G = \mathrm{GL}_N$. It is included primarily to motivate the twisted theory, but is also used in Proposition 6.3 further on. The twisted theory of endoscopy in Subsection 5.2 is a specialization of [CM18]*Section 5.4 to GL_N. In this subsection the ${}^\vee\mathrm{GL}_N$-equivariant sheaves of $K(X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}^\Gamma), \sigma)$ are recast as $({}^\vee\mathrm{GL}_N \rtimes \langle\sigma\rangle)$-equivariant sheaves. The endoscopic lifting map takes the form

$$\mathrm{Lift}_0 : K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}))^{\mathrm{st}} \to K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta).$$

The precursor to (16) is Proposition 5.3, where the Atlas extension is used instead of the Whittaker extension. The endoscopic lifting $\mathrm{Lift}_0(\eta^{\mathrm{ABV}}_{\psi_G})$ is described as an element $\eta^{\mathrm{ABV}+}_{\psi} \in K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$, which reduces to $\eta^{\mathrm{ABV}}_{\psi}$ when restricted to $\mathrm{GL}_N(\mathbb{R})$ (Theorem 5.6). The endoscopic lifting map is proved to be injective for GL_N-regular infinitesimal character ${}^\vee\mathcal{O}_G$.

In Section 6 we prove that for *any* A-parameter ψ of GL_N (not necessarily of the form (4)), there is only one irreducible character in the support of $\eta^{\mathrm{ABV}}_{\psi}$. This implies that $\Pi^{\mathrm{ABV}}_{\psi} = \{\pi_\psi\}$. It also implies that $\eta^{\mathrm{ABV}+}_{\psi}$ is supported on a single irreducible twisted character obtained by extension from π_ψ. The proof begins under the assumption that ψ is an A-parameter studied by Adams and Johnson ([AJ87]). Adams and Johnson defined A-packets for these parameters, and it is easily shown that their packets are singletons for GL_N. The anticipated equality of the Adams-Johnson packets with the ABV-packets is proven in [Ara19]. The proof that ABV-packets are singletons for arbitrary A-parameters ψ of GL_N follows from a decomposition of ψ in terms of Adams-Johnson A-parameters of smaller general linear groups, and an application of all standard endoscopic lifting from the direct product of these smaller general linear groups (Proposition 6.3).

The purpose of Section 7 is the proof of Equation (15(c)). This equation gives a precise relationship between the Whittaker and Atlas extensions of $\pi(\xi)$ in terms of the *integral lengths* $l^I(\xi)$ and $l^I_\vartheta(\xi)$ ([ABV92]*(16.6), (60), (61)). This identity is peculiar in that a Whittaker extension is inherently an analytic object, whereas an Atlas extension is inherently algebraic. When $\pi(\xi)$ is the Langlands quotient of a (standard) principal series representation, the differences between the two extensions may be attributed to differences in the extensions of quasicharacters of the diagonal subgroup H. This reduction for principal series furnishes an easy proof of (15(c)) (Lemma 7.1).

In some sense (see the proof of Proposition 7.3), $\pi(\xi)$ is furthest from a Langlands quotient of principal series when $\pi(\xi)$ is *generic*, *i.e.* has a Whittaker model. The proof of (15(c)) for generic representations is the key to the general proof, in that irreducible generic representations occur as subrepresentations of all standard representations (Lemma 7.2), and determine the Whittaker extensions of standard representations. If one knows the (signed) multiplicity with which an irreducible twisted generic character $\pi(\xi_0)^+$ appears in the decomposition of a twisted standard principal series representation $M(\xi)^+$, then one can use the knowledge of (15(c)) for $\pi(\xi)$ to prove (15(c)) for $\pi(\xi_0)$. This desired multiplicity is computed in Proposition 7.3, and the proof of (15(c)) for generic $\pi(\xi_0)$ occurring in the standard principal representation $M(\xi)$ is Proposition 7.4.

It is implicit in the previous paragraph that the parameters and representations are all ϑ-stable. However not every ϑ-stable generic representation $\pi(\xi_0)$ is a subrepresentation of a *ϑ-stable* principal series representation. Therefore, the strategy of the previous paragraph does not provide an exhaustive proof of (15(c)). Most of Section 7 is dedicated to the description of a ϑ-stable standard representation which plays the part of the principal series representation. In Lemma 7.6 we prove that every ϑ-stable generic representation $\pi(\xi_0)$ which has *integral infinitesimal character* is a subrepresentation of a ϑ-stable standard representation satisfying (15(c)). We remove the restriction of integrality on the infinitesimal character in Lemma 7.7. We then follow the strategy of the previous paragraph using the ϑ-stable standard representation of Lemma 7.7 to prove (15(c)) in general. This is the last result needed to apply the twisted pairing (18) to the computation of (17).

The theorems comparing $\eta^{\mathrm{Ar}}_{\psi_G}$ with $\eta^{\mathrm{ABV}}_{\psi_G}$ are to be found in Sections 8 and 9. Section 8 is presented under the assumption that the infinitesimal character ${}^\vee\mathcal{O}_G$ is regular in GL_N. This regularity condition is removed in Section 9 by applying the *Jantzen-Zuckerman translation principal* to the (twisted) characters and to the pairings. The main theorem, Theorem 9.3, states that

$$\eta^{\mathrm{Ar}}_{\psi_G} = \eta^{\mathrm{ABV}}_{\psi_G} \text{ and } \Pi^{\mathrm{Ar}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G} \tag{19}$$

when G is not isomorphic to SO_N for even N, and that

$$\eta^{\mathrm{Ar}}_{\psi_G} = \frac{1}{2}\left(\eta^{\mathrm{ABV}}_{\psi_G} + \eta^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}\right) \text{ and } \Pi^{\mathrm{Ar}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G} \cup \Pi^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G} \tag{20}$$

when $G \cong \mathrm{SO}_N$ with N even. In light of this theorem, we again look back to Problems B-E in Section 10. In particular, we show that the solution to Problem C in [ABV92] solves Problem C in Arthur's definition.

An unsolved problem, related to Problems B and C, is to determined the dimensions of the finite-dimensional representations $\tau_{\psi_G}(\pi)$. This has been explored in [MR20], [MR18], [MgR19] and in [Mœg11] for p-adic groups, where the dimensions have been shown to equal 1. Theorem 9.3 opens up the possibility of using techniques from microlocal geometry to settle this problem.

Our work also connects with the study of Adams-Johnson packets ([AJ87]). Adams-Johnson packets have been proven to equal Arthur's packets in [AMR18]. Moreover, Adams-Johnson packets are particular cases of our ABV-packets ([Ara19]). They are the ABV-packets with regular and integral infinitesimal character.

A natural question for future consideration is how the packets for quasisplit unitary groups, established by Mok [Mok15], compare with the microlocal packets of [ABV92]. The methods developed here appear to be equally applicable to the setting of quasisplit unitary groups. Furthermore, the context of pure inner forms in which we work, ought also to allow for easy comparison with the related work of [KMSW14] and [Art13]*Chapter 9.

Another natural question is whether similar comparisons between p-adic Arthur-packets can be made. In the p-adic context $\eta^{\mathrm{Ar}}_{\psi_G}$ is also defined in [Art13]. The beginnings of $\eta^{\mathrm{ABV}}_{\psi_G}$ in the p-adic context are to be found in [Vog93] and [**?**]. Low rank comparisons between the two stable distributions are made in [**?**]*Part 2.

The second and third authors would like to thank the developers of the Atlas of Lie groups software. It was a pleasure to see our early conjectures borne out by low rank computations.

2 The local Langlands correspondence

This section begins with a review of the local Langlands correspondence as conceived in [ABV92]. An important feature of this version of the correspondence is the notion of *strong real forms* and their representations. More recently, strong real forms have been supplanted by the equivalent notion of *strong involutions* ([AdC09]). We have chosen to use the language of strong involutions in our review.

Another difference in our review is in limiting ourselves to only *pure* strong involutions. In doing so, we limit the scope of [ABV92] to fewer real forms of G. This limitation is compensated for by not having to introduce covers of the dual group ${}^\vee G$. We still capture all of the information needed for the quasisplit form of G, while preserving a sense of how the theory applies to other real forms.

The objects parameterizing irreducible representations in [ABV92] have also been supplanted by newer parameters in [AdC09] and [AV15]. We call these newer parameters *Atlas parameters*. The advantages to Atlas parameters are their amenability to Vogan duality and Hecke algebra computations. These advantages are used in Section 4. Another advantage to using Atlas parameters is in defining canonical extensions of representations of $\mathrm{GL}_N(\mathbb{R})$ to representations of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. We call these canonical extensions *Atlas extensions*.

We conclude this section with a discussion of the Grothendieck groups of representations for connected groups G and for the disconnected group $\mathrm{GL}_N \rtimes \langle\vartheta\rangle$. In the connected case the Grothendieck group is isomorphic to the $\mathbb{Z}$-span of distribution characters. In the disconnected case we construct a quotient of the Grothendieck group which will be seen to be isomorphic to the $\mathbb{Z}$-span of twisted distribution characters.

2.1 Extended groups and complete geometric parameters

In this section G can be an arbitrary connected reductive complex group. We give a version of the local Langlands correspondence suitable to our application. We largely follow [ABV92], with modifications from the more recent papers [AdC09], [AV15] and [AvLTV20].

Our starting point is the connected reductive complex group G, together with a pinning

$$(B, H, \{X_\alpha\}) \tag{21}$$

in which B is a Borel subgroup, $H \subset B$ is a maximal torus and $\{X_\alpha\}$ is a set of simple root vectors relative to the positive root system $R^+(G,H) = R(B,H)$ of $R(G,H)$. Let ${}^\vee\rho = \frac{1}{2}\sum_{\alpha\in R^+(G,H)} {}^\vee\alpha$.

We fix an inner class of real forms for G. The inner class is determined by a unique algebraic involution δ_0 of G fixing the pinning ([AdC09]*Section 2). The involution defines the extended group

$$G^\Gamma = G \rtimes \langle\delta_0\rangle.$$

A *strong involution* of G^Γ is an element $\delta \in G^\Gamma - G$ such that δ^2 is central in G and has finite order ([AvLTV20]*Definition 12.3). Two strong involutions are equivalent if they are G-conjugate. There is a surjective map from (equivalence classes) of strong involutions to (isomorphism classes of) real forms. This map takes a strong involution δ to the real form $G(\mathbb{R},\delta)$ in the inner class with Cartan involution

$$\theta_\delta = \mathrm{Int}(\delta).$$

This map is bijective if G is adjoint, but is not injective in general.

There is also a well-known bijection between real forms in the inner class and the cohomology set $H^1(\mathbb{R}, G/Z(G))$ ([Spr98]*12.3.7). The domain of the quotient map

$$H^1(\mathbb{R}, G) \to H^1(\mathbb{R}, G/Z(G)) \tag{22}$$

defines the set of *pure inner forms* ([Vog93]*Section 2). Let $\sigma \in \Gamma$ be the nontrivial element of the Galois group. For any 1-cocycle $z \in Z^1(\mathbb{R}, G)$ one may define a strong involution by

$$z(\sigma)\exp(\pi i\,{}^\vee\rho)\delta_0 \in G^\Gamma$$

($\exp(\pi i\,{}^\vee\rho)\delta_0$ is the *large* involution in [AV15]*(11f)-(11h)). This sends classes in $H^1(\mathbb{R}, G)$ to G-conjugacy classes in $G^\Gamma - G$. The (equivalence classes of) *pure strong involutions* are defined as the image of this map. The quasisplit real form is pure in the sense that the fibre of the trivial cocycle in (22) is a singleton. The pure strong involution corresponding to the quasisplit pure real form is

$$\delta_q = \exp(\pi i\,{}^\vee\rho)\delta_0.$$

Given a strong involution δ we set K to be the fixed-point subgroup G^{θ_δ}. The real form $G(\mathbb{R},\delta)$ contains

$$K(\mathbb{R}) = G(\mathbb{R},\delta) \cap K \tag{23}$$

as a maximal compact subgroup and is determined by K ([AV15]*(5f)-(5g)). By a representation of $G(\mathbb{R},\delta)$ we usually mean an admissible $(\mathfrak{g}, K)$-module, although we will need admissible group representations in Section 7. A *representation of a strong involution* is a pair (π, δ) in which δ is a strong involution and π is an admissible $(\mathfrak{g}, K)$-module. There is a natural notion of equivalence of strong involutions ([AdC09]*Definition 6.1), and we let $\Pi(G(\mathbb{R},\delta))$ be the set of equivalence classes of irreducible representations (π', δ') of strong involutions in which δ' is equivalent to δ. Let

$$\Pi(G/\mathbb{R}) = \coprod_\delta \Pi(G(\mathbb{R},\delta))$$

be the disjoint union over the (equivalence classes of) pure strong involutions δ.

Let ${}^\vee G$ be the Langlands dual group of G together with a pinning

$$({}^\vee B, {}^\vee H, \{X_{{}^\vee\alpha}\}).$$

The pinning and the involution δ_0 fix an involution ${}^\vee\delta_0$ of ${}^\vee G$ as prescribed in [AV15]*(12). Following this prescription, ${}^\vee\delta_0$ is trivial if and only if δ_0 defines the inner class of the split form of G. The group

$$ {}^\vee G^\Gamma = {}^\vee G \rtimes \langle {}^\vee\delta_0 \rangle $$

is the L-group of our inner class.

Suppose λ is a semisimple element of the Lie algebra ${}^\vee\mathfrak{g}$. After conjugating by ${}^\vee G$ we may assume $\lambda \in {}^\vee\mathfrak{h}$. Using the canonical isomorphism ${}^\vee\mathfrak{h} \simeq \mathfrak{h}^*$ we identify λ with an element of $\mathfrak{h}^*$, and hence via the Harish-Chandra homomorphism, with an infinitesimal character for G. This construction only depends on the ${}^\vee G$-orbit of λ. We refer to a semisimple element $\lambda \in {}^\vee\mathfrak{g}$, or a ${}^\vee G$-orbit ${}^\vee\mathcal{O} \subset {}^\vee\mathfrak{g}$ of semisimple elements, as an *infinitesimal character* for G. Let

$$ \Pi({}^\vee\mathcal{O}, G/\mathbb{R}) \subset \Pi(G/\mathbb{R}) $$

be the representations (of pure strong involutions) with infinitesimal character ${}^\vee\mathcal{O}$.

Let $P\left({}^\vee G^\Gamma\right)$ be the set of *quasiadmissible* homomorphisms $\phi : W_\mathbb{R} \to {}^\vee G^\Gamma$ ([ABV92]*Definition 5.2). Associated to $\phi \in P({}^\vee G^\Gamma)$ is an infinitesimal character ([ABV92]*Proposition 5.6). Let

$$ P\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right) $$

be the set of quasiadmissible homomorphisms with infinitesimal character ${}^\vee\mathcal{O}$. The group ${}^\vee G$ acts on $P\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right)$ by conjugation.

2.2 The space $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$

We make frequent use of the complex variety $X\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right)$ of *geometric parameters* ([ABV92]*Definition 6.9, [Vog93]*Definition 6.9). Here we sketch a definition based on [ABV92]*Proposition 6.17 and state its main properties.

Write the Weil group $W_\mathbb{R}$ as $\mathbb{C}^\times \coprod j\mathbb{C}^\times$. Suppose $\phi \in P\left({}^\vee G^\Gamma\right)$. Define $\lambda, \gamma \in {}^\vee\mathfrak{g}$ by

$$ \phi(z) = z^\lambda \overline{z}^\gamma, \quad z \in \mathbb{C}^\times. \tag{24)(a} $$

Let ${}^\vee\mathfrak{n}(\lambda)$ be the sum of the positive integer eigenspaces of $\mathrm{ad}(\lambda)$ on ${}^\vee\mathfrak{g}$, and let ${}^\vee N(\lambda)$ be the connected unipotent subgroup of ${}^\vee G$ with Lie algebra ${}^\vee\mathfrak{n}(\lambda)$. Set

$$ \begin{aligned} {}^\vee G(\lambda) &= \mathrm{Cent}_{{}^\vee G}(\exp(2\pi i\lambda)) \\ {}^\vee L(\lambda) &= \mathrm{Cent}_{{}^\vee G}(\lambda) \subset {}^\vee G(\lambda) \end{aligned} \tag{24)(b} $$

and let

$$ {}^\vee P(\lambda) = {}^\vee L(\lambda) {}^\vee N(\lambda) \tag{24)(c} $$

a parabolic subgroup of ${}^\vee G(\lambda)$. Finally, write

$$ \begin{aligned} y &= \exp(\pi i\lambda)\phi(j) \\ {}^\vee K_y &= \mathrm{Cent}_{{}^\vee G}(y) \end{aligned} \tag{24)(d} $$

and let ${}^\vee N_y(\lambda)$ be the group generated by

$$ {}^\vee N(\lambda) \cap \mathrm{Int}(y)\big({}^\vee P(\lambda)\big) \quad \text{and} \quad {}^\vee P(\lambda) \cap \mathrm{Int}(y)\big({}^\vee N(\lambda)\big). $$

Define an equivalence relation on $P({}^\vee G^\Gamma)$ by

$$ \phi(y,\lambda) \sim \phi'(y',\lambda') \ \text{ if } \ y' = y \ \text{ and } \ \lambda' = n\cdot\lambda \ \text{ for some } \ n \in {}^\vee N_y(\lambda) \cap {}^\vee K_y $$

where the action is by the conjugation action of ${}^\vee G$. This equivalence relation preserves each subset $P({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \subset P({}^\vee G^\Gamma)$. We let $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ be the set of equivalence classes

$$ X\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right) = P\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right) / \sim \tag{25} $$

with the quotient topology. The element y of (24)(d) is constant on equivalence classes, so for $p \in X\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right)$ we define $y(p)$ accordingly.

There are ${}^\vee G$-equivariant maps

$$P\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right) \to X\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right) \to P\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right)/{}^\vee G \tag{26}$$

which are bijections on the levels of ${}^\vee G$-orbits ([ABV92]*Proposition 6.17). The more interesting and useful space is $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. It has finitely many ${}^\vee G$-orbits. Here is some information about its structure (see [ABV92]*Section 6).

Use the notation of (24)(a)-(24)(d), and suppose $p \in X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. Let $y = y(p)$. Note that ${}^\vee K_y$ is the fixed-point subgroup of the involution $\mathrm{Int}(y)$ in ${}^\vee G(\lambda)$. There is an open and closed, connected, smooth subvariety

$$X_y({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \subset X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$$

such that the ${}^\vee G$-orbits on $X_y({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ are in bijection with the ${}^\vee K_y$-orbits on the partial flag variety ${}^\vee G(\lambda)/{}^\vee P(\lambda)$. Furthermore, this bijection respects the closure relations between ${}^\vee G$ and ${}^\vee K_y$-orbits ([ABV92]*Proposition 6.16).

Now suppose $S \subset X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ is a ${}^\vee G$-orbit. If $p \in S$ let ${}^\vee G_p = \mathrm{Stab}_{{}^\vee G}(p)$. A *pure complete geometric parameter* for $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ is a pair (S, τ_S) where S is a ${}^\vee G$-orbit on $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ and τ_S is (an equivalence class of) an irreducible representation of the component group ${}^\vee G_p/({}^\vee G_p)^0$ ([ABV92]*Definitions 7.1 and 7.6). We denote the set of pure complete geometric parameters for $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ by $\Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$.

A special case of the local Langlands correspondence as stated in [ABV92]*Theorem 10.11 is a bijection

$$\Pi\left({}^\vee\mathcal{O}, G/\mathbb{R}\right) \longleftrightarrow \Xi\left({}^\vee\mathcal{O}, {}^\vee G^\Gamma\right) \tag{27}$$

between representations of pure strong involutions and pure complete geometric parameters. Recall from the previous section that the left-hand side of (27) contains the subset $\Pi({}^\vee\mathcal{O}, G(\mathbb{R}, \delta_q))$.

2.3 Extended groups for G and ${}^\vee G$

We specialize the results of the previous section to the groups GL_N, Sp_N and SO_N, providing further details.

For the group GL_N we fix the usual pinning (21) in which B is the upper-triangular subgroup, H is the diagonal subgroup, and X_α is a matrix with 1 in the entry corresponding to α and zeroes elsewhere. We fix the *split* inner class for GL_N. The split inner class consists of the split group $\mathrm{GL}_N(\mathbb{R})$, and, if N is even, also the quaternionic form $\mathrm{GL}_{N/2}(\mathbb{H})$.

There are two algebraic involutions of GL_N which fix the pinning: the identity, and ϑ (2). It is a coincidence that the strong involution corresponding to the split inner class is ϑ. Indeed,

$$G^\vartheta = \begin{cases} \mathrm{O}_N, & N \text{ odd} \\ \mathrm{Sp}_N, & N \text{ even} \end{cases}$$

which match the respective maximal compact subgroups of $\mathrm{GL}_N(\mathbb{R})$ and $\mathrm{GL}_{N/2}(\mathbb{H})$ ((23), [Kna96]*(1.123)). Thus, we define

$$\mathrm{GL}_N^\Gamma = \mathrm{GL}_N \rtimes \langle \delta_0 \rangle \tag{28}$$

where $\delta_0^2 = 1$ and δ_0 acts (by chance!) as ϑ on GL_N.

Lemma 2.1. *1. There is a unique conjugacy class of strong involutions in* $\mathrm{GL}_N \rtimes \langle \delta_0 \rangle$ *which maps to the (isomorphism class of the) split group* $\mathrm{GL}_N(\mathbb{R})$.

2. The strong involutions δ in this conjugacy class are characterized by

$$\delta^2 = (-1)^{N+1} = \exp(2\pi i\, {}^\vee\rho).$$

3. If N is odd this is the unique conjugacy class of strong involutions.

4. If N is even there is exactly one other class, whose elements square to 1*. This other class maps to the real form* $\mathrm{GL}_{N/2}(\mathbb{H})$.

5. *In both cases there is only only one class of pure strong involutions, and it corresponds to the split form* $\mathrm{GL}_N(\mathbb{R})$ *so that*

$$\Pi\left({}^\vee\mathcal{O}, \mathrm{GL}_N/\mathbb{R}\right) = \Pi\left({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})\right). \tag{29}$$

Proof. By [AdC09]*Proposition 12.19 (2) the conjugacy classes of strong involutions are parameterized by the H-conjugacy classes of the elements in

$$\left\{t\delta_0 \in H \rtimes \langle\delta_0\rangle : (t\delta_0)^2 \in Z(G)\right\}$$

modulo the action of a Weyl group. For GL_N it is straightforward to compute that up to conjugacy the representatives in this set are δ_0, with $\delta_0^2 = 1$, and, when N is even, $\delta = \exp(\pi i {}^\vee\rho)\delta_0$, with $\delta^2 = -1$. The real forms associated to these strong involutions are as stated ([Ada08]*Examples 7.6 and 7.8). We leave the identity $\exp(2\pi i\,{}^\vee\rho) = (-1)^{N+1}$ as an exercise. All that remains to be proven is that when N is even, the real form $\mathrm{GL}_N(\mathbb{H})$ is not pure. Looking back to (22), this follows from the fact that $H^1(\mathbb{R}, \mathrm{GL}_N) = \{1\}$ corresponds to a single real form, which is (quasi)split. $\square$

If δ is a strong involution of any group G, we say δ has *infinitesimal cocharacter* $g \in \mathfrak{h}$ if

$$\delta^2 = \exp(2\pi i g).$$

Lemma 2.1 tells us that the pure strong involutions of GL_N are exactly those with infinitesimal cocharacter ${}^\vee\rho$. Let $\delta_q = \delta_0$ when N is odd and $\delta_q = \exp(\pi i\,{}^\vee\rho)\delta_0$ when N is even. According to Lemma 2.1, the strong involutions δ in the GL_N-conjugacy class of δ_q form the set of pure involutions and these are the only strong involutions for which $\mathrm{GL}(\mathbb{R}, \delta) \cong \mathrm{GL}_N(\mathbb{R})$.

Since $\mathrm{GL}_N(\mathbb{R})$ is split, the L-group is

$${}^\vee\mathrm{GL}_N^\Gamma = {}^\vee\mathrm{GL}_N \times \langle{}^\vee\delta_0\rangle \simeq {}^\vee\mathrm{GL}_N \times \mathbb{Z}/2\mathbb{Z}. \tag{30}$$

We write ${}^\vee\mathrm{GL}_N$ instead of GL_N just to emphasize that the group is on the "dual side".

We also need the extended group

$$\mathrm{GL}_N \rtimes \langle\vartheta\rangle.$$

Although this is isomorphic to $\mathrm{GL}_N^\Gamma = \mathrm{GL}_N \rtimes \langle\delta_0\rangle$, it plays a very different role. The group GL_N^Γ plays a role in the ordinary (untwisted) Langlands correspondence by carrying strong involutions and thereby information about real forms. By contrast, the group $\mathrm{GL} \rtimes \langle\vartheta\rangle$ is the central object in the theory of twisted characters. We use this notation to distinguish the two roles. With this in mind, we have the group

$$\mathrm{GL}_N^\Gamma \rtimes \langle\vartheta\rangle = \langle\mathrm{GL}_N, \delta_0, \vartheta\rangle$$

in which ϑ and δ_0 commute. Similarly, we define

$${}^\vee\mathrm{GL}_N^\Gamma \rtimes \langle\vartheta\rangle = \langle{}^\vee\mathrm{GL}_N, {}^\vee\delta_0, \vartheta\rangle \tag{31}$$

in which ϑ and ${}^\vee\delta_0$ commute. See Section 5.2 for a discussion of the twisted endoscopic groups for $({}^\vee\mathrm{GL}_N^\Gamma, \vartheta)$.

Next we consider extended groups for Sp_N and SO_N. As in [Art13]*Section 1.2, we adopt the convention of expressing these groups in such a way that their upper-triangular subgroups are Borel subgroups. With this convention, their diagonal subgroups are maximal tori. When there is no confusion with the setting of GL_N, we will also denote these Borel subgroups by B and maximal tori by H. In fact, we will abusively imitate the notation for GL_N when the setting is clear. We arbitrarily fix a set of simple root vectors $\{X_\alpha\}$ for each simple root in $R(B, H)$.

Suppose first that $G = \mathrm{Sp}_{2n}$ or $G = \mathrm{SO}_{2n+1}$. Each of these groups has one inner class, which is the inner class of the split form. This allows us to choose δ_0 to act trivially on these groups and set

$$G^\Gamma = G \times \langle\delta_0\rangle$$

where $\delta_0^2 = 1$. Define $\delta_q = \delta_0$, so that $G(\mathbb{R}, \delta_q)$ is the split real form. The dual groups ${}^\vee G = {}^\vee\mathrm{Sp}_{2n} = \mathrm{SO}_{2n+1}$ and ${}^\vee G = {}^\vee\mathrm{SO}_{2n+1} = \mathrm{Sp}_{2n}$ have Borel subgroups and maximal tori as earlier. The L-group of G corresponding to the split inner class is

$${}^\vee G^\Gamma = {}^\vee G \times \langle{}^\vee\delta_0\rangle \cong {}^\vee G \times \mathbb{Z}/2\mathbb{Z}.$$

Finally, take $G = \mathrm{SO}_{2n}$. This group has two inner classes: one for the split form and the other for the quasisplit form $\mathrm{SO}(n+1, n-1)$, which is not split. The following definitions follow from [AV15]*(12c) and [Bou02]* Chapter VI §4.8 XI.

If the inner class contains the split form then ${}^\vee\delta_0$ acts trivially and

$$ {}^\vee\mathrm{SO}_{2n}^\Gamma = \mathrm{SO}_{2n} \times \langle {}^\vee\delta_0 \rangle. $$

If, in addition, n is even then δ_0 acts trivially and $\mathrm{SO}_{2n}^\Gamma = \mathrm{SO}_{2n} \times \langle \delta_0 \rangle$. On the on the other hand if n is odd then δ_0 acts by conjugation by an element in $\mathrm{O}_{2n} - \mathrm{SO}_{2n}$ which preserves the pinning. In this case SO_{2n}^Γ is a nontrivial semidirect product $\mathrm{SO}_{2n} \rtimes \langle \delta_0 \rangle$.

If we fix the inner class to be that of $\mathrm{SO}(n+1, n-1)$ then the L-group is the semidirect product

$$ {}^\vee\mathrm{SO}_{2n}^\Gamma = \mathrm{SO}_{2n} \rtimes \langle {}^\vee\delta_0 \rangle $$

in which ${}^\vee\delta_0$ acts by conjugation by an element in $\mathrm{O}_{2n} - \mathrm{SO}_{2n}$ which preserves the pinning. If n here is even then $\mathrm{SO}_{2n}^\Gamma = \mathrm{SO}_{2n} \rtimes \langle \delta_0 \rangle$ where δ_0 acts in the same way as ${}^\vee\delta_0$. On the other hand, if n is odd then $\mathrm{SO}_{2n}^\Gamma = \mathrm{SO}_{2n} \times \langle \delta_0 \rangle$.

For either of the two inner classes of SO_{2n} the strong involution $\delta_q = \exp(\pi i {}^\vee\rho)\delta_0$ corresponds to a quasisplit real form $G(\mathbb{R}, \delta_q)$ [AV15]*(11f).

2.4 Atlas parameters for GL_N

For our application we use a formulation of the local Langlands correspondence for $\mathrm{GL}_N(\mathbb{R})$ which is well suited to Vogan duality (see Section 4.1). The main references for this section are [AdC09] and [AV15]*Section 3.

We start by working in the context of the extended group (28): $\mathrm{GL}_N^\Gamma = \mathrm{GL}_N \rtimes \langle \delta_0 \rangle$. Let ${}^\vee\rho$ be the half-sum of the positive coroots for GL_N. Following [AV15]*Section 3 we set

$$ \mathcal{X}_{{}^\vee\rho} = \left\{ \delta \in \mathrm{Norm}_{\mathrm{GL}_N\delta_0}(H) \mid \delta^2 = \exp(2\pi i\, {}^\vee\rho) \right\} / H $$

where the quotient is by the conjugation action of H. This is a set of H-conjugacy classes of strong involutions with infinitesimal cocharacter ${}^\vee\rho$. By Lemma 2.1, these strong involutions are all pure and correspond to the split form $\mathrm{GL}_N(\mathbb{R})$.

Now we fix a ϑ-fixed, regular, integrally dominant element $\lambda \in {}^\vee\mathfrak{h}$ for GL_N. This means

$$ \begin{aligned} &\vartheta(\lambda) = \lambda \\ &\langle \lambda, {}^\vee\alpha \rangle \neq 0, \quad \alpha \in R(\mathrm{GL}_N, H) \\ &\langle \lambda, {}^\vee\alpha \rangle \notin \{-1, -2, -3, \ldots\}, \;\; \alpha \in R^+(\mathrm{GL}_N, H). \end{aligned} \tag{32} $$

This will be the infinitesimal character of our representations of $\mathrm{GL}_N(\mathbb{R})$. The assumption of integral dominance is harmless ([AV15]*Lemma 4.1). We shall remove the regularity assumption at the beginning of Section 9.

The action of δ_0 induces an action on the Weyl group $W(\mathrm{GL}_N, H)$. Consider the set

$$ \{ w \in W(\mathrm{GL}_n, H) : w\, \delta_0(w) = 1 \}. \tag{33} $$

If $x \in \mathcal{X}_{{}^\vee\rho}$ then the action (by conjugation) of x on H is equal to $w\delta_0$ for some w in the set (33). Define $p(x) = w$ accordingly. The map p is surjective. Let $\mathcal{X}^w_{{}^\vee\rho}$ be the fibre of p over w so that

$$ \mathcal{X}^w_{{}^\vee\rho} = \left\{ x \in \mathcal{X}_{{}^\vee\rho} : xhx^{-1} = w\delta_0 \cdot h, \;\; \text{for all } h \in H \right\}. \tag{34} $$

On the dual side we have an analogous set in which the infinitesimal cocharacter ${}^\vee\rho$ is replaced by an infinitesimal character λ, namely

$$ {}^\vee\mathcal{X}_\lambda = \left\{ {}^\vee\delta \in \mathrm{Norm}_{{}^\vee\mathrm{GL}_N\, {}^\vee\delta_0}({}^\vee H) \mid {}^\vee\delta^2 = \exp(2\pi i \lambda) \right\} / {}^\vee H. $$

Recall that $\mathrm{GL}_N(\mathbb{R})$ is split, so ${}^\vee\delta_0$ acts trivially on ${}^\vee\mathrm{GL}_N$, and we can safely identify this set with

$$ \left\{ {}^\vee\delta \in \mathrm{Norm}_{{}^\vee\mathrm{GL}_N}({}^\vee H) \mid {}^\vee\delta^2 = \exp(2\pi i \lambda) \right\} / {}^\vee H. $$

Since ${}^\vee\delta_0$ acts trivially, the analogue of (33) is

$$\left\{w \in W \mid w^2 = 1\right\}.$$

Let ${}^\vee\mathcal{X}^w_\lambda$ be the analogue of (34).

It is easily verified that

$$w\,\delta_0(w) = ww_0ww_0^{-1} = (ww_0)^2, \quad w \in W(\mathrm{GL}_N, H) \tag{35}$$

where $w_0 \in W(\mathrm{GL}_N, H)$ is the long Weyl group element. It follows that

$$w \mapsto ww_0$$

defines a bijection from (33) to $\{w \in W(\mathrm{GL}_N, H) : w^2 = 1\}$. This map allows us to pair any set $\mathcal{X}^w_{{}^\vee\rho}$ with the set ${}^\vee\mathcal{X}^{ww_0}_\lambda$.

The next result follows from [AdC09], [ABV92] and [AV15]*Theorem 3.11. We give the proof, which is much simpler in the case of GL_N than for other reductive groups.

Lemma 2.2. *There is a canonical bijection*

$$\coprod_{\{w: w\delta_0(w)=1\}} \mathcal{X}^w_{{}^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda \longleftrightarrow \Xi\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right).$$

Proof. First of all $|\mathcal{X}^w_{{}^\vee\rho}| = 1$ for all w. This follows from [AdC09]*Proposition 12.19(5): the dual inner class is the equal rank inner class, consisting of (products of) unitary groups $U(p, q)$, and it is well known that the Cartan subgroups of $U(p, q)$ are all connected. This is equivalent to the fact that all L-packets for $\mathrm{GL}_N(\mathbb{R})$ are singletons.

So the lemma comes down to the statement that there is a bijection

$$\coprod_{w\delta_0(w)=1} {}^\vee\mathcal{X}^{ww_0}_\lambda \longleftrightarrow \Xi\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right).$$

Recall the right-hand side is the set of pure complete geometric parameters (S, τ) where S is a ${}^\vee\mathrm{GL}_N$-orbit in $X\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right)$ and τ is an irreducible representation of the component group of the centralizer of a point in $X\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right)$. Since ${}^\vee\delta_0$ acts trivially on ${}^\vee\mathrm{GL}_N$, these centralizers are products of general linear groups, and are hence connected. Therefore we are further reduced to showing

$$\coprod_{w\delta_0(w)=1} {}^\vee\mathcal{X}^{ww_0}_\lambda \longleftrightarrow X\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right)/{}^\vee\mathrm{GL}_N.$$

Suppose $y \in {}^\vee\mathcal{X}^{ww_0}_\lambda$. This means that $y \in \mathrm{Norm}_{{}^\vee\mathrm{GL}_N}(H)$ (we can ignore the extension), y maps to $w \in W(\mathrm{GL}_N, H)$, and $y^2 = \exp(2\pi i\lambda)$. Define $\phi : W_\mathbb{R} \to {}^\vee\mathrm{GL}_N^\Gamma$ by

$$\begin{aligned}\phi(z) &= z^\lambda \overline{z}^{\mathrm{Ad}(y)(\lambda)}, \quad z \in \mathbb{C}^\times\\ \phi(j) &= \exp(-\pi i\lambda)y\end{aligned}$$

(compare (24)(a) and (d)). It is straightforward to see that ϕ is a quasiadmissible homomorphism (see the end of Section 2.3), and only a little more work to show that it induces the bijection indicated. See [AdC09]*Proposition 9.4.

□

Together with (27) this gives

Theorem 2.3. *Let ${}^\vee\mathcal{O}$ be the ${}^\vee\mathrm{GL}_N$-orbit of λ. There are canonical bijections:*

$$\coprod_{\{w: w\delta_0(w)=1\}} \mathcal{X}^w_{{}^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda \longleftrightarrow \Xi\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right) \longleftrightarrow \Pi\left({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})\right).$$

As in [AV15]*Theorem 3.11 the bijection of Theorem 2.3 is written as

$$\mathcal{X}^w_{^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda \ni (x,y) \mapsto J(x,y,\lambda) \tag{36}$$

We call the pair (x,y) on the left the *Atlas parameter* of the irreducible representation $J(x,y,\lambda)$. By Lemma 2.2, the Atlas parameter (x,y) is equivalent to a complete geometric parameter $\xi \in \Xi({}^\vee\mathcal{O}, \mathrm{GL}_N^\Gamma)$, and accordingly we define

$$\pi(\xi) = J(x,y,\lambda).$$

The representation $\pi(\xi)$ is the Langlands quotient of a standard representation which we denote by $M(\xi)$ or $M(x,y)$.

2.5 Twisted Atlas parameters for GL_N

Our next task is to describe the generalization of Theorem 2.3 to the ϑ-twisted setting. This involves certain irreducible representations of the extended group $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. We specialize the results of [AV15]*Sections 3-5 to this case. We are fortunate that some of the more complicated issues that arise in [AV15] do not occur for GL_N.

We continue with the hypotheses of (32). Recall that both ${}^\vee\rho$ and λ are fixed by ϑ. By Clifford theory, an irreducible representation of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ restricted to $\mathrm{GL}_N(\mathbb{R})$ is either an irreducible ϑ-fixed representation, or the direct sum of two irreducible representations which are exchanged by the action of ϑ. We only need representations of the first type.

It is a lengthy but straightforward task to show that the map (36) is ϑ-equivariant (*cf.* [CM18]*Theorem 4.1). Therefore $J(x,y,\lambda)$ is ϑ-stable if and only if $(x,y) \in \mathcal{X}^w_{^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda$ is fixed by ϑ. Let

$$\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \subset \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$$

be the subset of ϑ-fixed irreducible representations and set

$$W(\delta_0,\vartheta) = \{w \in w \mid w\delta_0(w) = 1, w = \vartheta(w)\}$$

(*cf.* (33)). By the ϑ-equivariance, Theorem 2.3 restricts to these sets and we obtain

Corollary 2.4. *Suppose λ satisfies the hypotheses of (32) and let ${}^\vee\mathcal{O}$ be its ${}^\vee G$-orbit. Then there is a canonical bijection*

$$\coprod_{\{w \in W(\delta_0,\vartheta)\}} \mathcal{X}^w_{^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda \longleftrightarrow \Pi\left({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})\right)^\vartheta$$

written $(x,y) \mapsto J(x,y,\lambda)$.

We now introduce the *extended parameters* of [AV15]*Sections 3-5, and summarize the facts that we need. Fix $w \in W(\delta_0,\vartheta)$. An *extended parameter* for w is a set

$$E = (\mathrm{\lambda}, \mathrm{\tau}, \ell, t), \quad \mathrm{\lambda}, \mathrm{\tau} \in X^*(H),\ \ell, t \in X_*(H) \tag{37}$$

satisfying certain conditions depending on w (see [AV15]*Definition 5.4).[1] There is a surjective map

$$E \mapsto (x(E), y(E)) \tag{38}$$

taking extended parameters for w to $\mathcal{X}^w_{^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda$. This map only depends on $\mathrm{\lambda}$ and ℓ. In addition,

$$J(x(E), y(E), \lambda) \in \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta,$$

and every ϑ-fixed irreducible representation arises this way. The remaining parameters $\mathrm{\tau}$ and t in E define an irreducible representation $J(E,\lambda)$ of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ satisfying

$$J(E,\lambda)|_{\mathrm{GL}_N(\mathbb{R})} = J(x(E), y(E), \lambda).$$

[1] Warning! The symbols $\mathrm{\lambda}$ and $\mathrm{\tau}$ here are not to be confused with symbols λ and τ appearing elsewhere. Note the slight difference in font. We have chosen to use $\mathrm{\lambda}$ and $\mathrm{\tau}$ for ease of comparison with [AV15].

The representation $J(x(E), y(E), \lambda)$ is determined by a quasicharacter of a Cartan subgroup of $\mathrm{GL}_N(\mathbb{R})$. The representation $J(E, \lambda)$ is determined by the semidirect product of this Cartan subgroup with an element $h\vartheta \in \mathrm{GL}_N \rtimes \vartheta$ ([AV15]*(24e)) and a choice of extension of the quasicharacter to the semidirect product. The value of the extended quasicharacter on the element $h\vartheta$ depends on a choice of sign [AV15]*Definition 5.2, and the square root of this sign is given by

$$z(E) = i^{\langle \tau, (1+w)t \rangle} (-1)^{\langle \lambda, t \rangle}. \tag{39}$$

The preceding discussion is a specialization of a general framework to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle \vartheta \rangle$. One of the special properties of $\mathrm{GL}_N(\mathbb{R})$ is that the preimage of any $(x, y) \in \mathcal{X}^w_{^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda$ under (38) has a *preferred extended parameter* of the form

$$(\lambda, \tau, 0, 0).$$

This comes down to the fact that $X^w_{^\vee\rho}$ is a singleton (see the proof of Lemma 2.2). By taking $t = 0$ we see $z(\lambda, \tau, 0, 0) = 1$, and this amounts to taking the aforementioned semidirect product of the Cartan subgroup with $h\vartheta = \vartheta$, and setting the value of the extended quasicharacter at ϑ equal to 1. In this way, the preferred extended parameter defines a canonical extension

$$J(x, y, \lambda)^+ = J((\lambda, \tau, 0, 0), \lambda) \tag{40}$$

of $J(x, y, \lambda)$ to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle \vartheta \rangle$. We call this extension the *Atlas extension* of $J(x, y, \lambda)$.

Going back to Theorem 2.3 and Corollary 2.4, we may formulate the result as follows.

Corollary 2.5. *There is a natural bijection of ϑ-fixed sets*

$$\coprod_{\{w \in W(\delta_0, \vartheta)\}} \mathcal{X}^w_{^\vee\rho} \times {}^\vee\mathcal{X}^{ww_0}_\lambda \longleftrightarrow \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta \longleftrightarrow \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$$

Furthermore, if $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ is identified with (x, y) under the first bijection then there is a canonical representation

$$\pi(\xi)^+ = J(x, y, \lambda)^+$$

extending $\pi(\xi)$ to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle \vartheta \rangle$.

The irreducible representation $\pi(\xi)^+$ is defined as the unique (Langlands) quotient of a representation $M(\xi)^+$ such that $M(\xi)^+_{|\mathrm{GL}_N(\mathbb{R})} = M(\xi)$. We call $\pi(\xi)^+$ and $M(\xi)^+$ the *Atlas extensions* of $\pi(\xi)$ and $M(\xi)$ respectively.

2.6 Grothendieck groups of characters

The setting for studying characters of reductive groups is the Grothendieck group of representations with a given central character. There is a corresponding notion in the twisted setting. In this section we establish notation for the objects that we need.

Fix a semisimple orbit ${}^\vee\mathcal{O} \subset {}^\vee\mathfrak{g}$, which we view as an infinitesimal character for G (cf. Section 2.1). Recall $\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$ is the set of equivalence classes of representations (π, δ) of pure strong involutions. We define $K\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$ to be the Grothendieck group of representations of pure strong involutions with infinitesimal character ${}^\vee\mathcal{O}$ (see [ABV92]*(15.5)-(15.6)). We identify this with the $\mathbb{Z}$-span of distribution characters of the irreducible representations in $\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$. We refer to elements of this space as virtual characters.

When G is Sp_N or SO_N we only need the subspace of stable characters, and only for the quasisplit form. So we define

$$K\Pi({}^\vee\mathcal{O}, G(\mathbb{R}, \delta_q))^{\mathrm{st}} \subset K\Pi({}^\vee\mathcal{O}, G(\mathbb{R}, \delta_q))$$

to be the subspace spanned by the (strongly) stable virtual characters. If we identify virtual characters with functions on $G(\mathbb{R}, \delta_0)$ these are the virtual characters η which satisfy $\eta(g) = \eta(g')$ whenever strongly regular semisimple elements $g, g' \in G(\mathbb{R}, \delta_q)$ are G-conjugate. See [She79]*Section 5 or [ABV92]*Definition 18.2.

2.7 Grothendieck groups of twisted characters

Here, we consider the split inner class of GL_N, equipped with the involution ϑ. Recall in this case $\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N/\mathbb{R}) = \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$ (*cf.* (29)), and so

$$K\Pi(\mathrm{GL}_N/\mathbb{R}) = K\Pi(\mathrm{GL}_N(\mathbb{R})).$$

We define

$$K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \subset K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$$

to be the submodule spanned by $\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$. This is not the Grothendieck group of ϑ-stable representations of $\mathrm{GL}_N(\mathbb{R})$, but we retain the "$K$" to help align the object with its ambient Grothendieck group. On the other hand we let

$$K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle) \tag{41}$$

be the Grothendieck group of admissible representations of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ with infinitesimal character ${}^\vee\mathcal{O}$.

We now discuss the $\mathbb{Z}$-module of twisted characters of $\mathrm{GL}_N(\mathbb{R})$. An irreducible character in $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle)$ is a distribution

$$f \mapsto \mathrm{Tr} \int_{\mathrm{GL}_N(\mathbb{R})} f(x)\pi(x)\, dx + \mathrm{Tr} \int_{\mathrm{GL}_N(\mathbb{R})} f(x\vartheta)\pi(x)\pi(\vartheta)\, dx,$$

where $f \in C_c^\infty(\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle)$ and π is an irreducible representation of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. The restriction of such a distribution character to the non-identity component $\mathrm{GL}_N(\mathbb{R}) \rtimes \vartheta$ has the form

$$f \mapsto \mathrm{Tr} \int_{\mathrm{GL}_N(\mathbb{R})} f(x\vartheta))\pi(x)\pi(\vartheta)\, dx, \quad f \in C_c^\infty(\mathrm{GL}_N(\mathbb{R}) \rtimes \vartheta). \tag{42}$$

When the resulting restricted distribution is non-zero, we define it to be an irreducible *twisted character* of $\mathrm{GL}_N(\mathbb{R}) \rtimes \vartheta$. We define

$$K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$$

to be the $\mathbb{Z}$-module generated by the irreducible twisted characters of $\mathrm{GL}_N(\mathbb{R}) \rtimes \vartheta$ of infinitesimal character ${}^\vee\mathcal{O}$.

As noted in Section 2.5, an irreducible representation of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ restricts either to an irreducible ϑ-fixed representation of $\mathrm{GL}_N(\mathbb{R})$, or to a direct sum $\pi \oplus (\pi \circ \vartheta)$ of inequivalent irreducible representations. In the second case the twisted character is 0, so we only need to consider the first case. The first case describes the irreducible representations in $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$. If $\pi \in \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$ then it has two extensions $\pi^\pm$ to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$, satisfying

$$\pi^-(\vartheta) = -\pi^+(\vartheta). \tag{43}$$

Consequently the twisted characters of $\pi^\pm$ agree up to sign. If we set $U_2 = \{\pm 1\}$ then it follows that the homomorphism

$$K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \otimes_\mathbb{Z} \mathbb{Z}[U_2] \to K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta),$$

which restricts the distribution character of $\pi(\xi)^+$ to the non-identity component, is surjective. By (43), the homomorphism passes to an isomorphism

$$K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \cong K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \otimes_\mathbb{Z} \mathbb{Z}[U_2]/\langle(\pi \otimes 1) + (\pi \otimes -1)\rangle \tag{44(a)}$$

where the quotient runs over $\pi \in \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$. The map taking $\pi(\xi) \in \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$ to the twisted character character

$$f \mapsto \mathrm{Tr} \int_{\mathrm{GL}_N(\mathbb{R})} f(x\vartheta)\, \pi(\xi)(x)\, \pi(\xi)^+(\vartheta)\, dx, \quad f \in C_c^\infty(\mathrm{GL}_N(\mathbb{R}) \rtimes \vartheta)$$

extends to an isomorphism

$$K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \simeq K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta. \tag{44(b)}$$

We should once again remind the reader that the $\mathbb{Z}$-modules appearing in (44) are not Grothendieck groups in any natural fashion, notwithstanding the appearance of the "K". Nevertheless it is helpful to use this notation, to help remind the reader of the origins of these modules.

3 Sheaves and Characteristic Cycles

Suppose ψ_G is an Arthur parameter for G as in (1). In this section we give more details on the definition of the ABV-packet $\Pi^{\mathrm{ABV}}_{\psi_G}$ and its stable virtual character $\eta^{\mathrm{ABV}}_{\psi_G}$ (10). The results apply in the more general context of complex connected reductive groups G ([ABV92]*Sections 19, 22). However, for this section G will be Sp_N, SO_N or GL_N, with the setup of Section 2. The definitions depend on a pairing between characters and sheaves.

We also define a pairing between *twisted* characters and *twisted* sheaves for GL_N [CM18]*Sections 5-6. The key properties of this twisted pairing are listed in this section and shall be proved in Section 4.

3.1 The pairing and the ABV-packets in the non-twisted case

Let ϕ_{ψ_G} be the Langlands parameter associated to ψ_G (9), ${}^\vee\mathcal{O}$ be the infinitesimal character of ϕ_{ψ_G}, and $S_{\psi_G} \subset X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ (25) be the corresponding orbit ([ABV92]*Proposition 6.17, (26)). Recall that $\Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ is the set of pure complete geometric parameters (see the end of Section 2.2). There is a bijection (27) between $\Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ and $\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$, the (equivalence classes of) irreducible representations of pure strong involutions of G.

Let $\mathcal{C}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$ be the category of ${}^\vee G$-equivariant constructible sheaves of complex vector spaces on $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. This is an abelian category and its simple objects are parameterized by the set of complete geometric parameters $\xi = (S, \tau_S) \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ as follows. Choose $p \in S$, let ${}^\vee G_p = \mathrm{Stab}_{{}^\vee G}(p)$, and choose a character τ_ξ of the component group of ${}^\vee G_p$ so that (p, τ_ξ) is a representative of ξ. Then τ_ξ pulled back to ${}^\vee G_p$ defines an algebraic vector bundle

$$ {}^\vee G \times_{{}^\vee G_p} V \to S. \tag{45} $$

The sheaf of sections of this vector bundle is, by definition, a ${}^\vee G$-equivariant local system on S ([ABV92]*Section 7, Lemma 7.3). Extend this local system to the closure $\bar{S}$ by zero and then take the direct image into $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ to obtain an irreducible (*i.e.* simple) ${}^\vee G$-equivariant constructible sheaf denoted by $\mu(\xi)$ ([ABV92]*(7.10)(c)).

Now let $\mathcal{P}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$ be the abelian category of ${}^\vee G$-equivariant perverse sheaves of complex vector spaces on $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ [BL94]*Section 5. The simple objects of $\mathcal{P}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$ are defined from $\xi = (S, \tau_S) \in \Xi({}^\vee G^\Gamma, {}^\vee\mathcal{O})$ and the algebraic vector bundle (45) by taking the *intermediate* extension [BBD82]*Section 2 to the closure $\bar{S}$ instead of the extension by zero. This is denoted $P(\xi)$ ([ABV92]*(7.10)(d)). It is an irreducible ${}^\vee G$-equivariant perverse sheaf on $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$.

The Grothendieck groups of the two categories $\mathcal{C}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$ and $\mathcal{P}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$ are canonically isomorphic ([BBD82], [ABV92]*Lemma 7.8). We identify the two Grothendieck groups via this isomorphism and denote them by $KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. This Grothendieck group has two natural bases

$$ \{\mu(\xi) \mid \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)\} \quad \text{and} \quad \{P(\xi) \mid \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)\}. $$

Suppose $\xi = (S, \tau) \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. We define two invariants associated to ξ. First, let $d(\xi)$ be the dimension of S_ξ. Second, associated to ξ is the representation $\pi(\xi)$ of a pure strong involution of G (27). Let $e(\xi) = \pm 1$ be the Kottwitz invariant of the underlying real form of this strong involution ([ABV92]*Definition 15.8).

As discussed in the introduction, we define a perfect pairing

$$ \langle\,\cdot,\cdot\,\rangle : K\Pi({}^\vee\mathcal{O}, G/\mathbb{R}) \times KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \to \mathbb{Z} \tag{46} $$

by

$$ \langle M(\xi), \mu(\xi')\rangle = e(\xi)\,\delta_{\xi,\xi'}. $$

The pairing also takes a simple form relative to the bases given by $\pi(\xi)$ and $P(\xi')$ ([ABV92]*Theorem 1.24, Sections 15-17). We state it as a theorem.

Theorem 3.1. *The pairing* (46) *satisfies*

$$ \langle \pi(\xi), P(\xi')\rangle = (-1)^{d(\xi)}\, e(\xi)\, \delta_{\xi,\xi'}, \quad \xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma). $$

This pairing allows us to regard elements of $K\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$ as $\mathbb{Z}$-linear functionals of $KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. The microlocal multiplicity maps χ_S^{mic} discussed in (8) are $\mathbb{Z}$-linear functionals on $KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. Before making the obvious connection with the pairing (46), we review some facts needed to define χ_S^{mic}. To begin, we consider the category of ${}^\vee G$-equivariant coherent $\mathcal{D}$-modules on $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. We denote this category by $\mathcal{D}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$. Here, $\mathcal{D}$ is the sheaf of algebraic differential operators on $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ ([BGK+87]*VIII.14.4, [ABV92]*Section 7). Convenient references for equivariant $\mathcal{D}$-modules are [HTT08] and [ST00].

The equivariant Riemann-Hilbert correspondence ([BGK+87]* Theorem VIII.14.4) induces an isomorphism

$$DR : K\mathcal{D}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)) \to KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma). \tag{47}$$

For simplicity we write $X = X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$, and $\mathcal{D}X = \mathcal{D}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$.

The sheaf $\mathcal{D}$ is filtered by the order of the differential operators, and the associated graded ring is canonically isomorphic to $\mathcal{O}_{T^*(X)}$, the coordinate ring of the cotangent bundle of X ([HTT08]*Section 1.1). Suppose $\mathcal{M} \in \mathcal{D}X$. Then $\mathcal{M}$ has a filtration such that the resulting graded sheaf $\mathrm{gr}\mathcal{M}$ is a coherent $\mathcal{O}_{T^*(X)}$-module ([HTT08]*Section 2.1).

The support of $\mathrm{gr}\mathcal{M}$ is a closed subvariety of $T^*(X)$ ([ABV92]*Definition 19.7). Each minimal ${}^\vee G$-invariant component of this closed subvariety is the closure of a conormal bundle $T^*_S(X)$, where $S \subset X$ is a ${}^\vee G$-orbit ([ABV92]*Proposition 19.12(c)). Therefore to each conormal bundle $T^*_S(X)$ we may attach a non-negative integer, denoted by $\chi_S^{\mathrm{mic}}(\mathcal{M})$, which (when nonzero) is the length of the module $\mathrm{gr}\mathcal{M}$ localized at $T^*_S(X)$ [HTT08]*Section 2.2.

The characteristic cycle of $\mathcal{M}$ is defined as

$$\mathrm{Ch}(\mathcal{M}) = \sum_{S \in X/{}^\vee G} \chi_S^{\mathrm{mic}}(\mathcal{M}) \ \overline{T^*_S(X)}.$$

For a given ${}^\vee G$-orbit S we may regard χ_S^{mic} as a function on $\mathcal{D}$-modules which is additive for short exact sequences ([ABV92]*Proposition 19.12(e)). It therefore defines a homomorphism $K\mathcal{D}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)) \to \mathbb{Z}$, called the *microlocal multiplicity* along S. Using the isomorphism (47), we interpret this as a homomorphism

$$\chi_S^{\mathrm{mic}} : KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \to \mathbb{Z}.$$

We now return to the pairing (46) and its relationship to χ_S^{mic}. This relationship defines $\eta_{\psi_G}^{\mathrm{ABV}}$. We first define $\eta_{\psi_G}^{\mathrm{mic}} \in K\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$ to be the element of $K\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$ corresponding via the pairing to the element χ_S^{mic} in the dual of $KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. Explicitly working through the identifications in the definition we see

$$\eta_{\psi_G}^{\mathrm{mic}} = \sum_{\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)} (-1)^{d(S_\xi) - d(S_{\psi_G})} \ \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi))\, \pi(\xi). \tag{48}$$

An important result of Kashiwara and Adams-Barbasch-Vogan is

Proposition 3.2 ([ABV92]*Theorem 1.31, Corollary 19.16). *$\eta_{\psi_G}^{\mathrm{mic}}$ is a stable virtual character.*

The *microlocal packet* $\Pi_{\psi_G}^{\mathrm{mic}}$ of ψ_G is defined to be the irreducible representations in the support of $\eta_{\psi_G}^{\mathrm{mic}}$. In other words

$$\Pi_{\psi_G}^{\mathrm{mic}} = \{\pi(\xi) : \xi \in \Xi({}^\vee\mathcal{O}{}^\vee G^\Gamma) \mid \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \neq 0\}.$$

This is a set of irreducible representations of pure strong involutions of G. We are primarily interested in the packet for the quasisplit strong involutions. We therefore define

$$\eta_{\psi_G}^{\mathrm{ABV}} = \eta_{\psi_G}^{\mathrm{mic}}(\delta_q) \tag{49}$$

to be the restriction of $\eta_{\psi_G}^{\mathrm{mic}}$ to the submodule of $K\Pi({}^\vee\mathcal{O}, G/\mathbb{R})$ generated by the representations in $\Pi({}^\vee\mathcal{O}, G(\mathbb{R}, \delta_q))$. The ABV-packet $\Pi_{\psi_G}^{\mathrm{ABV}}$ is defined as the support of $\eta_{\psi_G}^{\mathrm{ABV}}$, that is

$$\Pi_{\psi_G}^{\mathrm{ABV}} = \{\pi(\xi) : \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma), \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \neq 0, \pi(\xi) \in \Pi(G(\mathbb{R}, \delta_q))\}. \tag{50}$$

We conclude this section with a restatement of Theorem 3.1. Define the representation-theoretic transition matrix m_r by

$$M(\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)} m_r(\xi', \xi)\, \pi(\xi'). \tag{51}$$

Define the geometric "transition matrix" c_g by

$$P(\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)} (-1)^{d(\xi)}\, c_g(\xi', \xi)\, \mu(\xi'). \tag{52}$$

(see [ABV92]*(7.11)(c)). Then [ABV92]*Corollary 15.13 says

Proposition 3.3. *Theorem 3.1 is equivalent to the identity*

$$m_r(\xi', \xi) = (-1)^{d(\xi) - d(\xi')}\, c_g(\xi, \xi'). \tag{53}$$

This equation relates the decomposition of characters with the decomposition of sheaves.

3.2 The pairing in the twisted case

As discussed in the previous section, the pairing (46) plays a fundamental role in the definition of ABV-packets. We now discuss a twisted version of this pairing for GL_N.

We replace $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$ with the $\mathbb{Z}$-module $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ of twisted characters (44). Associated to $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ are an irreducible representation $\pi(\xi) \in \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta$ as well as a canonical extension $\pi(\xi)^+$ to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ (Corollary 2.5). The twisted character of $\pi(\xi)^+$ is an element of the space $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ of twisted characters, and this gives a basis of $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ parameterized by $\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. See (41) and the end of Section 2.5.

The twisted characters are to be paired with twisted sheaves which are elements in a $\mathbb{Z}$-module generalizing $KX({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$. The twisted objects for this pairing are given in [ABV92]*(25.7) (see also [CM18]*Section 5.4). We provide a short summary.

Let $s \in \mathrm{GL}_N$ be an element such that

$$\sigma = \mathrm{Int}(s) \circ \vartheta \tag{54}$$

is an automorphism of GL_N of finite order. Then σ acts on $X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ in a manner which is compatible with the ${}^\vee\mathrm{GL}_N$-action ([ABV92]*(25.1)), and so also acts on its ${}^\vee\mathrm{GL}_N$-equivariant sheaves.

Let $\mathcal{P}(X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma); \sigma)$ be the category of ${}^\vee\mathrm{GL}_N$-equivariant perverse sheaves with a compatible σ-action. An object in this category is a pair (P, σ_P) in which P is an equivariant perverse sheaf and σ_P is an automorphism of P which is compatible with σ ([CM18]*Section 5.4). Similarly, we define $\mathcal{C}(X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma); \sigma)$ to be the category of ${}^\vee\mathrm{GL}_N$-equivariant constructible sheaves with a compatible σ-action. An object in this category is a pair (μ, σ_μ) in which μ is an equivariant constructible sheaf and σ_μ is an automorphism of μ which is compatible with σ.

The Grothendieck groups of these two categories are isomorphic [CM18]*(35). We identify them and denote their Grothendieck groups by $K(X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma); \sigma)$. This is the sheaf-theoretic analogue of $K\Pi(\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle)$.

As with the representations (see (40)), we seek a canonical choice of extension of $P(\xi)$, *i.e.* an automorphism $\sigma_{P(\xi)}$ of $P(\xi)$.

Lemma 3.4. *Let ${}^\vee G = {}^\vee\mathrm{GL}_N$, $\xi = (S, \tau_S) \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$, $p \in S$, and (45) be the equivariant vector bundle representing $\mu(\xi)$.*

(a) Suppose $p' \in S$ and $p' = a \cdot p$ for some $a \in {}^\vee\mathrm{GL}_N$. Then the maps

$$\begin{aligned} (g, v) &\mapsto (ga^{-1}, v) \\ g \cdot p &\mapsto (ga^{-1}) \cdot p' \end{aligned} \tag{55}$$

define an isomorphism of equivariant vector bundles

$${}^\vee G \times_{{}^\vee G_p} V \to {}^\vee G \times_{{}^\vee G_{p'}} V. \tag{56}$$

which is independent of the choice of a.

(b) *There exist canonical choices of pairs*

$$\mu(\xi)^+ = (\mu(\xi), \sigma^+_{\mu(\xi)}) \in \mathcal{C}(X(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma); \sigma),$$

$$P(\xi)^+ = (P(\xi), \sigma^+_{P(\xi)}) \in \mathcal{P}(X(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma); \sigma)$$

such that if $p \in S$ *is fixed by* σ *then* $\sigma^+_{\mu(\xi)}$ *(and* $\sigma^+_{P(\xi)}$*) acts trivially on the stalk of* $\mu(\xi)$ *(and* $P(\xi) \in KX(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$*) at* p.

Proof. Let (p, τ) and (p', τ') be representatives of ξ. It is well-known that the component group ${}^\vee G_p/({}^\vee G_p)^0$ is trivial for the general linear group ([ABV92]*Lemma 7.5), and so τ is its trivial quasicharacter. For the same reason, τ' is the trivial quasicharacter of the trivial group. Both τ and τ' lift to the trivial representations of ${}^\vee G_p$ and ${}^\vee G_{p'} = a\,{}^\vee G a^{-1}$ respectively. By definition

$$(gh, v) = (g, \tau(h)v) = (g, v), \ (g, v) \in {}^\vee G \times_{{}^\vee G_p} V, \quad h \in {}^\vee G_p.$$

Applying (55) to the left-most element, we obtain

$$(gha^{-1}, v) = (ga^{-1}aha^{-1}, v) = (ga^{-1}, \tau'(aha^{-1})v) = (ga^{-1}, v)$$

in ${}^\vee G \times_{{}^\vee G_{p'}} V$. This proves that the map (55) is well-defined. The map is clearly a ${}^\vee\mathrm{GL}_N$-equivariant isomorphism. The element $a \in {}^\vee G$ is unique up to right-multiplication by an element in $a_1 \in {}^\vee G_p$. Since

$$(g(aa_1)^{-1}, v) = (ga^{-1}aa_1^{-1}a^{-1}, v) = (ga^{-1}, \tau'(aa_1^{-1}a^{-1})v) = (ga^{-1}, v)$$

in ${}^\vee G \times_{{}^\vee G_{p'}} V$, the isomorphism (55) is independent of the choice of a. This proves the first assertion.

Suppose $p' = \sigma(p) = a \cdot p \in S$. Then σ induces a bundle isomorphism

$${}^\vee G \times_{{}^\vee G_p} V \to {}^\vee G \times_{{}^\vee G_{p'}} V,$$

which when composed with the inverse of (56) yields a canonical automorphism which we set equal to $\sigma^+_{\mu(\xi)}$. To be explicit

$$\sigma^+_{\mu(\xi)}(g, v) = (\sigma(g)a, v), \quad (g, v) \in {}^\vee G \times_{{}^\vee G_p} V \tag{57}$$

We identify $\sigma^+_{\mu(\xi)}$ with the unique automorphism of $\mu(\xi)$ which it determines.

This choice of $\sigma^+_{\mu(\xi)}$ determines a canonical choice $\sigma^+_{P(\xi)}$ by virtue of the fact that $\mu(\xi)$ occurs in the decomposition of $P(\xi)$ in $KX(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ with multiplicity one ((7.11)(b) [ABV92], [CM18]*pp. 154-155).

Finally, suppose σ preserves p. Then $g = a = 1$ in (57) and the last assertion is proved. □

We now imitate the definition of $K\Pi(^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ (44) for the sheaves appearing in Lemma 3.4. Attached to $\xi \in \Xi(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ are perverse sheaves $P(\xi)^\pm$, where $P(\xi)^+$ is defined in Lemma 3.4, and $P(\xi)^-$ is the unique other choice of extension. Furthermore, the *microlocal traces* of $P(\xi)^\pm$ differ by sign ([ABV92]*(25.1)(j)). Similar comments apply to $\mu(\xi)^\pm$.

We are interested only in irreducible sheaves with non-vanishing microlocal trace. We consequently follow the definition of (44) in defining the quotient

$$KX(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma) = K(X(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma))^\sigma \otimes \mathbb{Z}[U_2]/\langle (P(\xi) \otimes 1) + (P(\xi) \otimes -1)\rangle \tag{58}$$

where the quotient runs over $\xi \in \Xi(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

This is the $\mathbb{Z}$-module which we shall pair with

$$K\Pi(^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \cong K\Pi(^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$$

in Section 4. We call the elements of this module *twisted sheaves*, and remind the reader that these modules are not naturally Grothendieck groups, even though we have kept the "K" in the notation.

For reasons that will only become clear in Section 7, the definition of our twisted pairing involves some additional signs. The signs depend on the integral lengths of parameters, which may be described as follows.

From now on we assume $\lambda \in {}^\vee\mathcal{O}$ satisfies the regularity condition (32). Let $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. Lemma 2.2 tells us that associated to ξ is an element $x \in \mathcal{X}_{{}^\vee\rho}$. Set $\theta_x = \mathrm{Int}(x) \in \mathrm{Norm}_G(H)$. Let

$$R(\lambda) = \{\alpha \in R(\mathrm{GL}_N, H) \mid \langle \lambda, {}^\vee\alpha\rangle \in \mathbb{Z}\} \tag{59)(a}$$

be the λ-integral roots, with positive λ-integral roots

$$R^+(\lambda) = \{\alpha \in R(\lambda) \mid \langle \lambda, {}^\vee\alpha\rangle > 0\}. \tag{59)(b}$$

Define the *integral length*, following [ABV92]*(16.16), as

$$l^I(\xi) = -\frac{1}{2}\left(|\{\alpha \in R^+(\lambda) : \theta_x(\alpha) \in R^+(\lambda)\}| + \dim(H^{\theta_x})\right). \tag{60}$$

The integral length takes values in the non-positive integers.

Furthermore define

$$R^+_\vartheta(\lambda) = \{\alpha \in R((\mathrm{GL}_N^\vartheta)^0, (H^\vartheta)^0) \mid \langle {}^\vee\alpha, \lambda\rangle \in \mathbb{Z}_{>0}\}.$$

We define the *ϑ-integral length* by

$$l^I_\vartheta(\xi) = -\frac{1}{2}\left(|\{\alpha \in R^+_\vartheta(\lambda) \mid \theta_x(\alpha) \in R^+_\vartheta(\lambda)\}| + \dim((H^\vartheta)^{\theta_x})\right). \tag{61}$$

This is the integral length for (the identity component of) the group GL_N^ϑ.

Now we define a perfect pairing (under the assumption (32)):

$$\langle\cdot,\cdot\rangle : K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \times KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma) \to \mathbb{Z} \tag{62}$$

by setting

$$\langle M(\xi)^+, \mu(\xi')^+\rangle = (-1)^{l^I(\xi) - l^I_\vartheta(\xi)}\,\delta_{\xi,\xi'} \tag{63}$$

for $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. The analogue of Theorem 3.1 is

Theorem 3.5. *Suppose $\lambda \in {}^\vee\mathcal{O}$ satisfies (32). Define the pairing (62) by (63). Then*

$$\langle \pi(\xi)^+, P(\xi')^+\rangle = (-1)^{d(\xi)}\,(-1)^{l^I(\xi) - l^I_\vartheta(\xi)}\,\delta_{\xi,\xi'}$$

where $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

The proof of this theorem is the primary purpose of Section 4. Its proof is modelled on the proof of Theorem 3.1 in [ABV92]*Sections 15-17.

The signs $(-1)^{l^I(\xi) - l^I_\vartheta(\xi)}$ appear in the pairing to account for the comparison of extensions given in Section 7. Note that if ϑ were taken to be the identity automorphism then the signs would disappear, and one would recover the ordinary pairing (46) for GL_N.

We conclude this section by giving a twisted analogue of Proposition 3.3. This analogue will only be needed in Sections 7 and 9, so the reader may wish to skip this discussion and return to it later.

For $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta$, define $m_r(\xi'_\pm, \xi_+)$ to be the multiplicity of the representation $\pi(\xi')^\pm$ in $M(\xi)^+$ in the Grothendieck group $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle)$ (Section 2.6). In other words

$$M(\xi)^+ = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta} m_r(\xi'_+, \xi_+)\,\pi(\xi')^+ + m_r(\xi'_-, \xi_+)\,\pi(\xi')^- + \cdots$$

where the omitted summands are irreducible representations of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ of the second type (Section 2.6). Define

$$m_r^{\vartheta}(\xi', \xi) = m_r(\xi'_+, \xi_+) - m_r(\xi'_-, \xi_+) \tag{64}$$

for $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta$ (*cf.* [AvLTV20]*(19.3d)). By construction, the image of $M(\xi)^+$ in $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ (44) decomposes as

$$M(\xi)^+ = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} m_r^{\vartheta}(\xi', \xi)\, \pi(\xi')^+. \tag{65}$$

Lemma 3.6. *The matrix given by*

$$m_r^{\vartheta}(\xi', \xi), \quad \xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta$$

(64) is invertible.

Proof. The invertibility of the matrix given by $m_r(\xi', \xi)$ in (51) follows since it is uni-upper-triangular with respect to the Bruhat order ([AvLTV20]*Definition 18.4, [Vog81]*Lemma 6.6.6). We show that $m_r^{\vartheta}(\xi'\xi)$ inherits the same two properties. By restricting to $\mathrm{GL}_N(\mathbb{R})$ we see

$$m_r(\xi', \xi) = m_r(\xi'_+, \xi_+) + m_r(\xi'_-, \xi_+)$$

(see the equation preceding [AvLTV20]*19.3c). Furthermore $m_r(\xi, \xi) = 1$ implies $m_r(\xi_+, \xi_+) = 1$ and $m_r(\xi_-, \xi_+) = 0$ or $m_r(\xi_+, \xi_+) = 0$ and $m_r(\xi_-, \xi_+) = 1$. Therefore $m_r^{\vartheta}(\xi, \xi) = \pm 1$ and by (64) we conclude $m_r^{\vartheta}(\xi', \xi)$ is upper-triangular with ± 1 along the diagonal. In particular, it is invertible. □

In a parallel fashion, we define $c_g(\xi'_\pm, \xi_+)$ for $\xi, \xi' \in ({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta$ by

$$P(\xi)^+ = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta} (-1)^{d(\xi')}\, c_g(\xi'_+, \xi_+)\, \mu(\xi')^+ + (-1)^{d(\xi')}\, c_g(\xi'_-, \xi_+)\, \mu(\xi')^- + \cdots \tag{66}$$

in the Grothendieck group $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N; \sigma)$ of Section 3.2. Setting

$$c_g^{\vartheta}(\xi', \xi) = c_g(\xi'_+, \xi_+) - c_g(\xi'_-, \xi_+). \tag{67}$$

we see that the image of $P(\xi)^+$ in $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N, \sigma)$ is

$$\sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta} (-1)^{d(\xi')}\, c_g^{\vartheta}(\xi', \xi)\, \mu(\xi')^+. \tag{68}$$

Just as Theorem 3.1 is equivalent to Proposition 3.3. We have the following equivalence.

Proposition 3.7. *Theorem 3.5 is equivalent to the identity*

$$m_r^{\vartheta}(\xi', \xi) = (-1)^{l^I_\vartheta(\xi) - l^I_\vartheta(\xi')}\ c_g^{\vartheta}(\xi, \xi') \tag{69}$$

for all $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Proof. Using Lemma 3.6 we compute

$$\begin{aligned}\langle \pi(\xi_1)^+, P(\xi_2)^+\rangle &= \sum_{\xi'_1, \xi'_2} (m_r^{\vartheta})^{-1}(\xi'_1, \xi_1)\ c_g^{\vartheta}(\xi'_2, \xi_2)\, (-1)^{d(\xi'_2)} \langle M(\xi'_1)^+, \mu(\xi'_2)\rangle \\ &= \sum_{\xi'_1} (m_r^{\vartheta})^{-1}(\xi'_1, \xi_1)\ c_g^{\vartheta}(\xi'_1, \xi_2)\, (-1)^{d(\xi'_1)}\, (-1)^{l^I(\xi'_1) - l^I_\vartheta(\xi'_1)}\end{aligned}$$

for $\xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. If Theorem 3.5 holds then this sum is equal to

$$(-1)^{d(\xi_1)}\, (-1)^{l^I(\xi_1) - l^I_\vartheta(\xi_1)} \delta_{\xi_1, \xi_2}$$

and so

$$m_r^\vartheta(\xi_1', \xi_1) = (-1)^{l^I(\xi_1)-d(\xi_1)}\,(-1)^{l^I(\xi_1')-d(\xi_1')}(-1)^{l^I_\vartheta(\xi_1)-l^I_\vartheta(\xi_1')}\; c_g^\vartheta(\xi_1, \xi_1').$$

By [AMR17]*Proposition B.1,

$$(-1)^{l^I(\xi_1)-d(\xi_1)} = (-1)^{l^I(\xi_1')-d(\xi_1')}$$

is a constant independent of any parameters ξ_1 and ξ_1'. Thus,

$$m_r^\vartheta(\xi_1', \xi_1) = (-1)^{l^I_\vartheta(\xi_1)-l^I_\vartheta(\xi_1')}\; c_g^\vartheta(\xi, \xi').$$

The process we have given may easily be reversed to prove the converse statement. □

4 The proof of Theorem 3.5

4.1 The Beilinson-Bernstein correspondence in the proof of Theorem 3.5

Our proof of Theorem 3.5 will follow the same strategy as the proof of Theorem 3.1 in [ABV92]*Sections 15-17. We recall some of the theory of KLV-polynomials in the non-twisted context first.

The basic tool in this theory is the Hecke algebra for $\mathrm{GL}_N(\mathbb{R})$ ([ABV92]*(16.10)). For Harish-Chandra modules of $\mathrm{GL}_N(\mathbb{R})$ of infinitesimal character ${}^\vee\mathcal{O}$, this is a free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebra $\mathcal{H}({}^\vee\mathcal{O})$, which comes equipped with a representation on the Hecke module

$$\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) = K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

This representation is actually transported from a Hecke algebra action on a module generated by constructible sheaves ([Vog82]*Proposition 12.5, [LV83]), using the Riemann-Hilbert (47) and Beilinson-Bernstein [BB81] correspondences.

It is the latter kind of Hecke algebra action which gives us a representation of $\mathcal{H}({}^\vee\mathcal{O})$ on

$$\mathcal{K}X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma) = KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

[ABV92]*Proposition 16.13. In order to describe the details of the Hecke action in the twisted case (Section 4.4), it is convenient to replace the space $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ with a space of characters of representations of certain inner forms of ${}^\vee\mathrm{GL}_N$. To be more specific, we define

$${}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$$

to be the set of irreducible characters obtained by applying the Riemann-Hilbert and Beilinson-Bernstein correspondences to the irreducible equivariant perverse sheaves on $X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$.

Here is some detail about ${}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$. Suppose $\xi = (S, \tau_S) \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ and write ϕ for the Langlands parameter with orbit S ([ABV92]*Proposition 6.17, (26)). Define λ and y by (24)(a), ${}^\vee\mathrm{GL}_N(\lambda)$ by (24)(b), and ${}^\vee K_y$ as in Equation (24)(d). It is easy to see that ${}^\vee\mathrm{GL}_N(\lambda)$ is a product of groups GL_{n_i}, and that the real group corresponding to ${}^\vee K_y$ is a product of indefinite unitary groups $\mathrm{U}(p_i, q_i)$ with $p_i + q_i = n_i$. Let ${}^\vee\rho_\lambda = \frac{1}{2}\sum_{\alpha \in R^+(\lambda)} {}^\vee\alpha$ (see (59)). Then ${}^\vee\rho - {}^\vee\rho_\lambda$ defines a two-fold cover of ${}^\vee K_y$ which we denote by ${}^\vee\widetilde{K}_y$ ([AV92a]*Definition 8.11). The set ${}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$ consists of $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-modules.

To summarize:

Proposition 4.1 ([Vog83]*Proposition 1.2, [ABV92]*Theorem 8.5)**.** *The Riemann-Hilbert and Beilinson-Bernstein correspondences define a bijection*

$$\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma) \longleftrightarrow {}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})).$$

In this correspondence $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ is sent to an irreducible $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module of infinitesimal character ${}^\vee\rho$. This correspondence induces an isomorphism of $\mathbb{Z}$-modules

$$KX\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right) \cong K{}^\vee\Pi\left({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})\right). \tag{70}$$

4.2 Vogan Duality for GL_N

We want to understand the $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-modules of Section 4.1 in terms of our parameters. Suppose $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$, and let (x, y) be the corresponding Atlas parameter in $\mathcal{X}_{{}^\vee\rho} \times {}^\vee\mathcal{X}_\lambda$ given by Lemma 2.2. As we shall see, the reversed pair (y, x) then defines an Atlas parameter for ${}^\vee\mathrm{GL}_N(\lambda)$ ([AV15]*Section 6.1). In the case of integral infinitesimal character this is an example of Vogan duality in the version of [AdC09]*Corollary 10.8.

Here are some details in our setting. Let $\sigma_w \in \mathrm{GL}_N$ be the Tits representative of an element $w \in W(\mathrm{GL}_N, H)$ [AV15]*Section 12, and $w_0 \in W(\mathrm{GL}_N, H)$ and $w_0' \in W({}^\vee\mathrm{GL}_N(\lambda), {}^\vee H)$ be the long elements in their respective Weyl groups. Set

$$\delta_0' = \sigma_{w_0'}\sigma_{w_0}^{-1}\delta_0 \in \mathrm{GL}_N \rtimes \langle\delta_0\rangle$$

(see (28)).

Lemma 4.2. *(a)* $(\delta_0')^2 = \exp(2\pi i({}^\vee\rho - {}^\vee\rho_\lambda)) \in Z(\mathrm{GL}_N(\lambda))$.

(b) $\mathrm{GL}_N(\lambda) \rtimes \langle\delta_0'\rangle$ *is an E-group for* ${}^\vee\mathrm{GL}_N(\lambda)$ *in the sense of [ABV92]**Definition 4.6, with second invariant* $\exp(2\pi i({}^\vee\rho - {}^\vee\rho_\lambda))$.

(c) The pair $(y, x) \in {}^\vee\mathcal{X}_\lambda \times \mathcal{X}_{{}^\vee\rho}$ *is naturally an Atlas parameter for an irreducible* $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$*-module.*

Proof. For part (a) we compute

$$\begin{aligned}(\delta_0')^2 &= \left(\delta_0'\sigma_{w_0'}(\delta_0')^{-1}\right)\left(\delta_0'\sigma_{w_0}^{-1}\delta_0\right)\\ &= \sigma_{\delta_0'(w_0)}\left(\sigma_{w_0'}\sigma_{w_0'}^{-1}\delta_0\sigma_{w_0}^{-1}\delta_0\right)\\ &= \sigma_{w_0'}\left(\sigma_{w_0'}\sigma_{w_0}^{-1}\sigma_{\delta_0(w_0)}^{-1}\right)\\ &= \sigma_{w_0'}^2\sigma_{w_0}^{-2}.\end{aligned}$$

using property [AV15]*(53g) twice. The final equality is a consequence of [AV15]*Proposition 12.1.

It is straightforward to show that conjugation by δ_0' preserves the pinning of $\mathrm{GL}_N(\lambda)$ obtained by restricting the usual pinning of GL_N. This is all that needs to be verified for part (b), once the definition of an E-group is recalled.

For part (c), suppose $(x, y) \in \mathcal{X}_{{}^\vee\rho}^w \times {}^\vee\mathcal{X}_\lambda^{ww_0}$ (Lemma 2.2). We must prove that

$$(y, x) \in {}^\vee\mathcal{X}_\lambda^{ww_0} \times \mathcal{X}_{{}^\vee\rho}^{ww_0w_0'}$$

relative to the extended groups

$${}^\vee\mathrm{GL}_N(\lambda) \rtimes \langle{}^\vee\delta_0\rangle \quad \text{and} \quad \mathrm{GL}_N(\lambda) \rtimes \langle\delta_0'\rangle. \tag{71}$$

It is a tautology that $y \in {}^\vee\mathcal{X}_\lambda^{ww_0}$. For the class x the corresponding statement follows from the fact that x acts on H as

$$w\delta_0 = ww_0w_0'\delta_0'.$$

The pair (y, x) now determines a $({}^\vee\mathfrak{h}, {}^\vee H^y)$-module of infinitesimal character ${}^\vee\rho_\lambda$ [AV15]*Corollary 3.9. This is equivalent to a $({}^\vee\mathfrak{h}, \widetilde{{}^\vee H^y})$-module of infinitesimal character ${}^\vee\rho$ ([KV95]*p. 719). The latter module then leads to a $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module following the prescription of [AV15]*(20). □

Suppose $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ corresponds to $(x, y) \in \mathcal{X}_{{}^\vee\rho}^w \times {}^\vee\mathcal{X}_\lambda^{ww_0}$ as in Lemma 2.2. We define

$${}^\vee\xi = (y, x) \in {}^\vee\mathcal{X}_\lambda^{ww_0} \times \mathcal{X}_{{}^\vee\rho}^{ww_0w_0'}. \tag{72}$$

By Lemma 4.2 (c), the Atlas parameter (y, x) defines an irreducible $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module, which we denote by $\pi({}^\vee\xi)$. The $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module $\pi({}^\vee\xi)$ is the Langlands quotient of a standard $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module ([AV15]*(20)), which we denote by $M({}^\vee\xi)$.

Proposition 4.3. *Under the bijection* (70) *we have:*

(a) $P(\xi) \mapsto \pi({}^\vee\xi)$

(b) $(-1)^{d(\xi)}\mu(\xi) \mapsto M({}^\vee\xi)$

Proof. This proposition holds in greater generality, but is simpler for $\mathrm{GL}_N(\mathbb{R})$. Suppose $\xi = (S, \tau_S)$ corresponds to (x, y) as in Lemma 2.2. The $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module corresponding to $P(\xi)$ under the Riemann-Hilbert and Beilinson-Bernstein correspondences is described by [ABV92]*Proposition 6.16 and [Vog83]*Corollary 2.2, Proposition 2.7. These results tell us that the $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module is determined by an $({}^\vee\mathfrak{h}, {}^\vee\widetilde{H^y})$-module. The character of ${}^\vee\widetilde{H^y}$ in this $({}^\vee\mathfrak{h}, {}^\vee\widetilde{H^y})$-module is completely determined by ${}^\vee\rho$ and τ_S. In our case the matter is simplified in that τ_S is the trivial representation of a trivial component group. This is also equivalent to the group ${}^\vee H^y$ being connected, or to the fact that all Cartan subgroups of $\mathrm{U}(p, q)$ are connected. In consequence the $({}^\vee\mathfrak{h}, {}^\vee\widetilde{H^y})$-module is determined entirely by the infinitesimal character ${}^\vee\rho$ specified on ${}^\vee\mathfrak{h}$.

On the other hand, according to the proof of Lemma 4.2 (c), the Atlas parameter (y, x) determines an irreducible $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module in terms of a $({}^\vee\mathfrak{h}, {}^\vee\widetilde{H^y})$-module with infinitesimal character ${}^\vee\rho$. Since ${}^\vee H^y$ is connected this $({}^\vee\mathfrak{h}, \widetilde{{}^\vee H^y})$-module is determined by ${}^\vee\rho$ alone, and is equal to the $({}^\vee\mathfrak{h}, {}^\vee\widetilde{H^y})$-module obtained from $P(\xi)$ above. This proves (a).

For (b) we recall (52) and apply [Vog83]*Theorem 1.6 to obtain

$$\pi({}^\vee\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)} c_g(\xi', \xi)\, M({}^\vee\xi'). \tag{73}$$

(The absence of signs in (73) is due to the fact that the sheaf on the left-hand side of [Vog83]*1.5 is equal to $(-1)^{d(\delta)}P(\delta)$ according to the definitions of [Vog83]*5.13 and [ABV92]*(7.10)(e), see also the proof of [ABV92]*Proposition 16.13). The matrix c_g is invertible and so (73) implies

$$M({}^\vee\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)} c_g^{-1}(\xi', \xi)\, \pi({}^\vee\xi'). \tag{74}$$

Similarly, by inverting the matrix c_g in (52), we obtain

$$(-1)^{d(\xi)}\mu(\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)} c_g^{-1}(\xi', \xi)\, P(\xi'). \tag{75}$$

By part (a) the Riemann-Hilbert and Beilinson-Bernstein correspondences carry the right-hand side of (75) to the right-hand side of (74). Therefore the left-hand sides correspond, which gives (b). □

Corollary 4.4. *The pairing*

$$\langle \cdot, \cdot \rangle : K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) \times K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) \to \mathbb{Z} \tag{76}$$

defined by

$$\langle M(\xi), M({}^\vee\xi') \rangle = (-1)^{l^I(\xi)}\delta_{\xi,\xi'}$$

satisfies

$$\langle \pi(\xi), \pi({}^\vee\xi') \rangle = (-1)^{l^I(\xi)}\delta_{\xi,\xi'}$$

Proof. By Proposition 4.3, Theorem 3.1 is equivalent to the assertion that if a pairing

$$\langle \cdot, \cdot \rangle' : K\Pi({}^\vee\mathcal{O}, G/\mathbb{R}) \times K^\vee\Pi({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \to \mathbb{Z}$$

is defined by

$$\langle M(\xi), M({}^\vee\xi') \rangle' = (-1)^{d(\xi)}\delta_{\xi,\xi'}$$

then

$$\langle \pi(\xi), \pi({}^\vee\xi') \rangle' = (-1)^{d(\xi)}\delta_{\xi,\xi'}. \tag{77}$$

By [AMR17]*Proposition B.1

$$(-1)^{d(\xi)}(-1)^{l^I(\xi)} = (-1)^{d(\xi)+l^I(\xi)} = (-1)^c \tag{78}$$

does not depend on ξ. Therefore, the pairing in (76) satisfies

$$\langle\cdot,\cdot\rangle = (-1)^c\,\langle\cdot,\cdot\rangle'.$$

The assertion of the corollary follows from Equation (77) and Equation (78). □

4.3 Vogan Duality for twisted GL_N

In the previous section we replaced the sheaf-theoretic module $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ with the isomorphic representation-theoretic module $K{}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$. We now wish to replace the twisted sheaf-theoretic module $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma)$ (58) with a space of twisted characters, and hence restate Theorem 3.5 with a statement about twisted representations analogous to Corollary 4.4. The main tool is Vogan duality for the disconnected group $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$, as discussed in [AV15]*Section 6.1.

By analogy with (58) we define

$$K{}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) = K{}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \otimes \mathbb{Z}[U_2]/\langle \pi({}^\vee\xi)\otimes 1) + (\pi({}^\vee\xi)\otimes -1)\rangle$$

where the complete geometric parameters ξ run over $\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Using Propositions 4.1 and 4.3, we define a bijection

$$P(\xi)^+ \mapsto \pi({}^\vee\xi)^+, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta.$$

The extended representation $\pi({}^\vee\xi)^+$ on the right is obtained by Vogan duality from $\pi(\xi)^+$ as in [AV15]*Corollary 6.4. The bijection yields an isomorphism

$$KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma) \cong K{}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta). \tag{79}$$

Proposition 4.5. *Under the isomorphism* (79)

$$(-1)^{d(\xi)}\mu(\xi)^+ \mapsto M({}^\vee\xi)^+, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta.$$

Proof. Define a $\mathbb{Z}$-linear map

$$B : KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta \otimes \mathbb{Z}[U_2] \to K{}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))^\vartheta \otimes \mathbb{Z}[U_2]$$

by setting

$$B(P(\xi)^+) = \pi({}^\vee\xi)^+ \text{ and } B(P(\xi)^-) = \pi({}^\vee\xi)^-, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta.$$

Recall Equation (66). The matrix c_g given by this equation is invertible. We may therefore invert Equation (66) by writing

$$(-1)^{d(\xi)}\mu(\xi)^+ = \sum_{\xi'\in\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g^{-1}(\xi'_+, \xi_+)\,P(\xi')^+ + c_g^{-1}(\xi'_-, \xi_+)\,P(\xi')^- + \cdots$$

The projection of this equation to $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta \otimes \mathbb{Z}[U_2]$ is

$$\sum_{\xi'\in\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g^{-1}(\xi'_+, \xi_+)\,P(\xi')^+ + c_g^{-1}(\xi'_-, \xi_+)\,P(\xi')^-. \tag{80}$$

Applying B to (80), we obtain

$$\begin{aligned} &B((-1)^{d(\xi)}\mu(\xi)^+) \\ &= B\left(\sum_{\xi'\in\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g^{-1}(\xi'_+, \xi_+)\,P(\xi')^+ + c_g^{-1}(\xi'_-, \xi_+)\,P(\xi')^-\right) \\ &= \sum_{\xi'\in\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g^{-1}(\xi'_+, \xi_+)\,\pi({}^\vee\xi')^+ + c_g^{-1}(\xi'_-, \xi_+)\,\pi({}^\vee\xi')^- \end{aligned} \tag{81}$$

The sum on the right is a formal sum of extensions of $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-modules (Lemma 4.2 (c)) to $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y \rtimes \langle\vartheta\rangle)$-modules. Since both $\pi({}^\vee\xi)^+$ and $\pi({}^\vee\xi)^-$ restrict to the same $({}^\vee\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_y)$-module $\pi({}^\vee\xi)$, we write the restriction of this sum as

$$B((-1)^{d(\xi)}\mu(\xi)^+)_{|K^\vee\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}))^\vartheta} = \sum_{\xi'\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (c_g^{-1}(\xi'_+,\xi_+) + c_g^{-1}(\xi'_-,\xi_+))\,\pi({}^\vee\xi'). \quad (82)$$

In a similar manner we apply to equation (80) the forgetful functor which takes $({}^\vee\mathrm{GL}_N \rtimes \langle\sigma\rangle)$-equivariant sheaves to ${}^\vee\mathrm{GL}_N$-equivariant sheaves. The result is

$$(-1)^{d(\xi)}\mu(\xi) = \sum_{\xi'\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (c_g^{-1}(\xi'_+,\xi_+) + c_g^{-1}(\xi'_-,\xi_+))\,P(\xi').$$

Comparing this equation with (75), we see that

$$c_g^{-1}(\xi'_+,\xi_+) + c_g^{-1}(\xi'_-,\xi_+) = c_g^{-1}(\xi',\xi).$$

Consequently, equation (82) takes the form

$$B((-1)^{d(\xi)}\mu(\xi)^+)_{|K^\vee\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}))^\vartheta} = \sum_{\xi'\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g^{-1}(\xi',\xi)\,\pi({}^\vee\xi'),$$

and by (74)

$$B((-1)^{d(\xi)}\mu(\xi)^+)_{|K^\vee\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}))^\vartheta} = M({}^\vee\xi).$$

The standard module $M({}^\vee\xi)$ has exactly two extensions $M({}^\vee\xi)^\pm$. We need to show

$$B((-1)^{d(\xi)}\mu(\xi)^+) = M({}^\vee\xi)^+$$

and for this it suffices to prove that $\pi({}^\vee\xi)^+$ occurs in $M({}^\vee\xi)^+$ as a (sub)quotient. Looking back to (81), the latter is equivalent to proving that $c_g^{-1}(\xi_+,\xi_+) \neq 0$. Looking a bit further back to (80) we see that this amounts to $P(\xi)^+$ appearing in the decomposition of $\mu(\xi)^+$, and this is true by definition (see the proof of Lemma 3.4). ☐

Using Proposition 4.5 we can restate Theorem 3.5.

Lemma 4.6. *Theorem 3.5 is equivalent to the following assertion. The pairing*

$$\langle\cdot,\cdot\rangle : K\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}),\vartheta) \times K^\vee\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}),\vartheta) \to \mathbb{Z} \quad (83)$$

defined by

$$\langle M(\xi)^+, M({}^\vee\xi')^+\rangle = (-1)^{l^I_\vartheta(\xi)}\,\delta_{\xi,\xi'}$$

satisfies

$$\langle \pi(\xi)^+, \pi({}^\vee\xi')^+\rangle = (-1)^{l^I_\vartheta(\xi)}\,\delta_{\xi,\xi'}$$

where $\xi,\xi' \in \Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Proof. We just need to notice that by Proposition 4.5, Theorem 3.5 is equivalent to the assertion that if a pairing

$$\langle\cdot,\cdot\rangle : K\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}),\vartheta) \times K^\vee\Pi({}^\vee\mathcal{O},\mathrm{GL}_N(\mathbb{R}),\vartheta) \to \mathbb{Z}$$

is defined by

$$\langle M(\xi)^+, M({}^\vee\xi')^+\rangle = (-1)^{d(\xi)}(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\,\delta_{\xi,\xi'}$$

then

$$\langle \pi(\xi)^+, \pi({}^\vee\xi')^+\rangle' = (-1)^{d(\xi)}(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\,\delta_{\xi,\xi'}.$$

The proof then follows exactly like that of Corollary 4.4, we leave the details to the reader. ☐

4.4 Twisted Hecke modules

The proof of Theorem 3.5 relies on a Hecke algebra and Hecke modules as in the ordinary, non-twisted setting of Sections 4.1-4.2. In the twisted setting, Lusztig and Vogan define a Hecke algebra which we denote by $\mathcal{H}(\lambda)$ [LV14]*Section 3.1. This Hecke algebra acts on the Hecke modules

$$\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) = K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

and

$$\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) = K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

as in [LV14]*Section 7. We shall extend the pairing (83) to these Hecke modules. Once the Hecke algebra action is supplemented with *Verdier duality* [LV14]*Section 2.4, we present special bases of the Hecke modules, essentially eigenvectors of Verdier duality. Theorem 3.5 will be seen to follow from a theorem expressing the values of the pairing on the special bases (Theorem 4.15).

We continue with a closer look at the Hecke algebra $\mathcal{H}(\lambda)$. Let κ be a ϑ-orbit on the set of simple roots of $R^+(\lambda)$. The orbit κ is equal to one of the following:

$$\begin{array}{lll} \text{one root } \{\alpha = \vartheta(\alpha)\} & & \text{(type 1)} \\ \text{two roots } \{\alpha, \beta = \vartheta(\alpha)\}, & \langle \alpha, {}^\vee\beta\rangle = 0 & \text{(type 2)} \\ \text{two roots } \{\alpha, \beta = \vartheta(\alpha)\}, & \langle \alpha, {}^\vee\beta\rangle = -1 & \text{(type 3).} \end{array} \tag{84}$$

Write $W(\lambda)$ for the Weyl group of the integral roots $R(\lambda)$, and let

$$W(\lambda)^\vartheta = \{w \in W(\lambda) : \vartheta(w) = w\}.$$

The group $W(\lambda)^\vartheta$ is a Coxeter group ([LV14]*Section 4.3) with generators

$$w_\kappa = \begin{cases} s_\alpha & \kappa \text{ type 1} \\ s_\alpha s_\beta & \kappa \text{ type 2} \\ s_\alpha s_\beta s_\alpha & \kappa \text{ type 3.} \end{cases} \tag{85}$$

The Hecke algebra $\mathcal{H}(\lambda)$ ([AV15]*Section 10, [LV14]*Section 4.7) is a free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebra with basis

$$\{T_w : w \in W(\lambda)^\vartheta\}.$$

It is a consequence of [LV14]*Equation 4.7 (a) that $\mathcal{H}(\lambda)$ is generated by the operators $T_\kappa := T_{w_\kappa}$, where κ is a ϑ-orbit as in (84).

Before we move to a discussion of $\mathcal{H}(\lambda)$-modules, we digress on how the ϑ-orbits κ are further categorized relative to a fixed parameter $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. The parameter $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ is equivalent to an Atlas parameter (x, y) as in Lemma 2.2. The adjoint action of x acts as an involution on $R(\lambda)$. This action separates the ϑ-orbits of roots into various types, *e.g.* real, imaginary, etc. Lusztig and Vogan combine this information with the types of (84) and also with the types defined by Vogan in [Vog81]*Section 8.3. The interested reader must be vigilant in distinguishing between these three kinds of types! The list of combined types may be found in [LV14]*Section 7 or [AV15]*Table 1.

Not all of the types that appear in this list are relevant for $\mathrm{GL}_N(\mathbb{R})$. For example the classification of roots in [Vog81]*Section 8.3 labels the roots as either type I or type II, and it is well-known that roots of $\mathrm{GL}_N(\mathbb{R})$ are all of type II. Another well-known fact is that $\mathrm{GL}_N(\mathbb{R})$ has no compact roots relative to x in the sense of [Kna86]*Section VI.3. Using these two facts, it is tedious, but simple, to verify that the only relevant types for $\mathrm{GL}_N(\mathbb{R})$ in [AV15]*Table 1 are labelled as

$$\begin{array}{c} \texttt{1C+}, \texttt{1C-}, \texttt{1i2f}, \texttt{1i2s}, \texttt{1r2}, \texttt{1rn}, \texttt{2C+}, \texttt{2C-}, \texttt{2Ci}, \\ \texttt{2Cr}, \texttt{2i22}, \texttt{2r22}, \texttt{2rn}, \texttt{3C+}, \texttt{3C-}, \texttt{3Ci}, \texttt{3r}, \texttt{3rn}. \end{array} \tag{86}$$

Any ϑ-orbit κ also has a type relative to the dual parameter ${}^\vee\xi$ (72). The dual parameter is equivalent to the Atlas parameter (y, x) and the adjoint action of y is essentially the negative of the adjoint action of x ([AV15]*Definition 3.10). In consequence it is easy to convert the types of

(86) into types for the Vogan dual group ${}^\vee\mathrm{GL}_N(\lambda) \rtimes \langle {}^\vee\delta_0 \rangle$ ((71), [AV15]*Section 11 and Table 5). They are

$$\begin{aligned} &\texttt{1C}-, \texttt{1C}+, \texttt{1r1f}, \texttt{1r1s}, \texttt{1i1}, \texttt{1ic}, \texttt{2C}-, \texttt{2C}+, \texttt{2Cr}, \\ &\texttt{2Ci}, \texttt{2r11}, \texttt{2i11}, \texttt{2ic}, \texttt{3C}-, \texttt{3C}+, \texttt{3Cr}, \texttt{3i}, \texttt{3ic}. \end{aligned} \tag{87}$$

Let us return to the subject of Hecke modules. In [LV14]*Section 4 and [AV15]*Section7 it is explained how $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ can be made into a Hecke module by defining the action of the operators T_κ on the generating set $\{M(\xi)^+ : \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta\}$. The actions are computed explicitly in a geometric setting in [LV14]*Section 7, and are presented in terms of extended Atlas parameters in [AV15]*Proposition 10.4. A case-by-case summary of the actions is given in [AV15]*Table 5, according to the categorization of (86).

The construction defining the Hecke algebra $\mathcal{H}(\lambda)$ and the Hecke module structure for the module $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ in [LV14], also defines a Hecke algebra ${}^\vee\mathcal{H}(\lambda)$ and a Hecke module structure for $\mathcal{K}{}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$. The Hecke module actions in this case are again given in [AV15]*Table 5 in terms of (87). The Hecke algebra ${}^\vee\mathcal{H}(\lambda)$ for the Vogan dual group (71) is generated by Hecke operators $T_{{}^\vee\kappa}$, where ${}^\vee\kappa$ runs over the simple coroots corresponding to κ. The bijection between the two sets of operators

$$\{T_\kappa : \kappa \in R^+(\lambda) \text{ simple}\} \longleftrightarrow \{T_{{}^\vee\kappa} : \kappa \in R^+(\lambda) \text{ simple}\}$$

extends to an isomorphism $\mathcal{H}(\lambda) \cong {}^\vee\mathcal{H}(\lambda)$. In this manner, we also regard $\mathcal{K}{}^\vee\Pi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N(\mathbb{R}), \vartheta)$ as an $\mathcal{H}(\lambda)$-module.

There is a partial order on $\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$, the *Bruhat order* which is defined geometrically ([LV14]*Section 5.1, [ABV92]*(7.11)(f)). The Bruhat order for the dual parameters ${}^\vee\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ is defined by the inverse order

$${}^\vee\xi \geq {}^\vee\xi' \longleftrightarrow \xi \leq \xi', \; \xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta. \tag{88}$$

We now return to the pairing (83) and extend it to a Hecke module pairing

$$\langle \cdot, \cdot \rangle : \mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \times \mathcal{K}{}^\vee\Pi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N(\mathbb{R}), \vartheta) \to \mathbb{Z}[q^{1/2}, q^{-1/2}], \tag{89}$$

by setting

$$\langle M(\xi)^+, M({}^\vee\xi')^+ \rangle = (-1)^{l^I_\vartheta(\xi)}\, q^{\left(l^I(\xi) + l^I({}^\vee\xi')\right)/2}\, \delta_{\xi,\xi'}$$

for all $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. In view of the Kronecker delta, the term $q^{1/2\left(l^I(\xi)+l^I({}^\vee\xi')\right)}$ in the pairing could be replaced by $q^{1/2\left(l^I(\xi)+l^I({}^\vee\xi)\right)}$ or $q^{1/2\left(l^I(\xi')+l^I({}^\vee\xi')\right)}$. In fact, both of the latter terms are independent of ξ or ξ', as may be seen by the following lemma.

Lemma 4.7. *Suppose* $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. *Then*

$$l^I(\xi) + l^I({}^\vee\xi) = -\frac{1}{2}\left(|R^+(\lambda)| + \dim(H)\right)$$

Proof. For simplicity let us identify the dual group ${}^\vee\mathrm{GL}_N(\lambda)$ with $\mathrm{GL}_N(\lambda)$. Similarly, we identify ${}^\vee H$ with H. Let (x, y) be the Atlas parameter of ξ as in (36). Then (y, x) is the Atlas parameter for ${}^\vee\xi$ (72), where the adjoint action of y on $\mathfrak{h}$ is the negative of the adjoint action of x on $\mathfrak{h}$ ([AV15]*Definition 3.10). From definition (60) we compute that

$$\begin{aligned} &l^I(\xi) + l^I({}^\vee\xi) \\ &= -\frac{1}{2}\left(|\{\alpha \in R^+(\lambda) : x\cdot\alpha \in R^+(\lambda)\}| + \dim(H^x)\right) - \frac{1}{2}\left(|\{\alpha \in R^+(\lambda) : y\cdot\alpha \in R^+(\lambda)\}| + \dim(H^y)\right) \\ &= -\frac{1}{2}\left(|\{\alpha \in R^+(\lambda) : x\cdot\alpha \in R^+(\lambda)\}| + |\{\alpha \in R^+(\lambda) : -(x\cdot\alpha) \in R^+(\lambda)\}| + \dim(H^x) + \dim(H^{-x})\right) \\ &= -\frac{1}{2}\left(|R^+(\lambda)| + \dim(H)\right). \end{aligned}$$

□

4.5 The Hecke module isomorphism

The extended pairing (89) induces a $\mathbb{Z}$-module isomorphism

$$\begin{aligned} \mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) &\to \mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^* \\ M(\xi)^+ &\mapsto \langle M(\xi)^+, \cdot \rangle \end{aligned} \tag{90}$$

We endow $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^*$ with the Hecke module structure given in [AV15]*Section 11. The main goal of this section is

Proposition 4.8. *The map (90) is an isomorphism of $\mathcal{H}(\lambda)$-modules.*

This is a generalization of [AV15]*Proposition 11.2, which is stated only in the case of integral infinitesimal character. In the course of the proof we correct a sign in [AV15]*Proposition 11.2.

We first describe the $\mathcal{H}(\lambda)$-action in more detail. Since $\mathcal{H}(\lambda)$ is not commutative, one cannot define a Hecke action on $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^*$ merely by transposing the action on $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$. One must include an anti-automorphism of $\mathcal{H}(\lambda)$ defined by

$$T_w \longmapsto (-1)^{l_\vartheta(w)} q^{l(w)} T_w^{-1}, \quad w \in W(\lambda)^\vartheta,$$

(*cf.* [AV15]*(50) (removing q on the left), [ABV92]*(17.15)(c)). Here, $l(w)$ is the length of w with respect to the simple reflections in $W(\mathrm{GL}_N, H)$, and $l_\vartheta(w)$ is the length of w with respect to the generators of (85). The $\mathcal{H}(\lambda)$-action on $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^*$ is defined by

$$\begin{aligned} T_w^* : \mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^* &\to \mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^* \\ T_w^* \cdot \mu &= (-1)^{l_\vartheta(w)} q^{l(w)} (T_w^{-1})^t \cdot \mu, \end{aligned} \tag{91}$$

where $(T_w^{-1})^t$ is the transpose of T_w^{-1}. According to [LV14]*Equation 7.2(a), for any ϑ-orbit κ of a simple root in $R(\lambda)$ we have

$$(T_{w_\kappa} + 1)(T_{w_\kappa} - q^{l(w_\kappa)}) = 0. \tag{92}$$

From this, the inverse of T_{w_κ} may be computed to be

$$T_{w_\kappa}^{-1} = q^{-l(w_\kappa)} T_{w_\kappa} + (q^{-l(w_\kappa)} - 1).$$

The Hecke action of (91) for a generator therefore takes the form

$$T_{w_\kappa}^* \cdot \mu = -(T_{w_\kappa})^t \cdot \mu + (q^{l(w_\kappa)} - 1)\mu. \tag{93}$$

The Hecke module structures of both the domain and codomain in the proposed isomorphism (90) are now established, and both are presented in [AV15]*Table 5 in terms of cross actions and Cayley transforms ([AV15]*Tables 2-4). Our next goal is to show that cross actions and Cayley transforms commute with Vogan duality (72). In the following two propositions we distinguish in notation between $\alpha \in R(\lambda) \subset R(\mathrm{GL}_N, H)$ and ${}^\vee\alpha \in R({}^\vee\mathrm{GL}_N(\lambda), {}^\vee H)$ although the reader may prefer to identify these two roots.

Proposition 4.9. *Fix a complete geometric parameter $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ and $w_\kappa \in W(\lambda)^\vartheta$ as in (85). Then*

(a) ${}^\vee(w_\kappa \times M(\xi)) = w_{{}^\vee\kappa} \times M({}^\vee\xi)$

(b) ${}^\vee(w_\kappa \times M(\xi)^+) = w_{{}^\vee\kappa} \times M({}^\vee\xi)^+$

Proof. We identify ξ with its equivalent Atlas parameter (x, y) of Lemma 2.2. The dual parameter ${}^\vee\xi$ is then identified with the Atlas parameter (y, x) (72). By the definition of cross action in [AdC09]*(9.11 f),

$$w_\kappa \times M(x, y) = M(\dot{w}x\dot{w}^{-1}, \dot{w}y\dot{w}^{-1}) \tag{94}$$

where $\dot{w} \in \mathrm{GL}_N(\lambda)$ is any representative for w_κ. By (72)

$${}^\vee(w_\kappa \times M(x, y)) = {}^\vee M(\dot{w}x\dot{w}^{-1}, \dot{w}y\dot{w}^{-1}) = M(\dot{w}y\dot{w}^{-1}, \dot{w}x\dot{w}^{-1}) = w_{{}^\vee\kappa} \times M(y, x)$$

This proves the first assertion of the proposition.

For part (b), we first claim that

$$w_\kappa \times M(\xi)^+ = (w_\kappa \times M(\xi))^+. \tag{95}$$

In other words the cross action carries an Atlas extension to an Atlas extension (40). When the ϑ-orbit κ of a simple root in $R(\lambda)$ is also comprised of simple roots in $R(\mathrm{GL}_N, H)$, this may be seen by noting that an extended parameter $(\lambda, \tau, 0, 0)$ for $M(\xi)^+$ (37) is carried to an extended parameter of the form $(\lambda', \tau', 0, 0)$ under all instances of cross action in [AV15]*Tables 2-4. For general κ this fact follows from [AV15]*Definition 10.4 and Tables 2-4. (This is a special property of the group $\mathrm{GL}_N \rtimes \langle \delta_0 \rangle$ which avoids "bad" roots such as those of type `2i12` in the tables.)

Taking the Vogan dual of (95), we obtain

$$^\vee(w_\kappa \times M(\xi)^+) = {}^\vee((w_\kappa \times M(\xi))^+) = ({}^\vee(w_\kappa \times M(\xi)))^+ = (w_{^\vee\kappa} \times M(^\vee\xi))^+.$$

Here, the second equality follows from the definition of Vogan dual for an Atlas extension ([AV15]*Corollary 6.4), and the final equality follows from the first assertion of the proposition.

To complete the second assertion, we must prove

$$(w_{^\vee\kappa} \times M(^\vee\xi))^+ = w_{^\vee\kappa} \times M(^\vee\xi)^+,$$

which is analogous to (95). However, unlike (95) this identity is to be proved using [AV15]*Tables 2-4 for the dual group rather than for $\mathrm{GL}_N \rtimes \langle \delta_0 \rangle$. Once again we turn to extended parameters. If $(\lambda, \tau, 0, 0)$ (37) is an extended parameter corresponding to $M(\xi)^+ = M(x,y)^+$ then an extended parameter corresponding to $M(^\vee\xi)^+ = M(y,x)^+$ may be chosen to have the form $(0, 0, \ell', t')$. Indeed, the zeros in the first two entries satisfy the requisite equations of [AV15]*Propositions 3.8 and 4.5 when regarding $M(y,x)$ as a $(\mathfrak{gl}_N(\lambda), {}^\vee\widetilde{K}_{y_\xi})$-module of infinitesimal character ρ_λ (Section 4.1). As [AV15]*Tables 2-4 indicate, any cross action from (87) applied to $(0, 0, \ell', t')$ yields an extended parameter with zeros in the first two entries. This means that $w_{^\vee\kappa} \times M(^\vee\xi)^+$ has an extended parameter with zero in its first entry. According to [AV15]*Lemma 5.3.1 this extended parameter corresponds to the Atlas extension $(w_{^\vee\kappa} \times M(^\vee\xi))^+$, and the proposition is complete. □

Proposition 4.10. *Fix a complete geometric parameter $\xi \in \Xi(^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ and suppose that κ as in (84) allows for a Cayley transform c_κ ([AV15]*(42e))*[2]. *Then*

(a) $^\vee\,(c_\kappa(M(\xi))) = c_{^\vee\kappa}(M(^\vee\xi))$

(b) $^\vee\,(c_\kappa(M(\xi)^+)) = c_{^\vee\kappa}(M(^\vee\xi)^+)$

Proof. We see no means for avoiding a case-by-case proof, the cases being according to the type of κ as given in (86). We will prove an illustrative case in detail, leaving the others to the reader. As in the previous proposition we identify ξ with its equivalent Atlas parameter (x,y) of (36), and identify the dual parameter $^\vee\xi$ with (y,x).

Suppose $\kappa = \{\alpha, \beta = \vartheta \cdot \alpha\}$ is of type `2r22`, *i.e.* $\alpha, \beta \in R^+(\lambda)$ are orthogonal roots which are real with respect to a(ny) strong involution in the class x ([AV15]*Proposition 3.4. Suppose further that α and β are simple in $R^+(\mathrm{GL}_N, H)$. Then c_κ is defined as the composition of the Cayley transforms c_α of α, and c_β of β (*cf.* [LV14]*7.6 (h)). The Cayley transform c_α is defined in terms of Atlas parameters in [AdC09]*Definitions 14.1 and 14.8 as follows. Let G_α be the derived group of the centralizer of $\ker(\alpha)$ in $G = \mathrm{GL}_N$, and let $H_\alpha \subset G_\alpha \cap H$ be the one-parameter subgroup corresponding to α. Then G_α is isomorphic to SL_2 and H_α is a Cartan subgroup of G_α. Let $\sigma_\alpha \in G_\alpha$ be the Tits representative ([AV15]*(53)) of the non-trivial Weyl group element in $W(G_\alpha, H_\alpha)$ and write $m_\alpha = \sigma_\alpha^2$. The same formalism applies with $G = {}^\vee\mathrm{GL}_N$ and $^\vee\alpha$, so that we have a Tits representative $\sigma_{^\vee\alpha}$ and $m_{^\vee\alpha} = (\sigma_{^\vee\alpha})^2$. Let $\delta \in x$ and $^\vee\delta \in y$ be representative strong involutions. Then the Atlas parameters of the representations in the image of $c_\alpha(M(x,y))$ are the classes of $(\sigma_\alpha\delta, \sigma_{^\vee\alpha}{}^\vee\delta)$ and $(m_\alpha\sigma_\alpha\delta, \sigma_{^\vee\alpha}{}^\vee\delta)$. In this case the two classes coincide (*cf.* type `1r2` [AV15]*Table 1) and therefore $c_\alpha(M(x,y))$ is single-valued.

[2]Contrary to custom, we leave κ in subscript regardless of whether the Cayley transform is made relative to real, complex or imaginary roots.

The same reasoning applied to c_β leads to a single representation in the image of $c_\kappa(M(x,y)) = c_\beta(c_\alpha(M(x,y)))$ and the Atlas parameter of this representation is the class of $(\sigma_\beta\sigma_\alpha\delta, \sigma_{^\vee\beta}\sigma_{^\vee\alpha}{}^\vee\delta)$. The Vogan dual of this Atlas parameter (72) is the class of $(\sigma_{^\vee\beta}\sigma_{^\vee\alpha}{}^\vee\delta, \sigma_\beta\sigma_\alpha\delta)$. Following the path delineated above, it is a straightforward exercise to compute that this is the Atlas parameter of $c_{^\vee\kappa}(M(y,x))$, where now ${}^\vee\kappa = \{{}^\vee\alpha, {}^\vee\beta\}$ is of type `2i11` with respect to the representative ${}^\vee\delta$ of y. This proves the first assertion of the proposition for the first example when α and β are simple in $R^+(\mathrm{GL}_N, H)$.

When α and β are merely simple in $R^+(\lambda)$ and not necessarily simple in $R^+(\mathrm{GL}_N, H)$ then the Cayley transform is defined by

$$c_\kappa(M(x,y)) = w^{-1} \times c_{w\kappa}(w \times M(x,y)) \tag{96}$$

where $w \in W(\mathrm{GL}_N, H)^\vartheta$ and $w\kappa$ is comprised of simple roots in $R^+(\mathrm{GL}_N, H)$ ([AV15]*Proposition 10.4). In this case the Atlas parameter of $c_\kappa(M(x,y))$ is the class of

$$\begin{aligned}&(\dot{w}^{-1}\sigma_{w\beta}\sigma_{w\alpha}\dot{w}\delta\dot{w}^{-1}\dot{w}, \dot{w}^{-1}\sigma_{w\,^\vee\beta}\sigma_{w\,^\vee\alpha}\dot{w}\,{}^\vee\delta\dot{w}^{-1}\dot{w})\\ &= (\dot{w}^{-1}\sigma_{w\beta}\sigma_{w\alpha}\dot{w}\delta, \dot{w}^{-1}\sigma_{w\,^\vee\beta}\sigma_{w\,^\vee\alpha}\dot{w}\,{}^\vee\delta),\end{aligned} \tag{97}$$

where $\dot{w}$ is any representative of w (*cf.* the proof of Proposition 4.9) and $\sigma_{w\alpha}$ etc. are the Tits representatives in $G_{w\alpha}$ etc. These Tits representatives, as well as the Tits representatives of $\sigma_\alpha \in G_\alpha$, etc. for the possibly non-simple roots, all have the form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

regarded as elements in SL_2. From this it is clear that one may choose $\dot{w}$ so that $\dot{w}^{-1}\sigma_{w\alpha}\dot{w} = \sigma_\alpha$, $\dot{w}^{-1}\sigma_{w\beta}\dot{w} = \sigma_\beta$, and then (97) reduces to

$$(\sigma_\beta\sigma_\alpha\delta, \sigma_{^\vee\beta}\sigma_{^\vee\alpha}{}^\vee\delta).$$

The class of this pair has the same form as the class in the case that κ is simple earlier on. Thus, the first assertion of the proposition follows for non-simple κ as in the simple case.

For the second assertion of the proposition we choose an extended parameter $(\lambda, \tau, 0, 0)$ for $M(\xi)^+ = M(x,y)^+$ and return to (96) in this extended setting. As noted in the proof of Proposition 4.9, $w \times M(\xi)^+$ has an extended parameter equivalent to $(\lambda_1, \tau_1, 0, 0)$. According to [AV15]*Table 3, $c_{w\kappa}(w \times M(\xi)^+)$ then also corresponds to an extended parameter of the form $(\lambda_2, \tau_2, 0, 0)$. Finally applying a cross action by w^{-1} yields an extended parameter of the form $(\lambda_3, \tau_3, 0, 0)$ for $c_\kappa(M(\xi)^+)$. The zeroes appearing in the two finally entries of this extended parameter imply that $c_\kappa(M(\xi)^+) = (c_\kappa(M(\xi))^+$ (*cf.* (40)). The sequence of equalities

$${}^\vee\left(c_\kappa(M(\xi)^+)\right) = {}^\vee\left((c_\kappa(M(\xi))^+\right) = ({}^\vee(c_\kappa(M(\xi)))^+ = (c_{^\vee\kappa}(M({}^\vee\xi)))^+ = c_{^\vee\kappa}(M({}^\vee\xi)^+)$$

follows using the same reasoning as given at the end of Proposition 4.9. □

We are now ready to prove the main result of this section.

Proof of Proposition 4.8: Recalling the dual $\mathcal{H}(\lambda)$-action (93), the proposition amounts to proving

$$\langle T_{w_\kappa} M(\xi_1)^+, M({}^\vee\xi_2)^+\rangle = \langle M(\xi_1)^+, -T_{w_\kappa} M({}^\vee\xi_2)^+ + (q^{l(w_\kappa)} - 1)M({}^\vee\xi_2)^+\rangle \tag{98}$$

for all $\xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ and w_κ as in (85). Looking back to the definition of (89), the left-hand side of (98) may be expressed as

$$\langle T_{w_\kappa} M(\xi_1)^+, M({}^\vee\xi_2)^+\rangle = (-1)^{l^I_\vartheta(\xi_2)} q^{(l^I(\xi_2)+l^I({}^\vee\xi_2))/2} \cdot (\text{the coefficient of } M(\xi_2)^+ \text{ in } T_{w_\kappa} M(\xi_1)^+).$$

Similarly, the right-hand side of (98) may be expressed as the product of $(-1)^{l^I_\vartheta(\xi_1)} q^{(l^I(\xi_1)+l^I({}^\vee\xi_1))/2}$ with

$$\text{the coefficient of } M({}^\vee\xi_1)^+ \text{ in } -T_{w_\kappa} M({}^\vee\xi_2)^+ + (q^{l(w_\kappa)} - 1)M({}^\vee\xi_2)^+.$$

By Lemma 4.7, Equation (98) is equivalent to proving that

$$\begin{aligned}(-1)^{l^I_\vartheta(\xi_2)-l^I_\vartheta(\xi_1)} &\cdot (\text{the coefficient of } M(\xi_2)^+ \text{ in } T_{w_\kappa}M(\xi_1)^+) \\ &= \text{ the coefficient of } M({}^\vee\xi_1)^+ \text{ in } -T_{w_\kappa}M({}^\vee\xi_2)^+ + (q^{l(w_\kappa)}-1)M({}^\vee\xi_2)^+.\end{aligned} \tag{99}$$

The proof of the proposition is a case-by-case verification of (99) according to the type of κ relative to x for $\xi_1 = (x, y)$ ((36), (86)). We prove a typical case in detail here, leaving the remaining cases to the reader.

Suppose that κ is a root of type `2i22` relative to ξ_1. Then

$$c_\kappa(M(\xi_1)^+) = \{M(\xi)^+, M(\xi')^+\}$$

is double-valued [AV15]*Table 1. According to [AV15]*Table 5, the coefficient of $M(\xi_2)^+$ in $T_{w_\kappa}M(\xi_1)^+$ is 1 when $\xi_2 = \xi_1, \xi, \xi'$ and 0 otherwise. As for the ϑ-integral lengths (61), we compute

$$\begin{aligned}l^I_\vartheta(\xi) &= -\frac{1}{2}\left(|\{\alpha \in R^+_\vartheta(\lambda) : w_\kappa x\cdot\alpha \in R^+_\vartheta(\lambda)\}| + \dim((H^\vartheta)^{w_\kappa x})\right) \\ &= -\frac{1}{2}\left(|\{\alpha \in R^+_\vartheta(\lambda) : x\cdot\alpha \in R^+_\vartheta(\lambda)\}| - 2 + \dim((H^\vartheta)^{x})\right) \\ &= l^I_\vartheta(\xi_1) + 1\end{aligned}$$

and so $(-1)^{l^I_\vartheta(\xi_1)-l^I_\vartheta(\xi)} = -1$. Similarly $(-1)^{l^I_\vartheta(\xi_1)-l^I_\vartheta(\xi')} = -1$. The left-hand side of (99) is therefore equal to 1 if $\xi_2 = \xi_1$, is equal to -1 if $\xi_2 = \xi, \xi'$, and is equal to 0 otherwise.

Let us consider the right-hand side of (99), in which ${}^\vee\kappa$ is of type `2r11` relative to ${}^\vee\xi_1$. According to [AV15]*Table 5, $M({}^\vee\xi_1)^+$ occurs in $T_{w_\kappa}M({}^\vee\xi_2)^+$ only if one of the following holds

1. $M({}^\vee\xi_1)^+ = M({}^\vee\xi_2)^+$
2. $M({}^\vee\xi_1)^+$ belongs to $c_{{}^\vee\kappa}(M({}^\vee\xi_2)^+)$
3. $M({}^\vee\xi_1)^+ = w_{{}^\vee\kappa} \times M({}^\vee\xi_2)^+$.

The third possibility reduces to the first. Indeed, the third possibility holds if and only if $M({}^\vee\xi_2)^+ = w_{{}^\vee\kappa} \times M({}^\vee\xi_1)^+$, and for ${}^\vee\kappa$ of type `2r11` relative to ${}^\vee\xi_1$ one may compute that $w_{{}^\vee\kappa} \times M({}^\vee\xi_1)^+ = M({}^\vee\xi_1)^+$ using (94). Hence, we need to compute the right-hand side of (99) only for the first two possibilities.

When $M({}^\vee\xi_1)^+ = M({}^\vee\xi_2)^+$, *i.e.* $\xi_1 = \xi_2$, the right-hand side of (99) is

$$-(q^2-2) + q^{l(w_\kappa)} - 1 = -(q^2-2) + q^2 - 1 = 1$$

[AV15]*Table 5, and this equals the left-hand side of (99).

In the second possibility, $M({}^\vee\xi_1)^+$ is a Cayley transform of $M({}^\vee\xi_2)^+$ and this is true if and only if $M({}^\vee\xi_2)$ is an inverse Cayley transform of $M({}^\vee\xi_1)^+$. The latter condition is equivalent to ${}^\vee\xi_2$ being equal to either ${}^\vee\xi$ or ${}^\vee\xi'$, and ${}^\vee\kappa$ is of type `2i11` relative to ${}^\vee\xi_2$. As [AV15]*Table 5 indicates, the right-hand side of (99) equals -1, which is also equal to the left-hand side of (99). □

4.6 Verdier duality

In proving Theorem 3.5 we have extended (62) to the pairing (89) of Hecke modules and discussed the related Hecke algebra actions. Ultimately, we must evaluate the pairing on special elements which recover the basis elements $\pi(\xi)$ and $\pi({}^\vee\xi)$ of Lemma 4.6. As already mentioned, the desired elements are essentially eigenvectors of *Verdier duality* ([LV14]*Section 2.4). We introduce the key properties of Verdier duality, define the "eigenvectors," and finally show that the Hecke algebra isomorphism of Proposition 4.8 is equivariant with respect to Verdier duality.

Verdier duality on $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ is a $\mathbb{Z}$-linear involution D satisfying

$$\begin{aligned}D(q^{1/2}M(\xi)^+) &= q^{-1/2}D((\xi)^+) \\ D((T_\kappa+1)M(\xi)^+) &= q^{-l(w_\kappa)}(T_\kappa+1)\,D(M(\xi)^+)\end{aligned} \tag{100}$$

See [LV14]*4.8(f).

Theorem 4.11 ([LV14]*Section 8.1)**.** *Define elements* $R(\xi',\xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ *by*

$$D(M(\xi)^+) = q^{-l^I(\xi)} \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{l^I(\xi)-l^I(\xi')} R(\xi',\xi)\, M(\xi')^+$$

in $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$*. Then*

(a) $R(\xi',\xi') = 1$,

(b) $R(\xi',\xi) \neq 0$ *only if* $\xi' \leq \xi$.

In addition, if D' *is any* $\mathbb{Z}$*-linear involution of* $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ *which satisfies the properties of this theorem and (100), then* $D = D'$.

The constructions of [LV14] apply equally well to the dual module, yielding a Verdier Duality ${}^\vee D$ on $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$. The Verdier duality ${}^\vee D$ satisfies the obvious analogue of Theorem 4.11.

We also a need a dual version of Verdier duality. Let $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ so that $\langle M(\xi)^+, \cdot\rangle$ belongs to $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^*$. As in [ABV92]*(17.15)(f), we define

$${}^\vee D^* : \mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^* \to \mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^*$$

by

$${}^\vee D^* \langle M(\xi)^+, \cdot\rangle = \overline{\langle M(\xi)^+,\, {}^\vee D(\cdot)\rangle},$$

where ${}^\vee D$ is the Verdier duality on $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ and

$$^- : \mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

is the unique automorphism sending $q^{1/2}$ to $q^{-1/2}$. Imitating the proof of [Vog82]*Lemma 13.4, it is straightforward to verify that ${}^\vee D^*$ satisfies the analogues of (100) and Theorem 4.11, so we are justified in calling ${}^\vee D^*$ a Verdier dual.

Proposition 4.12. *The Hecke module isomorphism (90) is equivariant with respect to* D *and* ${}^\vee D^*$*, that is*

$$\langle DM(\xi_1)^+, M({}^\vee\xi_2)^+\rangle = \overline{\langle M(\xi_1)^+,\, {}^\vee DM({}^\vee\xi_2)^+\rangle},$$

for all $\xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Proof. As already remarked, both D and ${}^\vee D^*$ satisfy (analogues of) (100) and Theorem 4.11. The resulting properties of D imply that the $\mathbb{Z}$-linear involution

$$\langle M(\xi_1)^+, \cdot\rangle \mapsto \langle DM(\xi_1)^+, \cdot\rangle$$

on $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)^*$ also satisfies the analogues of (100) and Theorem 4.11. The proposition therefore follows from the uniqueness statement in the dual analogue of Theorem 4.11. ☐

We now define the special basis elements alluded to at the beginning of this section. The special bases are defined in terms of the twisted KLV-polynomials $P^\vartheta(\xi',\xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ defined in [LV14]*Section 0.1.

Theorem 4.13 ([LV14]*Theorem 5.2)**.** *For every* $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$*, define*

$$C^\vartheta(\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{l^I(\xi)-l^I(\xi')}\, P^\vartheta(\xi',\xi)\ M(\xi')^+, \tag{101}$$

an element in $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$*. Then*

1. $D(C^\vartheta(\xi)) = q^{-l^I(\xi)}\, C^\vartheta(\xi)$
2. $P^\vartheta(\xi,\xi) = 1$
3. $P^\vartheta(\xi',\xi) = 0$ *if* $\xi' \not\leq \xi$

4. $\deg P^\vartheta(\xi',\xi) \le (l^I(\xi) - l^I(\xi') - 1)/2$ *if* $\xi' \le \xi$.

Conversely suppose $\{\underline{C}(\xi',\xi)\}$ *and* $\{\underline{P}(\xi',\xi)\}$ *satisfy* (101) *and (a-d). Then* $\underline{P}(\xi',\xi) = P^\vartheta(\xi',\xi)$ *and* $\underline{C}(\xi',\xi) = C^\vartheta(\xi',\xi)$ *for all* $\xi',\xi \in \Xi({}^\vee\mathcal{O}{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Properties 2-3 of Theorem 4.13 ensure that

$$\{C^\vartheta(\xi) : \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta\}$$

is a basis for $\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$.

[LV14]*Theorem 5.2 also applies to $\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$, so there is an obvious variant of Theorem 4.13 characterizing the dual KLV-polynomials ${}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)$ and the basis elements

$$C^\vartheta({}^\vee\xi) = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{l^I({}^\vee\xi) - l^I({}^\vee\xi')}\, {}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)\, M({}^\vee\xi')^+. \tag{102}$$

Proposition 4.14. *By specializing to* $q = 1$*, we obtain*

$$C^\vartheta(\xi)(1) = \pi(\xi)^+,$$
$$C^\vartheta({}^\vee\xi)(1) = \pi({}^\vee\xi)^+$$

for all $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Proof. The assertions of the proposition are given in purely representation-theoretic terms. However, in the second equality the representations $M({}^\vee\xi')$ occurring in (102) are representations of possibly different strong involutions of ${}^\vee\mathrm{GL}_N(\lambda)$, which complicates matters. It is therefore clearer if we transport the assertion back to the original context of constructible sheaves using (79). The assertion equivalent to

$$C^\vartheta({}^\vee\xi)(1) = \pi({}^\vee\xi)^+$$

in this context is that $P({}^\vee\xi)^+$ equals

$$\sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{d(\xi)}\, {}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)(1)\, \mu({}^\vee\xi')^+ \tag{103}$$

(recall Proposition 4.5 and (78)). It follows from the definition of the KLV-polynomials ([LV14]*Section 0.1), (67) and (78) that

$${}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)(1) = (-1)^{d(\xi) - d(\xi')}\, c_g^\vartheta(\xi',\xi) = (-1)^{l^I(\xi) - l^I(\xi')}\, c_g^\vartheta(\xi',\xi). \tag{104}$$

(Note that the definition of $P(\xi)$ in [LV14] differs from ours by a shift in degree $d(\xi)$ *cf.* [ABV92]*(7.10)(d).) This implies that (103), as an element in $KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma)$, is equal to

$$\begin{aligned}
&\sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{d(\xi')} \left(c_g(\xi'_+, \xi_+) - c_g(\xi'_-, \xi_+)\right) \mu({}^\vee\xi')^+ \\
&= \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g(\xi'_+, \xi_+)\, (-1)^{d(\xi')} \mu({}^\vee\xi)^+ - c_g(\xi'_-, \xi_+)\, (-1)^{d(\xi')} \mu({}^\vee\xi')^+ \\
&= \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} c_g(\xi'_+, \xi_+)\, (-1)^{d(\xi')} \mu({}^\vee\xi)^+ + c_g(\xi'_-, \xi_+)\, (-1)^{d(\xi')} \mu({}^\vee\xi')^-, \\
&= P(\xi)^+
\end{aligned}$$

where the final equation is (68). This proves the second assertion of the proposition. The first may be proved in the same manner. However, a purely representation-theoretic proof is also possible following [AvLTV20]*Theorem 19.4 □

4.7 The proof of Theorem 3.5

The main theorem of this section is

Theorem 4.15. *Pairing (89) satisfies*

$$\langle C^\vartheta(\xi), C^\vartheta({}^\vee\xi')\rangle = (-1)^{l^I_\vartheta(\xi)}\, q^{\left(l^I(\xi)+l^I({}^\vee\xi')\right)/2}\,\delta_{\xi,\xi'}. \tag{105}$$

for all $\xi,\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

To prove Theorem 4.15 we need the following lemma.

Lemma 4.16. *There are unique elements* $\underline{C}^\vartheta(\xi) \in \mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$, $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$, *satisfying*

$$\langle \underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi')\rangle = (-1)^{l^I_\vartheta(\xi')}\, q^{\left(l^I(\xi)+l^I({}^\vee\xi')\right)/2}\,\delta_{\xi,\xi'}.$$

More explicitly, let $\underline{P}(\xi',\xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ *be the entries of the matrix inverse and transpose to the matrix formed by the polynomials* ${}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)$ *given in (102),* i.e.

$$\sum_{\xi'\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} \underline{P}(\xi',\xi)\,{}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi'') = \delta_{\xi,\xi''}.$$

Then

$$\underline{C}^\vartheta(\xi) = \sum_{\xi'\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{l^I(\xi)-l^I(\xi')}(-1)^{l^I_\vartheta(\xi)-l^I_\vartheta(\xi')}\,\underline{P}(\xi',\xi)\; M(\xi')^+.$$

Proof. We just need to verify, for all $\xi, \xi'' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$, the equality

$$\left\langle \sum_{\xi'}(-1)^{l^I(\xi)-l^I(\xi')}(-1)^{l^I_\vartheta(\xi)-l^I_\vartheta(\xi')}\,\underline{P}(\xi',\xi)\; M(\xi')^+, C^\vartheta({}^\vee\xi'')\right\rangle = (-1)^{l^I_\vartheta(\xi)}\, q^{\left(l^I(\xi)+l^I({}^\vee\xi'')\right)/2}\,\delta_{\xi,\xi''}.$$

Let $k = -\frac{1}{2}\left(|R^+(\lambda)| + \dim(H)\right)$ as in Lemma 4.7. Applying (89), we compute

$$\begin{aligned}
&\left\langle \sum_{\xi'}(-1)^{l^I(\xi)-l^I(\xi')}(-1)^{l^I_\vartheta(\xi)-l^I_\vartheta(\xi')}\,\underline{P}(\xi',\xi)\; M(\xi')^+, C^\vartheta({}^\vee\xi'')\right\rangle\\
&= \left\langle \sum_{\xi'}(-1)^{l^I(\xi)-l^I(\xi')}(-1)^{l^I_\vartheta(\xi)-l^I_\vartheta(\xi')}\,\underline{P}(\xi',\xi)\; M(\xi')^+, \sum_{\xi_1}(-1)^{l^I({}^\vee\xi'')-l^I({}^\vee\xi_1)}\;{}^\vee P^\vartheta({}^\vee\xi_1, {}^\vee\xi'')\,M({}^\vee\xi_1)^+\right\rangle\\
&= (-1)^{l^I_\vartheta(\xi)}\,(-1)^k(-1)^{l^I(\xi)+l^I({}^\vee\xi'')}\,q^{k/2}\sum_{\xi'\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta}\underline{P}(\xi',\xi)\,{}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi'')\\
&= (-1)^{l^I_\vartheta(\xi)}\,q^{k/2}\delta_{\xi,\xi''}\\
&= (-1)^{l^I_\vartheta(\xi)}q^{\left(l^I(\xi)+l^I({}^\vee\xi'')\right)/2}\,\delta_{\xi,\xi''}.
\end{aligned}$$

□

Proof of Theorem 4.15. If one proves that the coefficient polynomials $\underline{P}(\xi',\xi)$ of $\underline{C}^\vartheta(\xi)$ satisfy properties 1-4 of Theorem 4.13, then the uniqueness statement of that theorem implies $\underline{C}^\vartheta(\xi) = C^\vartheta(\xi)$ and the theorem follows from Lemma 4.16. To show properties 1-4 we follow the proof of [Vog82]*Lemma 13.7. For the first property we apply Proposition 4.12

$$\begin{aligned}
\langle D\underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi')\rangle &= \overline{\langle \underline{C}^\vartheta(\xi),\, {}^\vee DC^\vartheta({}^\vee\xi')\rangle}\\
&= q^{l^I({}^\vee\xi')}\,\overline{\langle \underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi')\rangle}\\
&= (-1)^{l^I_\vartheta(\xi)}q^{l^I({}^\vee\xi')}q^{-1/2(l^I(\xi)+l^I({}^\vee\xi'))}\delta_{\xi,\xi'}\\
&= \langle q^{-l^I(\xi)}\,\underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi')\rangle.
\end{aligned}$$

Since the elements $C^\vartheta({}^\vee\xi')$ form a basis we conclude that

$$D\underline{C}^\vartheta(\xi) = q^{-l^I(\xi)}\,\underline{C}^\vartheta(\xi)$$

and the first property of Theorem 4.13 is proved.

The second and third properties of Theorem 4.13 follow for $\underline{P}(\xi',\xi)$ since it is defined in terms of the transpose and inverse of a unipotent matrix, and the map $\xi \mapsto {}^\vee\xi$ (72) is order-reversing (88).

The fourth property of Theorem 4.13 is proven by induction on the integral length of a parameter. This uses a straightforward reformulation of [Vog82]*(13.9) which is left to the reader.

The uniqueness statement in Theorem 4.13 now implies $\underline{P}^\vartheta(\xi,\xi') = (-1)^{l^I_\vartheta(\xi)-l^I_\vartheta(\xi')}P^\vartheta(\xi,\xi')$ and $\underline{C}^\vartheta(\xi) = C^\vartheta(\xi)$. Finally by Lemma 4.16, Equation (105) holds, completing the proof of the theorem.

□

The proof of Theorem 3.5 is now immediate.

Proof of Theorem 3.5. It is enough to prove Lemma 4.6. Let $\langle\cdot\,,\cdot\rangle$ be the pairing in (89). By Theorem 4.15 we have

$$\langle C^\vartheta(\xi), C^\vartheta({}^\vee\xi')\rangle = (-1)^{l^I_\vartheta(\xi)}\, q^{\left(l^I(\xi)+l^I({}^\vee\xi')\right)/2}\,\delta_{\xi,\xi'}.$$

Setting $q = 1$ and applying Proposition 4.14, we conclude

$$\langle \pi(\xi)^+, \pi({}^\vee\xi')^+\rangle = (-1)^{l^I_\vartheta(\xi)}\delta_{\xi,\xi'}$$

as required. □

4.8 Twisted KLV-polynomials for the dual of $\mathrm{GL}_N(\mathbb{R})$

This section is thematically related to the others in Section 4, although it admittedly has nothing to do with the pairings. The sole purpose of this section is to determine the polynomials ${}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ appearing in the definition of $C^\vartheta({}^\vee\xi)$ (102) under certain circumstances. In our application ξ will be the parameter of a generic representation, and the value of ${}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)$ will be used in Section 7 (Proposition 7.3 and Proposition 7.4) to compare the Atlas extensions with the so-called *Whittaker extensions.*

A block of parameters is defined in [Vog82]*Definition 1.14. In particular $P(\xi,\xi') \neq 0$ implies ξ,ξ' are in the same block.

Proposition 4.17. *Suppose ${}^\vee\xi_0$ is the unique maximal element of a block ${}^\vee\mathcal{B}$ with respect to the Bruhat order (88). Then ${}^\vee P^\vartheta({}^\vee\xi, {}^\vee\xi_0) = 1$ for all ${}^\vee\xi \in {}^\vee\mathcal{B}$.*

The proof of the Proposition 4.17 is algorithmic in nature and relies on computations with Hecke operators. A broad examination of the algorithms is presented in [Ada17] and [LV14]. We assemble a few facts from these references here. The facts are centred upon the characterization of eigenspaces of Hecke operators.

For each $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$, Lusztig and Vogan define what it means for (the Weyl group element of) a ϑ-orbit κ of a simple root in $R(\lambda)$ to be a *descent* for ${}^\vee\xi$ ([LV14]*Section 7.2). We leave the definition (which is equivalent to (108)) to the interested reader, being content merely to record the relevant properties. By definition, a ϑ-orbit κ is an *ascent* for ${}^\vee\xi$ if it is not a descent for ${}^\vee\xi$. To assist in indexing these ϑ-orbits we define $\boldsymbol{\tau}({}^\vee\xi)$ to be the set of ϑ-orbits of simple roots which are descent for ${}^\vee\xi$. Thus, κ is

- a *descent* for ${}^\vee\xi$, if $\kappa \in \boldsymbol{\tau}({}^\vee\xi)$, and
- an *ascent* for ${}^\vee\xi$, if $\kappa \notin \boldsymbol{\tau}({}^\vee\xi)$.

Recall that the $\mathcal{H}(\lambda)$-module

$$\mathcal{K}^\vee\Pi(^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) = K^\vee\Pi(^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

((79), Section 4.4) has a basis $\{M(^\vee\xi)^+ : \xi \in \Xi(^\vee\mathcal{O}, {^\vee\mathrm{GL}}_N^\Gamma)^\vartheta\}$. For each ϑ-orbit κ let

$$\widehat{T}_\kappa = q^{-l(w_\kappa)/2}(T_\kappa + 1).$$

Suppose $\kappa \notin \boldsymbol{\tau}(^\vee\xi)$, *i.e.* κ is an ascent for $^\vee\xi$. Then we define

$$B_\kappa(^\vee\xi) \subset \Xi(^\vee\mathcal{O}, {^\vee\mathrm{GL}}_N^\Gamma)^\vartheta$$

to be the set of parameters which indexes the non-zero summands in $\widehat{T}_\kappa M(^\vee\xi)^+$, that is

$$\widehat{T}_\kappa M(^\vee\xi)^+ = \sum_{\xi' \in B_\kappa(^\vee\xi)} a(^\vee\xi')\, M(^\vee\xi')^+, \quad 0 \neq a(^\vee\xi') \in \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

A case-by-case inspection of [AV15]*Table 5, or the formulas of [LV14]*7.5-7.7, confirm that $\xi \in B_\kappa(^\vee\xi)$, and $|B_\kappa(^\vee\xi)| \leq 3$. Let

$$\mathcal{M}_\kappa(^\vee\xi) = \mathbb{Z}[q^{1/2}, q^{-1/2}]\text{-span of } \{M(^\vee\xi')^+ : \xi' \in B_\kappa(^\vee\xi)\}.$$

Keeping in mind that $\kappa \notin \boldsymbol{\tau}(^\vee\xi)$, we write

$$^\vee\xi' \xrightarrow{\kappa} {^\vee\xi}$$

if $\kappa \in \boldsymbol{\tau}(^\vee\xi')$ and $\xi' \in B_\kappa(^\vee\xi)$. In this definition κ is an ascent for $^\vee\xi$ and a descent for $^\vee\xi'$.

The quadratic relation (92) gives

$$\widehat{T}_\kappa^2 = (q^{l(w_\kappa)/2} + q^{-l(w_\kappa)/2})\widehat{T}_\kappa.$$

An important consequence of this equation is that the image of $\widehat{T}_\kappa$ is contained in the $(q^{l(w_\kappa)/2} + q^{-l(w_\kappa)/2})$-eigenspace of $\widehat{T}_\kappa$ ([LV14]*7.2 (c)).

From a case-by-case inspection of [AV15]*Table 5 or the formulas of [LV14]*7.5-7.7, it follows that if $\kappa \notin \boldsymbol{\tau}(^\vee\xi)$ then the space $\mathcal{M}_\kappa(^\vee\xi)$ is $\widehat{T}_\kappa$-invariant, and the $(q^{l(w_\kappa)/2} + q^{-\ell(\kappa)/2})$-eigenspace of $\widehat{T}_\kappa$ on $\mathcal{M}_\kappa(^\vee\xi)$ is spanned by

$$\left\{\widehat{T}_\kappa M(^\vee\xi')^+ \;:\; {^\vee\xi'} \xrightarrow{\kappa} {^\vee\xi}\right\} = \left\{\widehat{T}_\kappa M(^\vee\xi')^+ \;:\; {^\vee\xi'} \in B_\kappa(^\vee\xi) \text{ and } \kappa \in \boldsymbol{\tau}(^\vee\xi')\right\}. \tag{106}$$

Now suppose $\kappa \in \boldsymbol{\tau}(^\vee\xi)$, *i.e.* is a descent for $^\vee\xi$. Then [AV15]*Table 5 tells us that if κ is not of type `1r2`, `2r22` and `2r21` with respect to $^\vee\xi$, then

$$\widehat{T}_\kappa M(^\vee\xi)^+ = a(^\vee\xi)\left(M(^\vee\xi)^+ + \sum_{\{\xi' : {^\vee\xi} \xrightarrow{\kappa} {^\vee\xi'}\}} M(^\vee\xi')^+\right) \tag{107}$$

for some $a(^\vee\xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. A quick glance at (87) affirms that the types `1r2, 2r22` and `2r21` do not occur, so (107) holds for every κ which is a descent. Another fact we need for $\kappa \in \boldsymbol{\tau}(^\vee\xi)$ is that $C^\vartheta(^\vee\xi) \in \mathcal{K}^\vee\Pi(^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ defined in (102) satisfies

$$\widehat{T}_\kappa C^\vartheta(^\vee\xi) = (q^{l(w_\kappa)/2} + q^{-l(w_\kappa)/2})\, C^\vartheta(^\vee\xi). \tag{108}$$

This is stated in the last paragraph of [LV14]*Section 7.2, where $C^\vartheta(^\vee\xi)$ is denoted as $\mathfrak{A}_\mathcal{L}$ (see [LV14]*Section 5).[3]

[3]There is a misprint in this paragraph of [LV14]. In our notation $C^\vartheta(^\vee\xi)$ belongs to the $q^{l(w_\kappa)}$-eigenspace of T_κ (not $T_\kappa + 1$), and $C^\vartheta(^\vee\xi)$ is in the $(q^{l(w_\kappa)/2} + q^{-l(w_\kappa)/2})$-eigenspace of $\widehat{T}_\kappa$.

Finally, fix an arbitrary ϑ-orbit κ. If κ is a descent for ${}^\vee\xi'$ and is not of type `1ic,2ic,3ic`, then [AV15]*Table 5 indicates that $\xi' \in B_\kappa({}^\vee\xi)$, where κ is an ascent for some ${}^\vee\xi$. From this it is easy to see that

$$\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) = \sum_{\{{}^\vee\xi\,:\,\kappa\notin\boldsymbol{\tau}({}^\vee\xi)\}} \mathcal{M}_\kappa({}^\vee\xi) \qquad \oplus \bigoplus_{\substack{\kappa \text{ 1ic, 2ic, 3ic} \\ \text{for } {}^\vee\xi''}} \mathbb{Z}[q^{1/2}, q^{-1/2}]\, M({}^\vee\xi'').$$

This decomposition and (106) imply that the $(q^{l(w_\kappa)/2} + q^{-l(w_\kappa)/2})$-eigenspace of $\widehat{T}_\kappa$ is spanned by

$$\{\widehat{T}_\kappa M({}^\vee\xi')^+ : \kappa \in \boldsymbol{\tau}({}^\vee\xi')\}. \tag{109}$$

Lemma 4.18. *Suppose κ is a ϑ-orbit of a simple root in $R^+(\lambda)$, and $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ satisfy $\kappa \in \boldsymbol{\tau}({}^\vee\xi)$ and $\kappa \notin \boldsymbol{\tau}({}^\vee\xi')$. Then*

$$ {}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi) = \sum_{\{\xi'' : {}^\vee\xi'' \xrightarrow{\kappa} {}^\vee\xi'\}} {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi).$$

Proof. Write $C^\vartheta({}^\vee\xi)$ as

$$C^\vartheta({}^\vee\xi) = \sum_{\{\xi'' : \kappa\in\boldsymbol{\tau}({}^\vee\xi'')\}} {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi)\, M({}^\vee\xi'')^+ + \sum_{\{\xi' : \kappa\notin\boldsymbol{\tau}({}^\vee\xi')\}} {}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi)\, M({}^\vee\xi')^+ \tag{110}$$

On the other hand by (108) and (109) we have

$$C^\vartheta({}^\vee\xi) = \sum_{\{\xi'' : \kappa\in\boldsymbol{\tau}({}^\vee\xi'')\}} b({}^\vee\xi'', {}^\vee\xi)\ \widehat{T}_\kappa M({}^\vee\xi'')^+$$

for some $b({}^\vee\xi'', {}^\vee\xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Inserting (107) into this equation yields

$$\begin{aligned} C^\vartheta({}^\vee\xi) = &\sum_{\{\xi'' : \kappa\in\boldsymbol{\tau}({}^\vee\xi'')\}} b({}^\vee\xi'', {}^\vee\xi) a({}^\vee\xi'')\, M({}^\vee\xi'')^+ \\ &+ \sum_{\{\xi'' : \kappa\in\boldsymbol{\tau}({}^\vee\xi'')\}} b({}^\vee\xi'', {}^\vee\xi) a({}^\vee\xi'') \sum_{\{\xi' : {}^\vee\xi'' \xrightarrow{\kappa} {}^\vee\xi'\}} M({}^\vee\xi')^+ \end{aligned} \tag{111}$$

Since $\kappa \notin \boldsymbol{\tau}({}^\vee\xi')$ for all ${}^\vee\xi'$ appearing in the final sum, we may compare the coefficients of $M({}^\vee\xi'')^+$ with $\kappa \in \boldsymbol{\tau}({}^\vee\xi'')$ in (110) and (111) to conclude

$$b({}^\vee\xi'', {}^\vee\xi) a({}^\vee\xi'') = {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi).$$

Therefore

$$\begin{aligned} C^\vartheta({}^\vee\xi) = &\sum_{\{\xi'' : \kappa\in\boldsymbol{\tau}({}^\vee\xi'')\}} {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi)\, M({}^\vee\xi'')^+ \\ &+ \sum_{\{\xi'' : \kappa\in\boldsymbol{\tau}({}^\vee\xi'')\}} {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi) \sum_{\{\xi' : {}^\vee\xi'' \xrightarrow{\kappa} {}^\vee\xi'\}} M({}^\vee\xi')^+ \end{aligned}$$

Changing the order of summation in the second sum, it becomes

$$\sum_{\{\xi' : \kappa\notin\boldsymbol{\tau}({}^\vee\xi')\}} \sum_{\{\xi'' : {}^\vee\xi'' \xrightarrow{\kappa} {}^\vee\xi'\}} {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi)\, M({}^\vee\xi')^+$$

Comparing with the second sum in (110) gives

$$ {}^\vee P^\vartheta({}^\vee\xi', {}^\vee\xi) = \sum_{\{\xi'' : {}^\vee\xi'' \xrightarrow{\kappa} {}^\vee\xi'\}} {}^\vee P^\vartheta({}^\vee\xi'', {}^\vee\xi).$$

This completes the proof. ∎

Lemma 4.18 simplifies even further in the present setting.

Corollary 4.19. *Suppose κ is a ϑ-orbit of a simple root in $R^{+}(\lambda)$, and $\xi, \xi' \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_{N}^{\Gamma})^{\vartheta}$ satisfy $\kappa \in \boldsymbol{\tau}({}^{\vee}\xi)$ and $\kappa \notin \boldsymbol{\tau}({}^{\vee}\xi')$. Then there exists exactly one $\xi'' \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_{N}^{\Gamma})^{\vartheta}$ such that ${}^{\vee}\xi'' \xrightarrow{\kappa} {}^{\vee}\xi'$. Furthermore,*

$$
{}^{\vee}P^{\vartheta}({}^{\vee}\xi', {}^{\vee}\xi) = {}^{\vee}P^{\vartheta}({}^{\vee}\xi'', {}^{\vee}\xi).
$$

Proof. Here are the formulas, derived from [LV14]*Section 7 or [AV15]*Table 5, for $\widehat{T}_{\kappa} M({}^{\vee}\xi')^{+}$ for the various types $\kappa \notin \boldsymbol{\tau}({}^{\vee}\xi')$ which arise in (87).

Type	$\widehat{T}_{\kappa} M({}^{\vee}\xi')^{+}$
`1C+`	$q^{-1/2}\ (M({}^{\vee}\xi')^{+} + w_{\kappa} \times M({}^{\vee}\xi')^{+})$
`1i1`	$q^{-1/2}\ (M({}^{\vee}\xi')^{+} + w_{\kappa} \times M({}^{\vee}\xi')^{+} + c_{\kappa} M({}^{\vee}\xi')^{+})$
`2C+`	$q^{-1}\ (M({}^{\vee}\xi')^{+} + w_{\kappa} \times M({}^{\vee}\xi')^{+})$
`2Ci`	$(q^{-1}+1)\ (M({}^{\vee}\xi')^{+} + c_{\kappa} M({}^{\vee}\xi')^{+})$
`2i11`	$q^{-1}\ (M({}^{\vee}\xi')^{+} + w_{\kappa} \times M({}^{\vee}\xi')^{+} + c_{\kappa} M({}^{\vee}\xi')^{+})$
`3C+`	$q^{-3/2}\ (M({}^{\vee}\xi')^{+} + w_{\kappa} \times M({}^{\vee}\xi')^{+})$
`3Ci`	$q^{-3/2}(q+1)\ (M({}^{\vee}\xi')^{+} + c_{\kappa} M({}^{\vee}\xi')^{+})$
`3i`	$q^{-3/2}(q+1)\ (M({}^{\vee}\xi')^{+} + c_{\kappa} M({}^{\vee}\xi')^{+})$

The "Cayley transforms" c_{κ} appearing here follow the notation of the references. They may include cross actions in their definition (see types `2ci` and `3ci` which are 7.6 (c′) and 7.7 (c′) in [LV14]). In any event, the c_{κ} appearing here are all single-valued. Once more a case-by-case inspection of the types in [LV14]*7.5-7.7 shows that in each entry of the second column there is exactly one summand whose parameter makes κ a descent. Consequently, the table indicates that $\{\xi'' : {}^{\vee}\xi'' \xrightarrow{\kappa} {}^{\vee}\xi'\}$ has exactly one element. The corollary now follows from Lemma 4.18. □

We are ready to provide the proof for Proposition 4.17

Proof. It follows from the maximality of ${}^{\vee}\xi_0$ that $l^{I}({}^{\vee}\xi_0)$ is maximal among all the integral lengths appearing from the representations in the block ([Vog82]*Lemma 12.10). If κ is a ϑ-orbit of a simple root in $R^{+}(\lambda)$ with $\kappa \notin \boldsymbol{\tau}({}^{\vee}\xi_0)$ then $l^{I}({}^{\vee}\xi'') > l^{I}({}^{\vee}\xi_0)$ for some $\xi'' \in B_{\kappa}({}^{\vee}\xi_0)$ ([LV14]*7.5-7.7)–a contradiction to the maximality of $l^{I}({}^{\vee}\xi_0)$. Therefore $\kappa \in \boldsymbol{\tau}({}^{\vee}\xi_0)$ for all ϑ-orbits κ.

If $\xi' \neq \xi_0$ then by the uniqueness hypothesis ${}^{\vee}\xi'$ is not maximal in the Bruhat order. By [Vog82]*Theorem 8.8, $M({}^{\vee}\xi')^{+}$ is equal to the cross action or Cayley transform of some representation in its block with higher integral length. Looking to the formulas in [LV14]*7.5-7.7, we see that this implies the existence of some $\kappa \notin \boldsymbol{\tau}({}^{\vee}\xi')$.

We now prove ${}^{\vee}P^{\vartheta}({}^{\vee}\xi', {}^{\vee}\xi_0) = 1$ by induction on $l^{I}({}^{\vee}\xi_0) - l^{I}({}^{\vee}\xi')$. If $l^{I}({}^{\vee}\xi_0) = l^{I}({}^{\vee}\xi')$ then the uniqueness hypothesis implies ${}^{\vee}\xi' = {}^{\vee}\xi_0$ and we are done by [LV14]*Theorem 5.2 (c). Otherwise, $l^{I}({}^{\vee}\xi_0) > l^{I}({}^{\vee}\xi')$ and we have shown above that the hypotheses of Corollary 4.19 are satisfied for some κ. Corollary 4.19 tells us that ${}^{\vee}P^{\vartheta}({}^{\vee}\xi', {}^{\vee}\xi_0) = {}^{\vee}P^{\vartheta}({}^{\vee}\xi'', {}^{\vee}\xi_0)$ for some ${}^{\vee}\xi'' \xrightarrow{\kappa} {}^{\vee}\xi'$. The condition ${}^{\vee}\xi'' \xrightarrow{\kappa} {}^{\vee}\xi'$ necessitates $l^{I}({}^{\vee}\xi'') > l^{I}({}^{\vee}\xi')$ and so

$$
{}^{\vee}P^{\vartheta}({}^{\vee}\xi', {}^{\vee}\xi_0) = {}^{\vee}P^{\vartheta}({}^{\vee}\xi'', {}^{\vee}\xi_0) = 1
$$

by induction. □

5 Endoscopic lifting for general linear groups following Adams-Barbasch-Vogan

In this section we review standard endoscopy and twisted endoscopy from the perspective of [ABV92], but restricted only to the particular case of the group GL_N. We shall be using all of the previously defined objects and work under the assumption of (32) for the infinitesimal character. The references for this review are [ABV92]*Section 26 and [CM18]*Section 5.

5.1 Standard endoscopy

Let ${}^\vee\mathrm{GL}_N^\Gamma = {}^\vee\mathrm{GL}_N \rtimes \langle {}^\vee\delta_0\rangle$ be as in (30). An *endoscopic datum* for ${}^\vee\mathrm{GL}_N^\Gamma$ is a pair

$$(s, {}^\vee G^\Gamma)$$

which satisfies

1. $s \in {}^\vee\mathrm{GL}_N$ is semisimple
2. ${}^\vee G^\Gamma \subset {}^\vee\mathrm{GL}_N^\Gamma$ is open in the centralizer of s in ${}^\vee\mathrm{GL}_N^\Gamma$
3. ${}^\vee G^\Gamma$ is an E-group for a group G ([ABV92]*Definition 4.6).

This is a specialization of [ABV92]*Definition 26.15 to ${}^\vee\mathrm{GL}_N^\Gamma$. The groups ${}^\vee G$ and G here are isomorphic to products of smaller general linear groups. Consequently, ${}^\vee G$ and ${}^\vee\mathrm{GL}_N$ share the diagonal maximal torus ${}^\vee H$, and G and GL_N share the diagonal maximal torus H. We shall abusively denote by δ_q the strong involution on both G and GL_N which correspond to the split real forms. The group G in this definition is called the *endoscopic group*.

We do not require the general concept of an E-group in this section. From now on we assume that ${}^\vee G^\Gamma = {}^\vee G \times \langle {}^\vee\delta_0\rangle$ where ${}^\vee\delta_0^2 = 1$. In other words, ${}^\vee G^\Gamma$ is an L-group for G.

There is a notion of equivalence for endoscopic data, and using this equivalence we may assume without loss of generality that $s \in {}^\vee H$. We fix $\lambda \in {}^\vee\mathfrak{h}$ satisfying the hypotheses of (32) so that λ is regular with respect to ${}^\vee\mathrm{GL}_N$. Let ${}^\vee\mathcal{O}_G$ be the ${}^\vee G$-orbit of λ and ${}^\vee\mathcal{O}$ be the ${}^\vee\mathrm{GL}_N$-orbit of λ. The second property of the endoscopic datum above allows us to define the inclusion

$$\epsilon:\ {}^\vee G^\Gamma \hookrightarrow {}^\vee\mathrm{GL}_N^\Gamma. \tag{112}$$

This inclusion induces another map ([ABV92]*Corollary 6.21), which we abusively also denote as

$$\epsilon:\ X\left({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma\right) \to X\left({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma\right). \tag{113}$$

It is easily verified that the ${}^\vee G$-action on $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ is compatibly carried under ϵ to the ${}^\vee\mathrm{GL}_N$-action on $X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ ([ABV92]*(7.17)). As a result, the map(s) ϵ induces maps on the orbits of the spaces $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ and $X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$, and also induces a homomorphism

$$A^{\mathrm{loc}}(\epsilon) : {}^\vee G_p/({}^\vee G_p)^0 \to ({}^\vee\mathrm{GL}_N)_{\epsilon(p)}/(({}^\vee\mathrm{GL}_N)_{\epsilon(p)})^0 \tag{114}$$

on the component groups. As we have seen, the component groups for GL_N, and therefore also for G, are trivial.

The inverse image functor of ϵ on equivariant constructible sheaves induces a homomorphism

$$\epsilon^* : K({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma) \to K({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma) \tag{115}$$

[ABV92]*Proposition 7.18. One may describe its values on irreducible constructible sheaves $\mu(\xi)$, $\xi = (S, \tau_\xi) \in \Xi({}^\vee\mathrm{GL}_N^\Gamma, {}^\vee\mathcal{O})$ as follows. If the orbit S is not the image of an orbit of $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ under ϵ then $\epsilon^*\mu(\xi) = 0$. Otherwise, S is the ${}^\vee\mathrm{GL}_N$-orbit of $\epsilon(p) \in S$ for some $p \in X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$. In this case $\mu(\xi)$ may be identified with (45) where $\tau_\xi = 1$ the trivial quasicharacter of the trivial group $({}^\vee\mathrm{GL}_N)_{\epsilon(p)}/(({}^\vee\mathrm{GL}_N)_{\epsilon(p)})^0$. The stalk of the constructible sheaf $\epsilon^*\mu(\xi)$ at p is then the stalk V of (45). The representation on V is given by the quasicharacter $\tau_\xi \circ A^{\mathrm{loc}}(\epsilon)$, which is again the trivial quasicharacter (on the trivial component group). In summary,

$$\epsilon^*\mu(\xi) = \epsilon^*\mu(S,1) = \sum_{\{S_1 : {}^\vee\mathrm{GL}_N\cdot\epsilon(S_1) = S\}} \mu(S_1, 1)$$

in which S_1 is a ${}^\vee G$-orbit in $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$. This sum will be seen to reduce to a single term in Proposition 5.1.

When ϵ^* is combined with the pairings of Theorem 3.1, we obtain a map

$$\epsilon_* : K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}) \to K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$$

defined on $\eta_G \in K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R})$ by

$$\langle \epsilon_*\eta_G, \mu(\xi)\rangle = \langle \eta_G, \epsilon^*\mu(\xi)\rangle_G\,, \quad \xi \in \Xi({}^\vee\mathrm{GL}_N^\Gamma, {}^\vee\mathcal{O}). \tag{116}$$

Here, $K_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{Z}} K$ and we have placed a subscript G beside the pairing on the right to distinguish it from the pairing for GL_N on the left.

The endoscopic lifting map is a restriction of ϵ_* to a subspace of $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R})$ which is perhaps best described in two steps. The first subspace is generated by the (equivalence classes of) representations of the quasisplit strong involution δ_q (Section 2.1). We denote this subspace by $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))$. Lemma 2.1 tells that

$$K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q)) = K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}),$$

but this will not be true when we look at twisted endoscopic groups in Section 5.2. Inside of $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))$ is the subspace generated by stable virtual characters of $G(\mathbb{R},\delta_q)$ (Section 2.6, [ABV92]*18.2). We denote this subspace by $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}}$. Again, since G is a product of general linear groups, stability is not an issue and we have

$$K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}} = K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}). \tag{117}$$

This equality will not hold for twisted endoscopic groups in Section 5.2. We define the endoscopic lifting

$$\mathrm{Lift}_0 : K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}} \to K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R})) \tag{118}$$

as the restriction of ϵ_* to $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}}$.

An argument of Shelstad ([She79], [ABV92]*Lemma 18.11) provides a basis for $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}}$. The basis elements are of the form

$$\eta^{\mathrm{loc}}_{S_1}(\delta_q) = \sum_{\tau_{S_1}} M(S_1,\tau_{S_1}), \tag{119}$$

where $(S_1,\tau_{S_1}) \in \Xi({}^\vee G^\Gamma, {}^\vee\mathcal{O}_G)$ runs over those complete geometric parameters which correspond to the strong involution δ_q under the local Langlands correspondence. As mentioned earlier, τ_{S_1} is trivial for G and so (119) reduces to

$$\eta^{\mathrm{loc}}_{S_1}(\delta_q) = M(S_1, 1),$$

a single standard representation.

Proposition 5.1. *In the setting of (32):*

(a) Suppose $S_1, S_2 \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ are ${}^\vee G$-orbits which are carried to the same ${}^\vee\mathrm{GL}_N$-orbit S under ϵ. Then $S_1 = S_2$.

(b) The endoscopic lifting map Lift_0 (118) is injective and sends $\eta^{\mathrm{loc}}_{S_1}(\delta_q) = M(S_1,1)$ to $\eta^{\mathrm{loc}}_{S} = M(S,1)$.

(c) The endoscopic lifting map Lift_0 is equal to the parabolic induction functor $\mathrm{ind}^{\mathrm{GL}_N(\mathbb{R})}_{G(\mathbb{R},\delta_q)}$ on $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}}$.

Proof. By [ABV92]*Definition 6.9 and [ABV92]*(6.4), we may take ${}^\vee G$-orbits

$$S_1 = {}^\vee G\cdot(y_1, \mathcal{F}(\lambda)),\ S_2 = {}^\vee G\cdot(y_2, \mathcal{F}(\mathrm{Ad}(g_1)\lambda)),$$

where $g_1 \in {}^\vee G$, $y_1, y_2 \in {}^\vee G^\Gamma$, and

$$\mathcal{F}(\lambda) = \mathrm{Ad}(P(\lambda))\lambda,\ \mathcal{F}(\mathrm{Ad}(g_1)\lambda) = \mathrm{Ad}(g_1)(\mathrm{Ad}(P(\lambda))\lambda)$$

for a solvable subgroup $P(\lambda) \subset {}^\vee G$ ([ABV92]*Lemma 6.3). By the hypothesis of the first assertion, there exists $g \in {}^\vee\mathrm{GL}_N$ such that $gy_2g^{-1} = y_1$ and

$$\mathrm{Ad}(gg_1)(\mathrm{Ad}(P(\lambda))\lambda) = \mathrm{Ad}(P(\lambda))\lambda.$$

In particular, $\mathrm{Ad}(gg_1)\lambda = \mathrm{Ad}(g_2)\lambda$ for some $g_2 \in P(\lambda)$. As $\lambda \in {}^\vee\mathfrak{h}$ is regular the resulting equation $\mathrm{Ad}(g_2^{-1}gg_1)\lambda = \lambda$ implies that $g_2^{-1}gg_1 \in {}^\vee H \subset {}^\vee G$. Since $g_1, g_2 \in {}^\vee G$, this also implies $g \in {}^\vee G$ and the first assertion is proven.

The second part of the second assertion is equivalent to $\epsilon_* \eta^{\mathrm{loc}}_{S_1}(\delta_q) = \eta^{\mathrm{loc}}_S$ and this is proved in [ABV92]*Proposition 26.7. Finally, to prove the claim of injectivity, we observe that by the first assertion Lift_0 sends the basis

$$\left\{\eta^{\mathrm{loc}}_{S_G}(\delta_q) : S_G \text{ a } {}^\vee G\text{-orbit of } X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)\right\}$$

of $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$ bijectively onto the linearly independent subset

$$\left\{\eta^{\mathrm{loc}}_{{}^\vee\mathrm{GL}_N \cdot S_G} : S_G \text{ a } {}^\vee G\text{-orbit of } X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)\right\}$$

of $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$.

We now prove the third assertion. Since the standard characters form a basis for (117) and Lift_0 is additive, it suffices to prove $\mathrm{Lift}_0\left(M(S_1, 1)\right) = \mathrm{ind}_{G(\mathbb{R},\delta_q)}^{\mathrm{GL}_N(\mathbb{R})} M(S_1, 1)$. By the second assertion, this is equivalent to proving

$$\mathrm{ind}_{G(\mathbb{R},\delta_q)}^{\mathrm{GL_N}(\mathbb{R})} M(S_1, 1) = M(S, 1). \tag{120}$$

Let us recall the definition of $M(S_1, 1)$ using the Langlands classification [Lan89]. The ${}^\vee G$-orbit S_1 corresponds to a unique ${}^\vee G$-orbit of an L-parameter ϕ_G for G ([ABV92]*Proposition 6.17). The image of ϕ_G is contained in a Levi subgroup ${}^\vee G_0 \subset {}^\vee G$ minimally ([Bor79]*Section 11.3). It follows that the L-parameter ϕ_G factors through an L-parameter ϕ_{G_0} for G_0, and ϕ_{G_0} corresponds to a unique ${}^\vee G_0$-orbit S_0 of geometric parameters for G_0. The standard characters are defined so that $M(S_1, 1) = \mathrm{ind}_{G_0(\mathbb{R},\delta_q)}^{G(\mathbb{R},\delta_q)} M(S_0, 1)$ and $M(S, 1) = \mathrm{ind}_{G_0(\mathbb{R},\delta_q)}^{\mathrm{GL}_N(\mathbb{R})} M(S_0, 1)$ ([ABV92]*(11.2), [AV92b]*(8.22)). Identity (120) is therefore a consequence of induction by stages. □

The proof that $\mathrm{Lift}_0\left(\eta^{\mathrm{loc}}_{S_1}\right)(\delta_q) = \eta^{\mathrm{loc}}_S$ in this proposition follows from an elementary computation of $\epsilon^* \eta^{\mathrm{loc}}_S$ ([ABV92]*Proposition 23.7). It is much more difficult to compute the value of Lift_0 on the stable virtual character $\eta^{\mathrm{mic}}_{\psi_G}$ given in (48). Let $\psi = \epsilon \circ \psi_G$. According to [ABV92]*Theorem 26.25

$$\mathrm{Lift}_0\left(\eta^{\mathrm{mic}}_{\psi_G}\right) = \sum_{\xi \in \Xi({}^\vee\mathrm{GL}_N^\Gamma, {}^\vee\mathcal{O})} (-1)^{d(S_\xi) - d(S_\psi)}\, \chi^{\mathrm{mic}}_{S_\psi}(P(\xi))\, \pi(\xi) = \eta^{\mathrm{mic}}_\psi. \tag{121}$$

Recall from (50) that the ABV-packets $\Pi^{\mathrm{ABV}}_{\psi_G}$ and Π^{ABV}_ψ are defined from $\eta^{\mathrm{mic}}_{\psi_G}$ and η^{mic}_ψ respectively. We shall see in Section 6 that these ABV-packets are singletons. Equation (121) implies that the endoscopic lift of $\Pi^{\mathrm{ABV}}_{\psi_G}$ is Π^{ABV}_ψ.

5.2 Twisted endoscopy

Following the path of the previous section, we define twisted endoscopic data, the twisted endoscopic version of Lift_0 (118), compute twisted variants of $\mathrm{Lift}_0(\eta^{\mathrm{loc}}_S)$ for $S \in X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$, and compute twisted variants of $\mathrm{Lift}_0(\eta^{\mathrm{mic}}_{\psi_G})$.

An *endoscopic datum* for $({}^\vee\mathrm{GL}_N^\Gamma, \vartheta)$ is a pair

$$(s,\ {}^\vee G^\Gamma)$$

which satisfies

1. $s \in {}^\vee\mathrm{GL}_N$ is ϑ-semisimple (see [KS99]*(2.1.3))
2. ${}^\vee G^\Gamma \subset {}^\vee\mathrm{GL}_N^\Gamma$ is open in the fixed-point set of $\sigma = \mathrm{Int}(s) \circ \vartheta$ in $\mathrm{GL}_N^\Gamma \rtimes \langle\vartheta\rangle$ (31)
3. ${}^\vee G^\Gamma$ is an E-group for a group G ([ABV92]*Definition 4.6).

This is a special case of [CM18]*Definition 5.1 to ${}^\vee\mathrm{GL}_N^\Gamma$. There is a notion of equivalence for these endoscopic data ([CM18]*Definition 5.6, [KS99]*(2.1.5)-(2.1.6)). Up to this equivalence the relevant elements $s \in {}^\vee\mathrm{GL}_N$ in Arthur's work are drawn from [Art13]*Section 1.2 (see [CM18]*Example

5.2 for the details in the present setting). We shall restrict our attention to at most two elements s from [Art13]*Section 1.2, namely the diagonal matrix

$$s = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \tilde{J}^{-1} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & (-1)^{N-1} \end{bmatrix}, \tag{122}$$

and, if N is even, the diagonal matrix

$$s = \begin{bmatrix} 0 & & & & -1 \\ & & & \iddots & \\ & & 1 \quad -1 & & \\ & \iddots & & & \\ 1 & & & & 0 \end{bmatrix} \tilde{J}^{-1}. \tag{123}$$

The resulting automorphisms $\sigma = \mathrm{Int}(s) \circ \vartheta$ are involutions, and the endoscopic groups G are simple to describe. For s as in (122) and odd N, the endoscopic group is Sp_{N-1}. For s as in (122) and even N, the endoscopic group is SO_N. For s as in (123), the endoscopic group is SO_{N+1}. In the first and last cases, the embeddings of ${}^\vee G^\Gamma$ in ${}^\vee \mathrm{GL}_N^\Gamma$ determine a real structure on G as a split group. In the second case there are two distinct embeddings of ${}^\vee G^\Gamma$ in ${}^\vee \mathrm{GL}_N^\Gamma$ which determine either a split real structure on $G = \mathrm{SO}_N$, or a non-split and quasisplit structure on SO_N ([Art13]*Section 1.2). We henceforth work only with s as in (122)-(123), the so-called *simple* endoscopic data. As in the previous section, we will not require the concept of an E-group for these classical endoscopic groups. From now on we assume that ${}^\vee G^\Gamma$ is an L-group as in the end of Section 2.3.

Unlike the previous section, we must distinguish between maximal tori in ${}^\vee G$ and ${}^\vee \mathrm{GL}_N$. We let ${}^\vee H$ be the diagonal maximal torus in ${}^\vee \mathrm{GL}_N$, and ${}^\vee H_G$ be a maximal torus in ${}^\vee G$. In this case ${}^\vee H$ is preserved by $\mathrm{Int}(s)$ as well as ϑ. We may and shall take

$${}^\vee H_G = ({}^\vee H)^\sigma = ({}^\vee H)^\vartheta.$$

We fix $\lambda \in {}^\vee \mathfrak{h}$ satisfying (32) so that λ is regular with respect to ${}^\vee \mathrm{GL}_N$. Let ${}^\vee \mathcal{O}_G$ be the ${}^\vee G$-orbit of λ and ${}^\vee \mathcal{O}$ be the ${}^\vee \mathrm{GL}_N$-orbit of λ.

The ϵ maps of (112)-(113) have obvious analogues and are equally valid in the twisted setting. The proof of the injectivity of ϵ on the level of ${}^\vee G$-orbits is more delicate in the twisted setting.

Proposition 5.2. *In the setting of (32), suppose $S_1, S_2 \subset X({}^\vee \mathcal{O}_G, {}^\vee G^\Gamma)$ are ${}^\vee G$-orbits which are carried to the same ${}^\vee \mathrm{GL}_N$-orbit under ϵ. Then $S_1 = S_2$.*

Proof. By [ABV92]*Proposition 6.17, the ${}^\vee G$-orbits of $X({}^\vee \mathcal{O}_G, {}^\vee G^\Gamma)$ are in one-to-one correspondence with ${}^\vee G$-orbits of L-parameters with infinitesimal character in ${}^\vee \mathcal{O}_G$. It is simpler to prove the first assertion in the more familiar territory of L-parameters. To this end let $\phi_1 : W_\mathbb{R} \to {}^\vee G^\Gamma$ and $\phi_2 : W_\mathbb{R} \to {}^\vee G^\Gamma$ be L-parameters with infinitesimal characters $\lambda_1 \in {}^\vee \mathcal{O}_G$ and $\lambda_2 \in {}^\vee \mathcal{O}_G$ respectively. Suppose further that

$$\phi_1 = \mathrm{Int}(g) \circ \phi_2 \tag{124}$$

for some $g \in {}^\vee \mathrm{GL}_N$. We shall prove that there exists $g' \in {}^\vee G$ such that $\phi_1 = \mathrm{Int}(g') \circ \phi_2$. Without loss of generality we may assume that $\lambda_1 = \lambda \in {}^\vee \mathfrak{h}$ and $\lambda_2 = \mathrm{Ad}(g_1)\lambda \in {}^\vee \mathfrak{h}$ for some $g_1 \in {}^\vee G$ normalizing ${}^\vee H$. Moreover, we may assume that $\phi_1(\mathrm{j})$ and $\phi_2(\mathrm{j})$ normalize ${}^\vee H$. As in [ABV92]*Proposition 5.6,

$$\phi_1(z) = z^\lambda \bar{z}^{\mathrm{Ad}(\phi_1(\mathrm{j}))\lambda}, \quad \phi_2(z) = z^{\mathrm{Ad}(g_1)\lambda} \bar{z}^{\mathrm{Ad}(\phi_2(\mathrm{j})g_1)\lambda}, \quad z \in \mathbb{C}^\times.$$

By hypothesis

$$z^{\mathrm{Ad}(gg_1)\lambda} \bar{z}^{\mathrm{Ad}(g\phi_2(\mathrm{j})g_1)\lambda} = \mathrm{Int}(g) \circ \phi_2(z) = \phi_1(z) = z^\lambda \bar{z}^{\mathrm{Ad}(\phi_1(\mathrm{j}))\lambda}.$$

It follows that $\mathrm{Ad}(gg_1)\lambda = \lambda$. Since λ is regular in ${}^\vee \mathrm{GL}_N$, the element gg_1 belongs to ${}^\vee H$ and $g = hg_1^{-1}$ for some $h \in {}^\vee H$. Returning to the identity (124), we obtain

$$hg_1^{-1}\phi_2(\mathrm{j})g_1h^{-1} = \phi_1(\mathrm{j}).$$

Set $n_1 = \phi_1(\mathrm{j})$ and $n_2 = g_1^{-1}\phi_2(\mathrm{j})g_1$, so that the previous equation becomes

$$hn_2h^{-1} = n_1.$$

As ϵ maps into the fixed-point set of $\sigma = \mathrm{Int}(s)\circ\vartheta$, both n_1, n_2 are fixed by σ. Since n_1 and n_2 normalize ${}^\vee H$, they represent involutive elements in the fixed-point subgroup $W({}^\vee\mathrm{GL}_N, {}^\vee H)^{s\vartheta} = W({}^\vee\mathrm{GL}_N, {}^\vee H)^{\vartheta}$. We seek an element $h' \in {}^\vee H^{\vartheta} \subset {}^\vee G$ such that

$$h'n_2\,(h')^{-1} = hn_2h^{-1} = n_1. \tag{125}$$

With this element in hand we may set $g' = h'g_1^{-1}$ and the proposition is proved. Since ${}^\vee H = {}^\vee H^{\vartheta}\,{}^\vee H^{-\vartheta}$, we may decompose $h = h_1h_2$, where $h_1 \in {}^\vee H^{\vartheta}$ and $h_2 \in {}^\vee H^{-\vartheta}$. We compute that

$$(h_1n_2h_1^{-1}n_2^{-1})(h_2n_2h_2^{-1}n_2^{-1}) = h_1(h_2n_2h_2^{-1}n_2^{-1})(n_2h_1^{-1}n_2^{-1}) = n_1n_2^{-1} \in {}^\vee H^{s\vartheta} = {}^\vee H^{\vartheta}. \tag{126}$$

In addition, since $n_1, n_2 \in W({}^\vee\mathrm{GL}_N, {}^\vee H)^{\vartheta}$, we have

$$\vartheta(h_2n_2h_2^{-1}n_2^{-1}) = \vartheta(h_2)n_2\vartheta(h_2^{-1})n_2^{-1} = h_2^{-1}n_2h_2n_2^{-1} = (h_2n_2h_2^{-1}n_2^{-1})^{-1}.$$

Therefore $h_2n_2h_2^{-1}n_2^{-1} \in {}^\vee H^{-\vartheta}$. Similarly, $h_1n_2h_1^{-1}n_2^{-1} \in {}^\vee H^{\vartheta}$. By (126) we have $h_2n_2h_2^{-1}n_2^{-1} \in {}^\vee H^{-\vartheta} \cap {}^\vee H^{\vartheta}$. It is now an elementary exercise to show that there exists h_2' in the finite 2-group ${}^\vee H^{-\vartheta} \cap {}^\vee H^{\vartheta}$ such that

$$h_2'n_2(h_2')^{-1}n_2^{-1} = h_2n_2h_2^{-1}n_2^{-1}.$$

Indeed, conjugation by n_2 is represented by a ϑ-invariant product of 2-cycles when $W({}^\vee\mathrm{GL}_N, {}^\vee H)^{\vartheta}$ is regarded as a subgroup of the symmetric group, and the elements in ${}^\vee H^{-\vartheta} \cap {}^\vee H^{\vartheta}$ are represented by diagonal elements with ± 1 as entries. Leaving the details of the exercise to the reader, we set $h' = h_1h_2'$ and (125) holds.

□

At this point the picture of twisted endoscopy is more or less the same as the picture of standard endoscopy. The new idea in the twisted setting is to include the action of $\sigma = \mathrm{Int}(s)\circ\vartheta$ into the objects pertinent to endoscopy. In particular we wish to extend the sheaf theory of [ABV92] for ${}^\vee\mathrm{GL}_N$ to the disconnected group ${}^\vee\mathrm{GL}_N \rtimes \langle\sigma\rangle$, where we identify the automorphism σ of (54) with the automorphism in the endoscopic datum. This mimics the extension of the representation theory of GL_N to the disconnected group $\mathrm{GL}_N \rtimes \langle\vartheta\rangle$ in Section 2.5. Rather than viewing the sheaves in $\mathcal{C}({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^{\Gamma}; \sigma)$ as ${}^\vee\mathrm{GL}_N$-equivariant with compatible σ-action (Section 3.2), we view them simply as $({}^\vee\mathrm{GL}_N \rtimes \langle\sigma\rangle)$-equivariant sheaves and apply the theory of [ABV92] which is valid in this generality [CM18]*Section 5.4.

Let $\xi = (S, 1) \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^{\Gamma})^{\vartheta}$ and $(p, 1)$ be a representative for the class ξ. Here, $p \in S$ and 1 is the trivial representation of the trivial group ${}^\vee(\mathrm{GL}_N)_p/({}^\vee(\mathrm{GL}_N)_p)^0$ with representation space $V \cong \mathbb{C}$ as in (45). We define 1^+ on

$${}^\vee(\mathrm{GL}_N)_p/({}^\vee(\mathrm{GL}_N)_p)^0 \times \langle\sigma\rangle \tag{127}$$

by

$$1^+(\sigma) = \sigma_{\mu(\xi)^+} = 1 \tag{128}$$

(*cf.* (57)). In this way, 1^+ defines the local system underlying the irreducible $({}^\vee\mathrm{GL}_N \rtimes \langle\sigma\rangle)$-equivariant constructible sheaf $\mu(\xi)^+$ (Lemma 3.4, [ABV92]**p.* 83).

In a similar, but completely vacuous, fashion we may include the trivial action of σ on $\mu(\xi_1) \in \mathcal{C}({}^\vee\mathcal{O}_G, {}^\vee G)$ with $\xi_1 = (S_1, \tau_1)$ and $p_1 \in S_1$. In other words, we may regard $\mu(\xi_1)$ as a $({}^\vee G \times \langle\sigma\rangle)$-equivariant sheaf whose underlying local system is defined by a quasicharacter τ_1^+ on

$${}^\vee G_{p_1}/({}^\vee G_{p_1})^0 \times \langle\sigma\rangle \tag{129}$$

by $\tau_1^+(\sigma) = 1$. This artifice allows us to define a homomorphism, as in (114), by

$$A^{\mathrm{loc}}(\epsilon) : {}^\vee G_{p_1}/({}^\vee G_{p_1})^0 \times \langle\sigma\rangle \to ({}^\vee\mathrm{GL}_N)_{\epsilon(p_1)}/(({}^\vee\mathrm{GL}_N)_{\epsilon(p_1)})^0 \times \langle\sigma\rangle.$$

The inverse image functor (115) in the twisted setting is defined on $({}^\vee\mathrm{GL}_N \rtimes \langle \sigma \rangle)$-equivariant sheaves (Section 3.2)

$$\epsilon^* : KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma) \to KX({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$$

in the following manner. If the orbit S in $\xi = (S, 1) \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ is not the image of an orbit of $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ under ϵ then $\epsilon^*\mu(\xi) = 0$. Otherwise S is the ${}^\vee\mathrm{GL}_N$-orbit of $\epsilon(p_1) \in S$ for some $p_1 \in S_1 \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$. By Proposition 5.2 the orbit S_1 is unique. The stalk of the constructible sheaf $\epsilon^*\mu(\xi)^+$ at p_1 is the stalk $V \cong \mathbb{C}$ of (45) at $p = \epsilon(p_1)$. The representation on V is given by the quasicharacter $1^+ \circ A^{\mathrm{loc}}(\epsilon)$, which is again the (abusively denoted) trivial quasicharacter 1^+ on the group (129). In summary,

$$\epsilon^*\mu(\xi)^+ = \epsilon^*\mu(S, 1^+) = \mu(S_1, 1^+). \tag{130}$$

By contrast $\mu(\xi)^-$ is characterized by the quasicharacter 1^- of (127) with $1^-(\sigma) = -1$. With some obvious substitutions we obtain

$$\epsilon^*\mu(\xi)^- = \epsilon^*\mu(S, 1^-) = \mu(S_1, 1^-).$$

As in standard endoscopy, we combine ϵ^* with a pairing, namely the pairing of Theorem 3.5, to define

$$\epsilon_* : K_\mathbb{C}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}) \to K_\mathbb{C}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta).$$

To be precise, the image of any $\eta \in K_\mathbb{C}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R})$ under ϵ_* is determined by

$$\left\langle \epsilon_*\eta, \mu(\xi)^+ \right\rangle = \left\langle \eta, \epsilon^*\mu(\xi)^+ \right\rangle_G, \quad \xi \in \Xi({}^\vee\mathrm{GL}_N^\Gamma, {}^\vee\mathcal{O})^\vartheta. \tag{131}$$

(*cf.* (116)). The twisted endoscopic lifting map

$$\mathrm{Lift}_0 : K_\mathbb{C}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}} \to K_\mathbb{C}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \tag{132}$$

is the restriction of ϵ_* to $K_\mathbb{C}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$, a proper subspace of $K_\mathbb{C}({}^\vee\mathcal{O}_G, G/\mathbb{R})$ (*cf.* (117)). The pairing on the right-hand side of (131) is determined by pairing representations of $G(\mathbb{R}, \delta_q)$ with elements of the form $\mu(S_1, \tau_1^\pm)$. This is defined by

$$\left\langle M(S_1, \tau_1), \mu(S_1, \tau_1^\pm) \right\rangle_G = \pm 1, \quad \text{and} \quad \left\langle M(S_1', \tau_1'), \mu(S_1, \tau_1^\pm) \right\rangle_G = 0$$

when $\tau_1' \neq \tau_1$. (For a more conceptual explanation of these pairings see [CM18]* *p.* 151.)

Now, we wish to compute Lift_0 on the basis elements (119) of $K_\mathbb{C}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$. To maintain ease of comparison with [ABV92] we compute them on the virtual representations $\eta_{S_1}^{\mathrm{loc}}(\sigma)(\delta_q)$ ([ABV92]**p.* 279). These virtual characters are defined by

$$\eta_{S_1}^{\mathrm{loc}}(\sigma)(\delta_q) = \sum_{\tau_1} \mathrm{Tr}(\tau_1^+(\sigma))\, M(S_1, \tau_1) = \sum_{\tau_1} M(S_1, \tau_1),$$

where τ_1 runs over all quasicharacters of ${}^\vee G_{p_1}/({}^\vee G_{p_1})^0$ as in (129) which correspond to the strong involution δ_q ([ABV92]*Definition 18.9). It is immediate from the definitions that

$$\eta_{S_1}^{\mathrm{loc}}(\sigma)(\delta_q) = \eta_{S_1}^{\mathrm{loc}}(\delta_q)$$

and so this virtual character is stable ([ABV92]*Lemma 18.10). (Although not needed for our purposes, one could adhere to the framework of [ABV92] further by extending [ABV92]*Definitions 26.10 and 26.13 through taking products with $\langle \sigma \rangle$, and then speak of σ-stability.)

Proposition 5.3. *Suppose $S_1 \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ is a ${}^\vee G$-orbit which is carried to a ${}^\vee\mathrm{GL}_N$-orbit S under ϵ. Then the endoscopic lifting map (132) sends $\eta_{S_1}^{\mathrm{loc}}(\sigma)(\delta_q)$ to $(-1)^{l^I(S,1) - l^I_\vartheta(S,1)}\, M(S, 1)^+$.*

Proof. We prove the proposition without the injectivity of orbits given in Proposition 5.2. Not assuming injectivity, instead of equation (130), we see that

$$\epsilon^*\mu(S, 1)^+ = \sum_{S_1'} \mu(S_1', 1) = \mu(S_1, 1) + \sum_{S_1' \neq S_1} \mu(S_1', 1),$$

where S_1' runs over the ${}^\vee G$-orbits in $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ carried to S' under ϵ (*cf.* [ABV92]*Proposition 23.7 (b)). Therefore, according to (63), when $\xi = (S, 1)$

$$\begin{aligned}\left\langle \mathrm{Lift}_0\left(\eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q)\right), \mu(\xi)^+\right\rangle &= \left\langle \eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q),\, \epsilon^*\mu(S,1)^+\right\rangle_G \\ &= \left\langle \sum_{\tau_1} M(S_1,\tau_1), \mu(S_1,1)\right\rangle_G \\ &= 1\end{aligned}$$

Now suppose $\xi = (S', 1)$ where S' is a ${}^\vee\mathrm{GL}_N$-orbit not equal to S. Then

$$\epsilon^*\mu(S',1)^+ = \sum_{S_1'} \mu(S_1', 1)$$

where $S_1' \neq S_1$. If the sum runs over the empty set then we interpret it to equal zero and compute that

$$\left\langle \mathrm{Lift}_0\left(\eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q)\right), \mu(\xi)^+\right\rangle = \left\langle \eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q), \epsilon^*\mu(\xi)^+\right\rangle_G = \left\langle \eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q), 0\right\rangle_G = 0.$$

Otherwise the index of the sum is not empty and by (63)

$$\begin{aligned}\left\langle \mathrm{Lift}_0\left(\eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q)\right), \mu(\xi)^+\right\rangle &= \left\langle \eta^{\mathrm{loc}}_{S_1}(\sigma)(\delta_q),\, \epsilon^*\mu(\xi)^+\right\rangle_G \\ &= \left\langle \sum_{\tau_1} M(S_1,\tau_1), \sum_{S_1'} \mu(S_1',1)\right\rangle_G \\ &= 0\end{aligned}$$

Looking back to the definition of the pairing (62), we see that we have proven the proposition. □

Proposition 5.4. *Under the hypothesis of (32), the twisted endoscopic lifting map (132) is injective.*

Proof. The proof follows from Proposition 5.2 as in the proof of Proposition 5.1. We need only to observe that according to Proposition 5.3, Lift_0 sends the basis

$$\left\{\eta^{\mathrm{loc}}_{S_G}(\delta_q) : S_G \text{ a } {}^\vee G\text{-orbit of } X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)\right\}$$

of $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$ bijectively onto the linearly independent subset

$$\left\{\eta^{\mathrm{loc}}_{{}^\vee\mathrm{GL}_N\cdot S_G} : S_G \text{ a } {}^\vee G\text{-orbit of } X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)\right\}$$

of $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}))$. □

The next and final goal of this section is to provide the twisted analogue of the endoscopic lifting of the virtual characters attached to A-parameters as in (121). As a guiding principle, it helps to remember that in moving from η^{loc}_S to $\eta^{\mathrm{loc}}_S(\sigma)(\delta_q)$ we extended the component groups by $\langle\sigma\rangle$ to obtain (129), and then extended the quasicharacters τ_1 defined on the original component groups. We shall follow the same process with $\eta^{\mathrm{mic}}_{\psi_G}$, doing our best to avoid the theory of microlocal geometry.

The stable virtual character (48) for the endoscopic group G is

$$\eta^{\mathrm{mic}}_{\psi_G} = \sum_{\xi\in\Xi({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)} (-1)^{d(S_\xi)-d(S_{\psi_G})}\, \chi^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))\, \pi(\xi) \in K\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R})^{\mathrm{st}}.$$

Here, $S_{\psi_G} \subset X({}^\vee\mathcal{O}_G, G^\Gamma)$ is the ${}^\vee G$-orbit determined by the L-parameter ϕ_{ψ_G}, and $\xi = (S_\xi, \tau_{S_\xi})$. For each such ξ, there is a representation $\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))$ of ${}^\vee G_{\psi_G}/({}^\vee G_{\psi_G})^0$, the component group of the centralizer in ${}^\vee G$ of the image of ψ_G, which satisfies the following properties

- $\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))$ represents a (possibly zero) ${}^\vee G$-equivariant local system $Q^{\mathrm{mic}}(P(\xi))$ of complex vector spaces. (133)
- The degree of $\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))$ is equal to $\chi^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))$. (134)
- If $\xi = (S_{\psi_G}, \tau_{S_{\psi_G}})$ then $\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi)) = \tau_{S_{\psi_G}} \circ i_{S_{\psi_G}}$, where (135)
$$i_{S_{\psi_G}} : {}^\vee G_{\psi_G}/({}^\vee G_{\psi_G})^0 \to {}^\vee G_p/({}^\vee G_p)^0$$
is a surjective homomorphism for $p \in S_{\psi_G}$.

([ABV92]*Theorem 24.8, Corollary 24.9, Definition 24.15). By (134), we may rewrite $\eta^{\mathrm{mic}}_{\psi_G}$ as

$$\eta^{\mathrm{mic}}_{\psi_G} = \sum_{\xi \in \Xi({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)} (-1)^{d(S_\xi)-d(S_{\psi_G})} \operatorname{Tr}\left(\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))(1)\right) \pi(\xi). \tag{136}$$

Next, we extend ${}^\vee G_{\psi_G}/({}^\vee G_{\psi_G})^0$ trivially to

$${}^\vee G_{\psi_G}/({}^\vee G_{\psi_G})^0 \times \langle \sigma \rangle, \tag{137}$$

and extend $\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))$ trivially to (137) by defining $\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))(\sigma)$ to be the identity map. We define

$$\begin{aligned}
\eta^{\mathrm{mic}}_{\psi_G}(\sigma) &= \sum_{\xi \in \Xi({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)} (-1)^{d(S_\xi)-d(S_{\psi_G})} \operatorname{Tr}\left(\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))(\sigma)\right) \pi(\xi) \\
&= \sum_{\xi \in \Xi({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)} (-1)^{d(S_\xi)-d(S_{\psi_G})} \dim\left(\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))\right) \pi(\xi).
\end{aligned}$$

Clearly

$$\eta^{\mathrm{mic}}_{\psi_G}(\sigma) = \eta^{\mathrm{mic}}_{\psi_G}. \tag{138}$$

Finally, define

$$\begin{aligned}
\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q) &= \sum_{(S_\xi, \tau_{S_\xi})} (-1)^{d(S_\xi)-d(S_{\psi_G})} \operatorname{Tr}\left(\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))(\sigma)\right) \pi(\xi) \\
&= \sum_{(S_\xi, \tau_{S_\xi})} (-1)^{d(S_\xi)-d(S_{\psi_G})} \dim\left(\tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))\right) \pi(\xi)
\end{aligned} \tag{139}$$

in which the sum runs over only those $\xi = (S_\xi, \tau_{S_\xi}) \in \Xi({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ in which τ_{S_ξ} corresponds to the strong involution δ_q. Therefore

$$\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q) = \eta^{\mathrm{mic}}_{S_{\psi_G}}(\delta_q) = \eta^{\mathrm{ABV}}_{S_{\psi_G}}$$

(49). The virtual character $\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q)$ is a summand of the stable virtual character $\eta^{\mathrm{mic}}_{\psi_G}$ and is therefore also stable ([ABV92]*Theorem 18.7). Consequently, $\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q)$ lies in the domain of Lift_0. In addition, the ABV-packet $\Pi^{\mathrm{ABV}}_{\psi_G}$ consists of the irreducible characters in the support of $\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q)$ (50).

What we have done for $\eta^{\mathrm{mic}}_{\psi_G}$ we begin to do for $\eta^{\mathrm{mic}+}_{\psi}$, which we define as

$$\eta^{\mathrm{mic}+}_{\psi} = \sum_{\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee \mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{d(S_\xi)-d(S_\psi)} \operatorname{Tr}(\chi^{\mathrm{mic}}_{S_\psi}(P(\xi)))\,(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi)^+ \tag{140}$$

for

$$\psi = \epsilon \circ \psi_G. \tag{141}$$

The main difference now is that σ does not act trivially on ${}^\vee\mathrm{GL}_N$ and so the extensions require more attention. Properties (133)-(135) hold for ψ and GL_N as they do for ψ_G and G.

The first step is writing

$$\eta_{\psi}^{\mathrm{mic}+} = \sum_{\xi\in\Xi({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}} (-1)^{d(S_\xi)-d(S_\psi)}\ \mathrm{Tr}(\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))(1))\,(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi)^+.$$

This holds from (134) as (136) did for the endoscopic group G. What is new and simpler here is that the component group $({}^{\vee}\mathrm{GL}_N)_\psi/(({}^{\vee}\mathrm{GL}_N)_\psi)^0$ is trivial ([Art84]*Section 2.3). It follows that $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ is either the trivial representation or zero.

Let us digress briefly to examine property (135) for $\xi = (S_\psi, \tau_{S_\psi})$. Since the component group $({}^{\vee}\mathrm{GL}_N)_p/(({}^{\vee}\mathrm{GL}_N)_p)^0$ is trivial, the quasicharacter $\tau_{S_\psi} = 1$ is trivial. It follows that

$$\tau_{S_\psi}^{\mathrm{mic}}(P(S_\psi,\tau_{S_\psi})) = \tau_{S_\psi}\circ i_{S_\psi} = 1\circ i_{S_\psi} = 1 \neq 0. \tag{142}$$

In particular, $\pi(S_\psi, 1)$ is in the support of η_ψ^{mic} and belongs to Π_ψ^{ABV}. In the next section we will prove that this is the only representation in Π_ψ^{ABV}.

Returning to the matter of extensions, there is an obvious extension

$$({}^{\vee}\mathrm{GL}_N)_\psi/(({}^{\vee}\mathrm{GL}_N)_\psi)^0 \times \langle\sigma\rangle$$

of the trivial component group, as σ fixes the image of ψ. We wish to extend the representation $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ to this group for $\xi \in \Xi({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}$. The action of σ on $P(\xi) \in \mathcal{P}({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma};\sigma)$ determines an action on the stalks of the local system $Q^{\mathrm{mic}}(P(\xi))$ as in (133) ([ABV92]*(25.1)). [ABV92]*Proposition 26.23 (b) allows us to choose a stalk over a σ-fixed point p' (related to S_ψ) in the topological space of $Q^{\mathrm{mic}}(P(\xi))$. This places us in the same setting as Lemma 3.4, with τ_S replaced by $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ and S replaced by the ${}^{\vee}\mathrm{GL}_N$-orbit of p'. As a result, σ determines a canonical isomorphism of the stalk at p' equal to 1. In short, we define

$$\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)(\sigma) = 1 \tag{143}$$

and extend $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ to a quasicharacter $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)$. The quasicharacter $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)$ represents the $({}^{\vee}\mathrm{GL}_N\rtimes\langle\sigma\rangle)$-equivariant local system of the restriction of $Q^{\mathrm{mic}}(P(\xi))$ to the orbit of p'. We may extend i_{S_ψ} in (135) to include the products with $\langle\sigma\rangle$. Definitions (128) and (143) are compatible in that

$$\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))^+ = 1^+\circ i_{S_\psi}.$$

Finally, we define

$$\eta_{\psi}^{\mathrm{mic}+}(\sigma) = \sum_{\xi\in\Xi({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}} (-1)^{d(S_\xi)-d(S_\psi)}\ \mathrm{Tr}(\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)(\sigma))\,(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi)^+. \tag{144}$$

It is clear from definition (140) that $\eta_\psi^{\mathrm{mic}+}(\sigma) = \eta_\psi^{\mathrm{mic}+}$.

The obvious definition of the quasicharacter $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^-)$ is to take $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^-)(\sigma) = -1$. With this definition in place the following proposition is a consequence of [ABV92]*Corollary 24.9.

Proposition 5.5. *The functor $\tau_{S_\psi}^{\mathrm{mic}}(\cdot)$, from $({}^{\vee}\mathrm{GL}_N\rtimes\langle\sigma\rangle)$-equivariant perverse sheaves to representations of ${}^{\vee}G_\psi/({}^{\vee}G_\psi)^0\times\langle\sigma\rangle$, induces a map from the Grothendieck group $K(X({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma});\sigma)$ to the space of virtual representations. Furthermore the* microlocal trace *map*

$$\mathrm{Tr}\left(\tau_{S_\psi}^{\mathrm{mic}}(\cdot)(\sigma)\right)$$

induces a homomorphism from $K(X({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma}),\sigma)$ (as in (58)) to $\mathbb{C}$.

A similar statement is true for $\tau_{S_{\psi_G}}^{\mathrm{mic}}$ and the $({}^{\vee}G\times\langle\sigma\rangle)$-equivariant sheaves defined earlier.

Theorem 5.6. *(a) As a function on $K(X({}^{\vee}\mathcal{O},{}^{\vee}\mathrm{GL}_N^{\Gamma}),\sigma)$ we have*

$$\left\langle \eta_\psi^{\mathrm{mic}+}(\sigma),\cdot\right\rangle = (-1)^{d(S_\psi)}\,\mathrm{Tr}\left(\tau_{S_\psi}^{\mathrm{mic}}(\cdot)(\sigma)\right).$$

(b) The stable virtual character $\eta_{\psi}^{\mathrm{mic}+}(\sigma)$ *is equal to*

$$(-1)^{d(S_\psi)} \sum_{\xi\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} \mathrm{Tr}\left(\tau_{S_\psi}^{\mathrm{mic}}(\mu(\xi)^+)(\sigma)\right)(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\, M(\xi)^+.$$

(c) $\mathrm{Lift}_0\left(\eta_{\psi_G}^{\mathrm{mic}}(\sigma)(\delta_q)\right)=\eta_{\psi}^{\mathrm{mic}+}(\sigma)$.

Proof. The first two assertions follow from Theorem 3.5 and the computation

$$\left\langle \eta_{\psi}^{\mathrm{mic}+}(\sigma)(\delta_q), P(\xi)^+\right\rangle = (-1)^{d(S_\psi)}\,\mathrm{Tr}\left(\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)(\sigma)\right)$$

(*cf.* [ABV92]*Lemma 26.9).

For the final assertion, we compute

$$\begin{aligned}\left\langle \epsilon_*\eta_{\psi_G}^{\mathrm{mic}}(\sigma)(\delta_q), \mu(\xi)^+\right\rangle &= \left\langle \eta_{\psi_G}^{\mathrm{mic}}(\sigma)(\delta_q), \epsilon^*\mu(\xi)^+\right\rangle_G\\ &= \left\langle \eta_{\psi_G}^{\mathrm{mic}}(\sigma), \epsilon^*\mu(\xi)^+\right\rangle_G\\ &= (-1)^{d(S_{\psi_G})}\,\mathrm{Tr}\left(\tau_{S_{\psi_G}}^{\mathrm{mic}}(\epsilon^*\mu(\xi)^+)(\sigma)\right)\end{aligned}$$

using (130) and [ABV92]*Lemma 26.9 for $\eta_{\psi_G}^{\mathrm{mic}}(\sigma)$. By the deep result [ABV92]*Theorem 25.8, and the first assertion of the theorem, we may continue with

$$\begin{aligned}&= (-1)^{d(S_\psi)}\mathrm{Tr}\left(\tau_{S_\psi}^{\mathrm{mic}}(\mu(\xi)^+)(\sigma)\right)\\ &= \left\langle \eta_{\psi}^{\mathrm{mic}+}(\sigma), \mu(\xi)^+\right\rangle\end{aligned}$$

and the theorem is proven.

□

6 ABV-packets for general linear groups

In this section we prove that any ABV-packet for $\mathrm{GL}_N(\mathbb{R})$ consists of a single (equivalence class of an) irreducible representation. This implies that such an ABV-packet is equal to its corresponding L-packet ([ABV92]*Theorem 22.7 (a)). From the classification of the unitary dual of $\mathrm{GL}_N(\mathbb{R})$ we may deduce that the single representation in the packet is unitary.

In this section we let

$$\psi : W_{\mathbb{R}}\times \mathrm{SL}_2 \to \mathrm{GL}_N^\Gamma$$

be an arbitrary A-parameter for $\mathrm{GL}_N(\mathbb{R})$. The description of the ABV-packet Π_ψ^{ABV} will be achieved in three steps. First, we treat the case of an *irreducible* A-parameter. Second, we compute the ABV-packet for a Levi subgroup of GL_N, whose dual group contains the image of ψ minimally. The final result is obtained from the second step by considering the Levi subgroup as an endoscopic group of GL_N and applying the endoscopic lifting (121).

Following the description of [Art13]*Equation (1.4.1), all A-parameters ψ for $\mathrm{GL}_N(\mathbb{R})$ may be represented as formal direct sums of irreducible representations of $W_{\mathbb{R}}\times\mathrm{SL}_2$

$$\psi = \ell_1(\mu_1\boxtimes\nu_{n_1})\boxplus\cdots\boxplus\ell_r(\mu_r\boxtimes\nu_{n_r}). \tag{145}$$

Here, ν_{n_j} is the unique irreducible representation of SL_2 of dimension n_j, and μ_j is an irreducible representation of $W_{\mathbb{R}}$ with bounded image. The representations μ_j are of dimension one or two [Kna94]*Section 3. The parameter ψ in (145) is said to be *irreducible* if $r=1$ and $\ell_1=1$.

Proposition 6.1. *Suppose ψ is an irreducible A-parameter of* GL_N. *Then* Π_ψ^{ABV} *consists of a single unitary representation.*

Proof. We begin with the case of $\psi = \mu \boxtimes \nu_N$, in which μ is a one-dimensional representation of $W_{\mathbb{R}}$. Since ν_N is irreducible, the image of SL_2 under ψ contains a principally unipotent (*i.e.* regular and unipotent) element of GL_N. [Ara19]*Theorem 4.11 (d) (a generalization of [ABV92]*Theorem 27.18) therefore implies that Π_ψ^{ABV} consists of a single unitary character.

Let us now suppose that $\psi = \mu \boxtimes \nu_n$ is an A-parameter in which μ is a two-dimensional irreducible representation of $W_{\mathbb{R}}$ (*i.e.* $N = 2n$). The restriction of ψ to $\mathbb{C}^\times$ may be taken to have the form

$$\psi(z) = z^{\lambda_1} \bar{z}^{\lambda_2}, \quad z \in \mathbb{C}^\times \tag{146}$$

where λ_1, $\lambda_2 \in {}^\vee\mathfrak{h}$ are semisimple elements with $\exp(2\pi i(\lambda_1 - \lambda_2)) = 1$ (*cf.* [ABV92]*Proposition 5.6). Let $\mathcal{L}$ be the centralizer of $\psi(\mathbb{C}^\times)$ in ${}^\vee\mathrm{GL}_N$. Following [Ta7]*4.2.2, it is straightforward to verify the following technical conditions

AJ1. The identity component of the centre of the centralizer of $\psi(j)$ in $\mathcal{L}$ is contained in the centre of GL_N.

AJ2. $\psi(\mathrm{SL}_2)$ contains a principally unipotent element of $\mathcal{L}$.

AJ3. $\langle \lambda_1 + {}^\vee\rho_{\mathcal{L}}, \alpha \rangle \neq 0$ for all roots $\alpha \in R\left({}^\vee\mathrm{GL}_N, {}^\vee H\right)$.

These three conditions place ψ among the family of A-parameters studied by Adams and Johnson in [AJ87] [Art89]*Section 5. According to [Ara19]*Corollary 4.18,

$$\Pi_\psi^{\mathrm{ABV}} = \Pi_\psi^{\mathrm{AJ}},$$

where Π_ψ^{AJ} denotes the packet of cohomologically induced representations introduced in [AJ87]*Definition 2.11. The set Π_ψ^{AJ} is in bijection with a set of parabolic subgroups ([AMR18]*Section 8.2), which in this case reduces to a single parabolic subgroup (with Levi subgroup isomorphic to $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$). □

Let us go back to the case of a general A-parameter ψ as in Equation (145). Let

$$\psi = \boxplus_{i=1}^r \ell_i \psi_i, \quad \psi_i = \mu_i \boxtimes \nu_{n_i},$$

be its decomposition into irreducible A-parameters ψ_i. Let N_i be the dimension of ψ_i and define

$${}^\vee G = \prod_{i=1}^r \left({}^\vee G_i\right)^{\ell_i} \cong \prod_{i=1}^r (\mathrm{GL}_{N_i})^{\ell_i} \tag{147}$$

to be the obvious Levi subgroup of ${}^\vee\mathrm{GL}_N$ containing the image of ψ. Let ${}^\vee G^\Gamma = {}^\vee G \times \Gamma$, a subgroup of ${}^\vee\mathrm{GL}_N^\Gamma$. It is immediate that ψ factors through an A-parameter

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}_2 \xrightarrow{\psi_G} {}^\vee G^\Gamma \hookrightarrow {}^\vee\mathrm{GL}_N^\Gamma, \tag{148}$$

where $\psi_G = \times_{i=1}^r \ell_i \psi_{G,i}$ and each $\psi_{G,i}$ is an irreducible A-parameter of ${}^\vee G_i^\Gamma = {}^\vee G_i \times \Gamma$. The description of the ABV-packet corresponding to ψ_G is a fairly straightforward consequence of Proposition 6.1. We must only remind ourselves that the direct product of (147) translates into a tensor product of ABV-packets as it passes through the process defining the packets in Section 3.1.

Corollary 6.2. *The ABV-packet* $\Pi_{\psi_G}^{\mathrm{ABV}}$ *consists of a single unitary representation* $\pi(S_{\psi_G}, 1)$.

Proof. Let ${}^\vee\mathcal{O}_G \subset {}^\vee\mathfrak{g}$ be the ${}^\vee G$-orbit of the infinitesimal character determined by the L-parameter ϕ_{ψ_G}. This orbit has an obvious decomposition ${}^\vee\mathcal{O} = \prod_{i=1}^r ({}^\vee\mathcal{O}_i)^{\ell_i}$ into ${}^\vee G_i$-orbits, and the variety $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ of geometric parameters decomposes as

$$X\left({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma\right) = \prod_{i=1}^r \left(X\left({}^\vee\mathcal{O}_i, {}^\vee G_i^\Gamma\right)\right)^{\ell_i}.$$

As a consequence, all complete geometric parameters $\xi = (S, \tau_S) = (S, 1)$ in $\Xi\left({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma\right)$ decompose as

$$\xi = \left(\prod_{i=1}^{r}(S_i)^{\ell_i}, \bigotimes_{i=1}^{r}\tau_i^{\otimes \ell_i}\right),$$

for ${}^\vee G_i$-orbits $S_i \subset X\left({}^\vee\mathcal{O}_i, {}^\vee G_i^\Gamma\right)$, and trivial quasicharacters $\tau_i = 1$. Furthermore, the algebra of differential operators $\mathcal{D}_{X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)}$, its graded sheaf, and the conormal bundle $T^*_{{}^\vee H}(X\left({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma\right))$ all decompose as direct products. Irreducible $\mathcal{D}_{X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)}$-modules $D(\xi)$ ([ABV92]*(7.10)(e), (47)) are therefore tensor products

$$D(\xi) = \bigotimes_{i=1}^{r} D(\xi_i)^{\otimes \ell_i}, \quad \xi_i = (S_i, \tau_i),$$

of irreducible $\mathcal{D}_{X\left({}^\vee\mathcal{O}_i, {}^\vee G_i^\Gamma\right)}$-modules, and the same can be said about their corresponding irreducible graded modules. Consequently, the *singular support* $\mathrm{SS}(D(\xi))$ of the graded sheaf $\mathrm{gr}D(\xi)$ ([ABV92]*Definition 19.7) decomposes as

$$\mathrm{SS}(D(\xi)) = \prod_{i=1}^{r}(\mathrm{SS}(D(\xi_i)))^{\ell_i},$$

and in particular

$$T^*_{S_{\psi_G}}\left(X\left({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma\right)\right) \subset \mathrm{SS}(D(\xi))$$

if and only if

$$T^*_{S_{\psi_{G,i}}}\left(X\left({}^\vee\mathcal{O}_i, {}^\vee G_i^\Gamma\right)\right) \subset \mathrm{SS}(D(\xi_i))$$

for all $1 \leq i \leq r$. This is equivalent to saying

$$\chi^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi)) \neq 0 \quad \Longleftrightarrow \quad \chi^{\mathrm{mic}}_{S_{\psi_{G,i}}}(P(\xi_i)) \neq 0, \quad \forall\ 1 \leq i \leq r.$$

In other words, $\pi(\xi) \in \Pi^{\mathrm{ABV}}_{\psi_G}$ if and only if $\pi(\xi_i) \in \Pi^{\mathrm{ABV}}_{\psi_{G,i}}$ for all $1 \leq i \leq r$ (50). By Proposition 6.1, each ABV-packet $\Pi^{\mathrm{ABV}}_{\psi_{G,i}}$ consists of a single unitary representation. It is a consequence of (142) that each such unitary representation is of the form $\pi(S_{\psi_{G,i}}, 1)$. A look back to the definition of ABV-packets reveals that

$$\Pi^{\mathrm{ABV}}_{\psi_G} = \{\otimes_{i=1}^{r}\pi(S_{\psi_{G,i}}, 1)\} = \{\pi(S_{\psi_G}, 1)\}.$$

□

We are ready for the final step of describing the ABV-packets for $\mathrm{GL}_N(\mathbb{R})$.

Proposition 6.3. *Let ψ be an A-parameter for GL_N as in (145). Then the ABV-packet $\Pi^{\mathrm{ABV}}_{\psi}$ consists of a single unitary representation $\pi(S_\psi, 1)$.*

Proof. Define ${}^\vee G$ as in (147). Take $s \in Z({}^\vee G) \subset {}^\vee\mathrm{GL}_N$ to be as regular as possible so that its centralizer in ${}^\vee\mathrm{GL}_N$ is equal to ${}^\vee G$. Set ${}^\vee G^\Gamma = {}^\vee G \times \Gamma$, so that $(s, {}^\vee G^\Gamma)$ is an endoscopic datum (Section 5.1). Write $\psi_G : W_\mathbb{R} \times \mathrm{SL}_2 \to {}^\vee G^\Gamma$ for the A-parameter for G, satisfying $\psi = \epsilon \circ \psi_G$ ((112), (148)). According to (121), Corollary 6.2, and Proposition 5.1 (c)

$$\eta^{\mathrm{mic}}_{\psi} = \mathrm{Lift}_0\left(\eta^{\mathrm{mic}}_{\psi_G}\right) = \mathrm{Lift}_0\left(\pi(S_{\psi_G}, 1)\right) = \mathrm{ind}^{\mathrm{GL}_N(\mathbb{R})}_{G(\mathbb{R}, \delta_q)}\pi(S_{\psi_G}, 1).$$

The proposition now follows from the fact that parabolic induction for general linear groups takes irreducible unitary representations to irreducible unitary representations ([Tad09]*Proposition 2.1, Sections 4-5). □

Corollary 6.4. *The stable virtual character $\eta^{\mathrm{mic}+}_{\psi}(\sigma)$ defined in (144) is equal to $(-1)^{l^I(\xi) - l^I_\vartheta(\xi)}\pi(\xi)^+$, where $\xi = (S_\psi, 1)$. In particular,*

$$\mathrm{Lift}_0\left(\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q)\right) = (-1)^{l^I(\xi) - l^I_\vartheta(\xi)}\pi(\xi)^+$$

Proof. By Proposition 6.3, $\tau^{\mathrm{mic}}_{S_\psi}(P(\xi))$ is non-zero only for $\xi = (S_\psi, 1)$. By definition (143), $\tau^{\mathrm{mic}}_{S_\psi}(P(\xi)^+)(\sigma) = 1$ when $\xi = (S_\psi, 1)$ and is zero otherwise. The first assertion follows. The second assertion is a consequence of the first and Theorem 5.6. □

7 Whittaker extensions and their relationship to Atlas extensions

Thus far we have been working with preferred extensions of irreducible representations, from $\mathrm{GL}_N(\mathbb{R})$ to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. These are the Atlas extensions of (40). Arthur uses a different choice of canonical extension in [Art13], which we call the *Whittaker extension.* After reviewing the definition of Whittaker extensions, we will compute the sign giving the difference from the Atlas extensions. We conclude by rewriting the pairing of Theorem 3.5 using Whittaker extensions. Written in this manner, the pairing becomes simpler (Corollary 7.9).

The review of Whittaker extensions which we are about to give may be found in [Art13]*Section 2.2. We fix a unitary character χ on the upper-triangular unipotent subgroup $U(\mathbb{R}) \subset B(\mathbb{R})$ which satisfies $\chi \circ \vartheta = \vartheta$. In this manner (U, χ) is a ϑ-fixed Whittaker datum. We work under the hypothesis of (32) on an infinitesimal character $\lambda \in {}^\vee\mathfrak{h}$ and set ${}^\vee\mathcal{O}$ to be its ${}^\vee\mathrm{GL}_N$-orbit. Let $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ so that $\pi(\xi)$ is (an infinitesimal equivalence class of) an irreducible representation of $\mathrm{GL}_N(\mathbb{R})$. Here, and whenever we define Whittaker extensions, we must work with a *bona fide* admissible group representation in this equivalence class which we also denote $(\pi(\xi), V)$. If $\pi(\xi)$ is tempered then up to a scalar there is a unique Whittaker functional $\omega : V \to \mathbb{C}$ satisfying

$$\omega(\pi(\xi)(u)v) = \chi(u)\,\omega(v), \quad u \in U(\mathbb{R}), \tag{149}$$

for all smooth vectors $v \in V$. It follows that there is a unique operator $\mathcal{I}^\sim$ which intertwines $\pi(\xi) \circ \vartheta$ with $\pi(\xi)$ and also satisfies $\omega \circ \mathcal{I}^\sim = \omega$. We extend $\pi(\xi)$ to a representation $\pi(\xi)^\sim$ of $(\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$ by setting $\pi(\xi)^\sim(\vartheta) = \mathcal{I}^\sim$. We call this extension $\pi(\xi)^\sim$ the *Whittaker extension* of $\pi(\xi)$.

If $\pi(\xi)$ is not tempered then we express it as the Langlands quotient of a representation induced from an essentially tempered representation of a Levi subgroup. The ϑ-stability of $\pi(\xi)$ and the uniqueness statement in the Langlands classification together imply the ϑ-stability of the essentially tempered representation. The earlier argument for tempered representations has an obvious analogue for the essentially tempered representation of the Levi subgroup. We may argue as above to extend the essentially tempered representation to the semi-direct product of the Levi subgroup with $\langle\vartheta\rangle$. One then induces this extended representation to $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle$. The unique irreducible quotient of this representation is the canonical extension of $\pi(\xi)$, namely the *Whittaker extension* $\pi(\xi)^\sim$ of $\pi(\xi)$. If one omits the Langlands quotient in this argument then we obtain the Whittaker extension $M(\xi)^\sim$ of the standard representation $M(\xi)$.

We now turn to the question of how $\pi(\xi)^\sim$ differs from $\pi(\xi)^+$. The operators $\pi(\xi)^\sim(\vartheta)$ and $\pi(\xi)^+(\vartheta)$ are involutive, and both intertwine $\pi(\xi) \circ \vartheta$ with $\pi(\xi)$. Therefore they differ by a sign, *i.e.*

$$\pi(\xi)^\sim(\vartheta) = \pm\,\pi(\xi)^+(\vartheta). \tag{150}$$

Lemma 7.1. *Suppose $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ and $M(\xi)$ is a ϑ-stable principal series representation of $\mathrm{GL}_N(\mathbb{R})$. Then the Whittaker and Atlas extensions of $M(\xi)$ are equal,* i.e. $M(\xi)^\sim = M(\xi)^+$.

Proof. We abusively identify the equivalence class $M(\xi)$ with one of its representatives $M(\xi) = \mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0$. Define an operator $\mathcal{I}$ on functions f in the space of $M(\xi)$ by

$$\mathcal{I}f(g) = f(\vartheta(g)), \quad g \in \mathrm{GL}_N(\mathbb{R}).$$

It is easily verified that $\mathcal{I}$ intertwines $(\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0) \circ \vartheta$ with $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi_0 \circ \vartheta)$. By the ϑ-stability of $M(\xi)$ we have

$$\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0 \cong (\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0) \circ \vartheta \cong \mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi_0 \circ \vartheta).$$

The uniqueness statement in the Langlands quotient theorem may be applied to the equivalence above to conclude that $\pi_0 = \pi_0 \circ \vartheta$. Therefore $\mathcal{I}$ intertwines $M(\xi) \circ \vartheta$ with $M(\xi)$. We will prove the lemma by showing that $M(\xi)^\sim(\vartheta) = \mathcal{I} = M(\xi)^+(\vartheta)$.

Given a(ny) Whittaker functional ω satisfying (149), $M(\xi)^\sim(\vartheta)$ is defined by $\omega \circ M(\xi)^\sim(\vartheta) = \omega$. A convenient Whittaker functional to work with is

$$\omega(f) = \int_{U(\mathbb{R})} f(\dot{w}_0 u)\overline{\chi}(u)\,du \tag{151}$$

[Sha80]*Section 2. Here f is a smooth function in the space of $M(\xi)$ and $\dot{w}_0 = \tilde{J}$ is a representative of the long Weyl group element in $W(\mathrm{GL}_N, H)$. We compute

$$\begin{aligned}
\omega \circ \mathcal{I}(f) &= \int_{U(\mathbb{R})} \mathcal{I}f(\dot{w}_0 u)\overline{\chi}(u)\, du \\
&= \int_{U(\mathbb{R})} f(\vartheta(\dot{w}_0 u))\overline{\chi}(u)\, du \\
&= \int_{U(\mathbb{R})} f(\dot{w}_0 u)\overline{\chi}(\vartheta(u))\, du \\
&= \int_{U(\mathbb{R})} f(\dot{w}_0 u)\overline{\chi}(u)\, du \\
&= \omega(f).
\end{aligned}$$

This proves $\omega \circ \mathcal{I} = \omega$ so that $M(\xi)^{\sim}(\vartheta) = \mathcal{I}$.

To prove $M(\xi)^{+}(\vartheta) = \mathcal{I}$ we recall the definition of $M(\xi)^{+}$ as in (40). The complete geometric parameter ξ corresponds to an Atlas parameter (x, y) (Lemma 2.2), where x is the equivalence class of a strong involution

$$\delta = \exp(\pi i^{\vee}\rho)\dot{w}_0\delta_0 = \exp(\pi i^{\vee}\rho)\tilde{J}\delta_0$$

as in [AV15]*Proposition 3.2 (see the proof of Lemma 7.6). This strong involution negates every positive root in $R(B, H)$. It follows that the underlying $(\mathfrak{gl}_N, K)$-module of $M(\xi)^{+}$ is the representation [AV15]*(20), in which the Borel subalgebra $\mathfrak{b}$ is real. This implies that $M(\xi)^{+} = \mathrm{ind}_{B(\mathbb{R})\rtimes\langle\vartheta\rangle}^{\mathrm{GL}(N,\mathbb{R})\rtimes\langle\vartheta\rangle} \pi_0^{+}$, where $\pi_0^{+}(\vartheta) = 1$, since the value of the function z given in (39) is one. Suppose f is a function in the space of $\mathrm{ind}_{B(\mathbb{R})\rtimes\langle\vartheta\rangle}^{\mathrm{GL}(N,\mathbb{R})\rtimes\langle\vartheta\rangle} \pi_0^{+}$. We compute

$$\begin{aligned}
\left(M(\xi)^{+}(\vartheta) f\right)(g) &= f(g\vartheta) \\
&= f(\vartheta\, \vartheta(g)) \\
&= \pi_0^{+}(\vartheta)\, f(\vartheta(g)) \\
&= f(\vartheta(g)) \\
&= (\mathcal{I}f)(g).
\end{aligned}$$

This proves that $M(\xi)^{+}(\vartheta) = \mathcal{I}$. □

Our next goal is to make a link between the signs in (150) and the twisted multiplicity polynomials $m_r^{\vartheta}(\xi', \xi)$ appearing in (65). By Proposition 3.7 and (104), we have the alternative formulations

$$\begin{aligned}
m_r^{\vartheta}(\xi', \xi) &= (-1)^{l_{\vartheta}^{I}(\xi') - l_{\vartheta}^{I}(\xi)} c_g^{\vartheta}(\xi, \xi') \\
&= (-1)^{l^{I}(\xi') - l_{\vartheta}^{I}(\xi') + l^{I}(\xi) - l_{\vartheta}^{I}(\xi)}\ {}^{\vee}P^{\vartheta}({}^{\vee}\xi, {}^{\vee}\xi')(1).
\end{aligned} \tag{152}$$

If $\pi(\xi')$ is a subquotient of $M(\xi)$ occurring with multiplicity one then it is easily verified that $m_r^{\vartheta}(\xi', \xi) = 1$ if and only if $\pi(\xi')^{+}$ is a subquotient of $M(\xi)^{+}$. Similarly, $m_r^{\vartheta}(\xi', \xi) = -1$ if and only if $\pi(\xi')^{-}$ is a subquotient of $M(\xi)^{+}$. In this sense $m_r^{\vartheta}(\xi', \xi)$ is a signed multiplicity.

There is a special irreducible subquotient of $M(\xi)$ which is *generic*, *i.e.* admits a non-zero Whittaker functional as in (149).

Lemma 7.2. *Suppose $\xi \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}$. Then*

(a) (up to infinitesimal equivalence) there is a unique irreducible ϑ-stable generic representation $\pi(\xi_0) = M(\xi_0)$ which occurs in $M(\xi)$ as a subquotient with multiplicity one;

(b) (any representative in the class of) $\pi(\xi_0)$ embeds as a subrepresentation of (any representative in the class of) $M(\xi)$;

(c) (any representative in the class of) $\pi(\xi_0)^{\sim}$ embeds as a subrepresentation of (any representative in the class of) $M(\xi)^{\sim}$.

Proof. A result due to Vogan and Kostant states that every standard representation $M(\xi)$ contains a unique generic irreducible subquotient occurring with multiplicity one ([Kos78]*Theorems E and L, [Vog78]*Corollary 6.7). In the rest of the proof we write $\pi(\xi_0)$ for the actual generic representation (not the equivalence class) for some $\xi_0 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$. It is straightforward to verify that $\pi(\xi_0) \circ \vartheta$ satisfies (149), just as $\pi(\xi_0)$ does. Therefore $\pi(\xi_0) \circ \vartheta$ is the unique irreducible generic subquotient of $M(\xi) \circ \vartheta \cong M(\xi)$. By uniqueness, $\pi(\xi_0) \circ \vartheta \cong \pi(\xi_0)$ and so $\xi_0 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

The statements about $\pi(\xi_0)$ occurring as a subrepresentation of $M(\xi)$ and $\pi(\xi_0) = M(\xi_0)$ may be found in [Vog78]*Theorem 6.2 and [CS98]*Theorem 6.2.

For part (c) we consider the standard representation $M(\xi)$, which has a Whittaker functional ω ([Sha80]*Proposition 3.2). The functional ω restricts to a non-zero Whittaker functional on $\pi(\xi_0)$. By definition, $M(\xi)^\sim(\vartheta)$ is the intertwining operator which satisfies $\omega \circ M(\xi)^\sim(\vartheta) = \omega$. Restricting this equation to the subrepresentation $\pi(\xi_0)$ yields in turn that

$$\pi(\xi_0)^\sim(\vartheta) = M(\xi)^\sim(\vartheta)_{|\pi(\xi_0)} \text{ and } \pi(\xi_0)^\sim \hookrightarrow M(\xi)^\sim. \tag{153}$$

□

Lemma 7.2 tells us that the multiplicity of $\pi(\xi_0)^\sim$ in $M(\xi)^\sim$ is one. On the other hand $m_r^\vartheta(\xi_0, \xi)$ tells us about the "signed multiplicity" of $\pi(\xi_0)^+$ in $M(\xi)^+$. We investigate $m_r^\vartheta(\xi_0, \xi)$ further before comparing the two multiplicities.

Proposition 7.3. *Suppose $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ and $\pi(\xi_0)$ is the generic subrepresentation of $M(\xi)$ (Lemma 7.2). Then*

$$m_r^\vartheta(\xi_0, \xi) = (-1)^{l^I(\xi) - l^I_\vartheta(\xi) + l^I(\xi_0) - l^I_\vartheta(\xi_0)}. \tag{154}$$

Proof. We see from (152) that the proposition is equivalent to

$${}^\vee P^\vartheta({}^\vee\xi, {}^\vee\xi_0)(1) = 1.$$

This equation follows from Proposition 4.17 once we establish that ${}^\vee\xi_0$ is the unique maximal parameter in the block of $\pi({}^\vee\xi)$ in the (dual) Bruhat order. This is equivalent to establishing that ξ_0 is the unique minimal parameter in the block of $\pi(\xi_0)$ ((88), [Vog82]*Theorem 1.15). We use [ABV92]*Proposition 1.11 to convert the Bruhat order for the representations of $\mathrm{GL}_N(\mathbb{R})$ into a closure relation between ${}^\vee\mathrm{GL}_N$-orbits of $X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$. Moreover, this proposition implies that the minimality of $\xi_0 = (S_0, 1)$ is equivalent to the ${}^\vee\mathrm{GL}_N$-orbit $S_0 \subset X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ being maximal and therefore open. The uniqueness of the generic parameter follows from the fact that there is a unique open orbit in each component of $X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ (*cf.* [ABV92]*p. 19). □

The signed multiplicities $m_r^\vartheta(\xi', \xi)$ of Atlas extensions may be compared to the multiplicities of Whittaker extensions when there is a representation in the block for which the two extensions agree. This is the case when the block contains a ϑ-fixed principal series (Lemma 7.1). Under these circumstances, we obtain a formula for the signs in (150).

Proposition 7.4. *Suppose $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ and $\pi(\xi_0)$ is the generic representations of Lemma 7.2. If $\pi(\xi_0)^+$ occurs in the decomposition of a principal series representation $M(\xi_p)^+ \in K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ then*

$$M(\xi)^\sim(\vartheta) = (-1)^{l^I(\xi) - l^I_\vartheta(\xi)}\ M(\xi)^+(\vartheta)$$

and

$$\pi(\xi)^\sim(\vartheta) = (-1)^{l^I(\xi) - l^I_\vartheta(\xi)}\ \pi(\xi)^+(\vartheta).$$

Proof. Suppose that $\xi_p \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta$ is the complete geometric parameter of a ϑ-stable principal series representation $M(\xi_p)^+$ as in the hypothesis. It is straightforward to show $l^I(\xi_p) = l^I_\vartheta(\xi_p) = 0$. By Lemma 7.1, (65) and (154)

$$\begin{aligned} M(\xi_p)^\sim &= m_r^\vartheta(\xi_0, \xi_p)\, \pi(\xi_0)^+ \ + \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi_p)\, \pi(\xi')^+ \\ &= (-1)^{l^I(\xi_0) - l^I_\vartheta(\xi_0)}\, \pi(\xi_0)^+ \ + \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi_p)\, \pi(\xi')^+. \end{aligned}$$

According to (153), with $\xi = \xi_p$, and the observations immediately preceding Lemma 7.2, this equation implies

$$(-1)^{l^I(\xi_0)-l^I_\vartheta(\xi_0)}\pi(\xi_0)^+(\vartheta) = \pi(\xi_0)^\sim(\vartheta). \tag{155}$$

Thus, the proposition holds for $\xi = \xi_0$.

It remains to prove that the proposition holds when $\xi \neq \xi_0$. We compute, using (154) and (155), that

$$\begin{aligned} M(\xi)^+ &= \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi',\xi)\,\pi(\xi')^+ + m_r^\vartheta(\xi_0,\xi)\,\pi(\xi_0)^+ \\ &= \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi',\xi)\,\pi(\xi')^+ + (-1)^{l^I(\xi)-l^I_\vartheta(\xi)+l^I(\xi_0)-l^I_\vartheta(\xi_0)}\,\pi(\xi_0)^+ \\ &= \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi',\xi)\,\pi(\xi')^+ + (-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi_0)^\sim. \end{aligned}$$

This equation and the observations before Lemma 7.2 imply

$$(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi_0)^\sim(\vartheta) = M(\xi)^+(\vartheta)_{|\pi(\xi_0)}.$$

Combining this equation with (153), we see in turn that

$$(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}M(\xi)^+(\vartheta)_{|\pi(\xi_0)} = \pi(\xi_0)^\sim(\vartheta) = M(\xi)^\sim(\vartheta)_{|\pi(\xi_0)}$$

and $(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}M(\xi)^+(\vartheta) = M(\xi)^\sim(\vartheta)$. By taking Langlands quotients of the last equation we obtain $(-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi)^+(\vartheta) = \pi(\xi)^\sim(\vartheta)$. □

Proposition 7.4 describes the sign appearing in (150) explicitly. Unfortunately, the hypotheses of the Proposition do not always hold. It is not true that every generic representation $\pi(\xi_0)^+$ appears in the decomposition of a ϑ-stable principal series representation. This may already be seen for GL_2. It is instructive to examine and remedy this special case.

For every positive integer m let $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m$ be the principal series representation with

$$\pi_m = \begin{cases} |\cdot|^{(m-1)/2} \otimes |\cdot|^{(-m-1)/2}, & m \text{ even} \\ |\cdot|^{(m-1)/2} \otimes sgn(\cdot)|\cdot|^{-(m-1)/2}, & m \text{ odd} \end{cases} \tag{156}$$

Let D_m be the relative (limit of) discrete series representation which is the unique subrepresentation of $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m$.

For even m both D_m and $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m$ are ϑ-stable. This may be seen by computing that the linear map on the space of $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m$ defined by

$$f(x) \mapsto f(\vartheta(x)), \quad x \in \mathrm{O}(2), \tag{157}$$

intertwines $(\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m)\circ\vartheta$ with $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m$. Thus, for even m we have an obvious embedding of D_m into a ϑ-stable principal series representation exhibited by an explicit intertwining operator.

For odd m the map (157) intertwines $(\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m)\circ\vartheta$ with $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\pi_m\circ\vartheta)$, where

$$\pi_m \circ \vartheta = sgn(\cdot)|\cdot|^{(m-1)/2} \otimes |\cdot|^{-(m-1)/2}.$$

Unfortunately, for odd $m > 1$ the representation $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\pi_m\circ\vartheta)$ is not equivalent to $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\pi_m$. This may be deduced from the uniqueness statement in the Langlands Classification. Consequently, instead of embedding D_m into a ϑ-stable principal series representation, we must seek another means of finding an operator which intertwines $D_m\circ\vartheta$ with D_m. For this we look to facts about the discrete series of $\mathrm{SL}_2(\mathbb{R})$.

Lemma 7.5. *Suppose m is a positive integer. Then D_m is ϑ-stable and $D_m^\sim \cong D_m^+$.*

Proof. It is well-known that the restriction of D_m to $\mathrm{SL}_2(\mathbb{R})$ decomposes as a direct sum $D_{m+} \oplus D_{m-}$ of irreducible (limits of) discrete series representations ([Kna86]**pp.* 471-472). In this decomposition D_{m+} is the unique representation which is generic with respect to $(U \cap \mathrm{SL}_2(\mathbb{R}), \chi)$. Let ω_+ be its Whittaker functional. The representation D_m is equivalent to the tensor product of the trivial central character $1_{Z(\mathrm{GL}_2(\mathbb{R}))}$ with $\mathrm{ind}_{\mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2^{\pm}(\mathbb{R})} D_{m+}$. The direct sum $D_{m+} \oplus D_{m-}$ also occurs as the unique subrepresentation of the principal series representation $\mathrm{ind}_{B\cap \mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2(\mathbb{R})} \left((\pi_m)_{|H\cap \mathrm{SL}_2(\mathbb{R})}\right)$ ([Kna86]*(2.19)). It is easily verified that this principal series representation is ϑ-stable. Lemma 7.1 and Proposition 7.4 apply equally well for representations of $\mathrm{SL}_2(\mathbb{R})$ and so we conclude that D_{m+} is ϑ-stable with $D_{m+}^{\sim} \cong D_{m+}^{+}$. Let ω be the linear functional on the space of D_m obtained by composing ω_+ with the orthogonal projection P_+ onto the space of D_{m+}. We compute that for all $u \in U(\mathbb{R})$ and $f = (1-P_+)f + P_+ f$ in the space of D_m

$$\begin{aligned} \omega(D_m(u)f) &= \omega_+(P_+(D_{m-}(u)(1-P_+)f + D_{m+}(u)P_+f)) \\ &= \omega_+(D_{m+}(u)P_+f) \\ &= \chi(u)\,\omega_+(P_+f) \\ &= \chi(u)\,\omega(f). \end{aligned}$$

This proves that ω is a Whittaker functional for D_m and implies

$$D_m^{\sim} \cong 1_{Z(\mathrm{GL}_2(\mathbb{R}))} \otimes \mathrm{ind}_{\mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2^{\pm}(\mathbb{R})} D_{m+}^{\sim}.$$

The lemma now follows from

$$\begin{aligned} D_m^{\sim} &\cong 1_{Z(\mathrm{GL}_2(\mathbb{R}))} \otimes \mathrm{ind}_{\mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2^{\pm}(\mathbb{R})} D_{m+}^{\sim} \\ &\cong 1_{Z(\mathrm{GL}_2(\mathbb{R}))} \otimes \mathrm{ind}_{\mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2^{\pm}(\mathbb{R})} D_{m+}^{+} \\ &\cong D_m^{+} \end{aligned}$$

in which we appeal to [AV15]*(20) for the construction of D_m^+ and use [KV95]*Proposition 2.77 for the induction of finite index. □

In the following two lemmas we see that although a ϑ-stable irreducible generic representation need not be a subrepresentation of a ϑ-stable principal series representation, it is at worst a subrepresentation of a ϑ-stable standard representation which is essentially induced from D_m. We deal with the tempered representations first.

Lemma 7.6. *Suppose π_{gen} is a ϑ-stable irreducible tempered representation of $\mathrm{GL}_N(\mathbb{R})$ with integral infinitesimal character $\lambda \in {}^\vee\mathfrak{h}$ chosen as in (32)* i.e.

$$\vartheta(\lambda) = \lambda \quad \text{and} \quad \langle \lambda, {}^\vee\alpha \rangle \in \{1, 2, \cdots\},\ \alpha \in R^+(\mathrm{GL}_N, H).$$

(a) *Relative to the standard basis of the diagonal Lie algebra $\mathfrak{h}$, λ has coordinates of the form*

$$(\lambda_1, \ldots, \lambda_{N/2}, -\lambda_{N/2}, \ldots, -\lambda_1)$$

when N is even, and of the form

$$(\lambda_1, \ldots, \lambda_{(N-1)/2}, 0, -\lambda_{(N-1)/2}, \ldots, -\lambda_1)$$

when N is odd. The coordinates appear in strictly decreasing order.

(b) *Suppose N is odd. Then the coordinates λ_j are all integers, and π_{gen} embeds into a ϑ-stable principal series representation.*

(c) *Suppose N is even. Then the coordinates λ_j are either all integers or all half-integers (elements in $\mathbb{Z} + \frac{1}{2}$). In the latter case, π_{gen} embeds into a ϑ-stable principal series representation. In the former case, π_{gen} embeds into a ϑ-stable principal series representation when*

N is divisible by 4. When N is not divisible by 4 in the former case, π_{gen} embeds into the representation π which is parabolically induced from the representation

$$|\cdot|^{\lambda_1}\otimes\cdots\otimes|\cdot|^{\lambda_{\frac{N}{2}-1}}\otimes D_{2\lambda_{N/2}+1}\otimes|\cdot|^{-\lambda_{\frac{N}{2}-1}}\otimes\cdots\otimes|\cdot|^{-\lambda_1}. \tag{158}$$

The representation π is ϑ-stable and $\pi^{\sim}=\pi^{+}$.

Proof. The first assertion is an immediate consequence of

$$\lambda=\vartheta(\lambda)=-\mathrm{Ad}(\tilde{J})(\lambda).$$

Let $\alpha_1,\ldots,\alpha_{N-1}$ be the simple roots of GL_N relative to the Borel subgroup B.

Suppose first that N is odd. Then

$$\langle\lambda,{}^{\vee}\alpha_{(N-1)/2}\rangle=\lambda_{(N-1)/2}-0$$

is a positive integer by the integrality hypothesis. The remaining λ_j for $j<(N-1)/2$ are then seen to be positive integers by applying the integrality hypothesis to $\langle\lambda,{}^{\vee}\alpha_j\rangle$ successively.

Our strategy in showing that π_{gen} embeds into a ϑ-stable principal series representation is to express π_{gen} in terms of its Atlas parameter (Corollary 2.4) and to apply Cayley transforms and cross actions to its Atlas parameter [AV15]*Section 7. The resulting parameters correspond to ϑ-stable standard representations in the block of π_{gen} ([Vog82]*Definition 1.14, [Vog82]*Theorem 8.8, [AV15]*Section 7). For $\mathrm{GL}_N(\mathbb{R})$, π_{gen} is the unique irreducible generic representation in its block (Lemma 7.2). Furthermore, every standard representation in the block contains the irreducible generic representation π_{gen}, as its unique irreducible subrepresentation ([Vog78]*Theorem 6.2, [Vog78]*Corollary 6.7, [CS98]*Theorem 6.2). In consequence, it suffices to find Cayley transforms and cross actions which carry the Atlas parameter of π_{gen} to a ϑ-stable principal series representation.

By Corollary 2.4, the representation π_{gen} corresponds to an element $w\in W(\mathrm{GL}_N,H)$ satisfying $w\delta_0(w)=(ww_0)^2=1$ ((33), (35)). Such Weyl group elements are called *δ_0-twisted involutions* [AV15]*Section 3. We begin by determining the δ_0-twisted involution $w\in W(\mathrm{GL}_N,H)$ attached to π_{gen}. The δ_0-twisted involution $w\in W(\mathrm{GL}_N,H)$ determines an involutive automorphism $w\delta_0$ on H ([AV15]*(14e)). This automorphism also acts on ${}^{\vee}\mathfrak{h}$, and the integral length (60) of π_{gen} is equal to

$$-\frac{1}{2}\left(|\{\alpha\in R^{+}(\mathrm{GL}_N,H):w\delta_0\cdot\alpha\in R^{+}(\mathrm{GL}_n,H)\}|+\dim(H^{w\delta_0})\right). \tag{159}$$

By [Vog82]*Lemma 12.10 and the arguments of the proof of Proposition 7.3, the length of π_{gen} is minimal among the lengths of all representations in its block. It is not difficult to see that (159) is minimized at $w=1$. Moreover, there exists a representation in the block of π_{gen} corresponding to $w=1$ ([Vog82]*Theorem 8.8 and [AdC09]*Section 14). It follows that π_{gen} is in fact this representation and that the δ_0-twisted involution w for π_{gen} is trivial.

According to [Car72]*Lemma 5, there are orthogonal positive roots, $\beta_1,\ldots\beta_m$ such that

$$s_{\beta_1}\cdots s_{\beta_m}=ww_0=w_0.$$

From this we compute $\beta_1=\alpha_1+\cdots+\alpha_{N-1}$, $\beta_2=\alpha_2+\cdots+\alpha_{N-2}$, $\ldots$, $\beta_{(N-1)/2}=\alpha_{(N-1)/2}+\alpha_{(N+1)/2}$ and $m=(N-1)/2$. By Corollary 2.4 there is an element $y\in{}^{\vee}\mathcal{X}_{\lambda}^{s_{\beta_1}\cdots s_{\beta_m}}={}^{\vee}\mathcal{X}_{\lambda}^{w_0}$ and an element $x\in\mathcal{X}_{\rho^{\vee}}^{1}$ such that $J(x,y,\lambda)=\pi_{\mathrm{gen}}$.

Before listing Cayley transforms and cross actions to apply to the parameter (x,y), it is worthwhile to describe the δ_0-twisted involution $w'\in W(\mathrm{GL}_N,H)$ attached to a principal series representation. A principal series representation is parabolically induced from a real Borel subgroup. In the Atlas parameterization this is equivalent to the automorphism $w'\delta_0$ carrying all positive roots of $R(\mathrm{GL}_N,H)$ to negative roots, making B a real Borel subgroup ([KV95]*Proposition 4.76). Since δ_0 preserves the set of positive roots, this forces $w'=w_0$, the long Weyl group element.

We wish to apply Cayley transforms and cross actions to the parameter $(x,y)\in\mathcal{X}_{\rho^{\vee}}^{1}\times{}^{\vee}\mathcal{X}_{\lambda}^{w_0}$ in order to arrive at a parameter $(x',y')\in\mathcal{X}_{\rho^{\vee}}^{w_0}\times{}^{\vee}\mathcal{X}_{\lambda}^{1}$ and a ϑ-stable representation $\pi(x',y')=J(x',y',\lambda)$ (Corollary 2.4). Recall from Section 4.5 that Cayley transforms and cross actions are made relative to ϑ-orbits of simple roots κ (84). A Cayley transform c_κ applied to a parameter in $\mathcal{X}^w_{\rho^\vee} \times {}^\vee\mathcal{X}^{ww_0}_\lambda$ produces parameters in $\mathcal{X}^{w_\kappa w}_{\rho^\vee} \times {}^\vee\mathcal{X}^{w_\kappa ww_0}_\lambda$ ([AdC09]*Definition 14.1), where w_κ is prescribed in (85). A cross action $\kappa\times$ applied to a parameter in $\mathcal{X}^w_{\rho^\vee} \times {}^\vee\mathcal{X}^{ww_0}_\lambda$ results in a parameter in $\mathcal{X}^{w_\kappa ww_\kappa^{-1}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{w_\kappa ww_\kappa^{-1}w_0}_\lambda$ ([AdC09]*(9.11f)). In short, Cayley transforms left-multiply a δ_0-twisted involution w by w_κ, and cross actions conjugate w by w_κ. We wish to move from the δ_0-twisted involution 1 to the ξ_0-twisted involution $w_0 = s_{\beta_1}\cdots s_{\beta_{(N-1)/2}}$ using these operations.

We first describe how to move from 1 to s_{β_1}. We define the symbol $\rightsquigarrow$ to mean "takes a parameter in the set on the left to parameters in the set on the right". Let κ_j be the ϑ-orbit of α_j. It is straightforward to verify (using the *Atlas of Lie Groups and Representations* software)

$$\begin{aligned}
\mathcal{X}^1_{\rho^\vee} \times {}^\vee\mathcal{X}^{w_0}_\lambda &\overset{c_{\kappa_{(N-1)/2}}}{\rightsquigarrow} \mathcal{X}^{s_{\beta_m}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\beta_m}w_0}_\lambda \\
&\overset{\kappa_{(N-3)/2}\times}{\rightsquigarrow} \mathcal{X}^{s_{\beta_{m-1}}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\beta_{m-1}}w_0}_\lambda \\
&\overset{\kappa_{(N-5)/2}\times}{\rightsquigarrow} \mathcal{X}^{s_{\beta_{m-2}}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\beta_{m-2}}w_0}_\lambda \\
&\vdots \\
&\overset{\kappa_1\times}{\rightsquigarrow} \mathcal{X}^{s_{\beta_1}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\beta_1}w_0}_\lambda.
\end{aligned} \tag{160}$$

To move from s_{β_1} to $s_{\beta_1}s_{\beta_2}$ we repeat this procedure, but terminating with $\overset{\kappa_2\times}{\rightsquigarrow}$. Repeating this procedure in the obvious fashion, we arrive to $w_0 = s_{\beta_1}\cdots s_{\beta_{(N-1)/2}}$, as desired. This proves part (b).

We now continue under the assumption that N is even. By the integrality hypothesis

$$\langle \lambda, {}^\vee\alpha_{N/2}\rangle = \lambda_{N/2} - (-\lambda_{N/2}) = 2\lambda_{N/2}$$

is a positive integer. Therefore $\lambda_{N/2} \in \frac{1}{2}\mathbb{Z}$. If $\lambda_{N/2} \in \mathbb{Z}+\frac{1}{2}$ then the integrality of $\langle \lambda, {}^\vee\alpha_{(N/2)-1}\rangle = \lambda_{(N/2)-1} - \lambda_{N/2}$ implies $\lambda_{(N/2)-1} \in \mathbb{Z}+\frac{1}{2}$. Similar computations with the remaining simple roots $\alpha_{(N/2)-1},\ldots,\alpha_1$ then imply that all λ_j are half-integers. If $\lambda_{N/2}$ is an integer then the same argument proves that all λ_j are integers.

The approach to embedding π_{gen} into standard representations for even N is the same as in the odd case. In particular, $\pi_{\mathrm{gen}} = J(x,y,\lambda)$ where $(x,y) \in \mathcal{X}^1_{\rho^\vee} \times {}^\vee\mathcal{X}^{w_0}_\lambda$. We wish to apply Cayley transforms and cross actions to (x,y) in order to arrive to standard representations of the desired form. There are three cases to consider.

In the case that all coordinates of λ are half-integers the Cayley transform $c_{\kappa_{N/2}} = c_{\alpha_{N/2}}$ may be used to replace $c_{\kappa_{(N-1)/2}}$ in (160) in order to obtain the same conclusion.

However, in the case that all coefficients of λ are integers the Cayley transform $c_{\alpha_{N/2}}$ does not yield parameters which are ϑ-stable and must therefore be ignored ($\alpha_{N/2}$ is a root of type `1i2s` in [AV15]*Tables 1-2). In the special case of $\mathrm{GL}_4(\mathbb{R})$ one may circumvent this obstacle as follows

$$\begin{aligned}
\mathcal{X}^1_{\rho^\vee} \times {}^\vee\mathcal{X}^{w_0}_\lambda &\overset{c_{\kappa_1}}{\rightsquigarrow} \mathcal{X}^{s_{\alpha_1}s_{\alpha_3}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\alpha_1}s_{\alpha_3}w_0}_\lambda \\
&\overset{\kappa_2\times}{\rightsquigarrow} \mathcal{X}^{s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}w_0}_\lambda \\
&\overset{c_{\kappa_1}}{\rightsquigarrow} \mathcal{X}^{s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}w_0}_\lambda \\
&= \mathcal{X}^{s_{\beta_1}s_{\beta_2}}_{\rho^\vee} \times {}^\vee\mathcal{X}^1_\lambda.
\end{aligned} \tag{161}$$

More generally, if N is divisible by four one may replace the first step of (160) with the appropriate analogue of (161) and then continue by performing cross actions as in (160) to arrive at a parameter in $\mathcal{X}^{s_{\beta_1}s_{\beta_2}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\beta_1}s_{\beta_2}w_0}_\lambda$. Iterating this process, one arrives at a parameter in $\mathcal{X}^{w_0}_{\rho^\vee} \times {}^\vee\mathcal{X}^1_\lambda$ which corresponds to a ϑ-stable principal series representation.

In the last case where the coordinates of λ are integers and N has remainder two when divided by four one may only iterate the process just described to arrive at a parameter $(x',y') \in \mathcal{X}^{s_{\beta_1}\cdots s_{\beta_{(N/2)-1}}}_{\rho^\vee} \times {}^\vee\mathcal{X}^{s_{\beta_{N/2}}}_\lambda$. The involution corresponding to this parameter acts on roots by

$$s_{\beta_1}\cdots s_{\beta_{(N/2)-1}}\delta_0 = -s_{\alpha_{N/2}}. \tag{162}$$

In the Atlas parameterization this implies that $\alpha_{N/2}$ is an imaginary root and all simple roots orthogonal to $\alpha_{N/2}$ are real. Let us again identify the ϑ-stable representation

$$\pi = \pi(x', y') = J(x', y', \lambda)$$

with its underlying $(\mathfrak{gl}_N, K)$-module ([AV15]*(20)). In the language of [KV95]*Section 11 this module is the unique irreducible quotient of

$$M(x', y') = {}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}).$$

Here, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is the upper-triangular Borel subalgebra, $K = K_\delta$,

$$\delta = \exp(\pi i^\vee\rho)\sigma_{\alpha_1+\cdots+\alpha_{N-1}} \cdots \sigma_{\alpha_{\frac{N}{2}-1}+\alpha_{\frac{N}{2}+1}}\delta_0$$

([AV15]*Proposition 3.2), and $Z(\mathfrak{b}) = \mathbb{C}(y', \lambda) \otimes \wedge^{\mathrm{top}}(\mathfrak{n})$ as in [AV15]*(20). By [KV95]*Corollary 11.86, we may write

$$ {}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}) = {}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{p}, M\cap K}\, {}^u\mathcal{R}^{\mathfrak{m}, K\cap M}_{\mathfrak{b}\cap\mathfrak{m}, H\cap K} Z(\mathfrak{b}) = {}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{p}, M\cap K}\pi_M \tag{163}$$

where $\mathfrak{p} = \mathfrak{m}\oplus\mathfrak{u} \supset \mathfrak{b}$ is the parabolic subalgebra corresponding to $\alpha_{N/2}$, and $\pi_M = {}^u\mathcal{R}^{\mathfrak{m}, K\cap M}_{\mathfrak{b}\cap\mathfrak{m}, H\cap K} Z(\mathfrak{b})$. The induction functor ${}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{p}, M\cap K}$ on the right may be identified with parabolic induction ([KV95]*Proposition 11.57) and π_M is the underlying module of (158) ([KV95]*Theorem 11.178).

It remains to prove that the Whittaker extension of π equals its Atlas extension. Using the formula for the Whittaker functional of a parabolically induced representation ([Sha81]*Proposition 3.2), we have

$$({}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}))^\sim = ({}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{p}, (M\cap K)}\pi_M)^\sim = {}^u\mathcal{R}^{\mathfrak{gl}_N, K\rtimes\langle\vartheta\rangle}_{\mathfrak{p}, (M\cap K)\rtimes\langle\vartheta\rangle}\pi_M^\sim.$$

By definition of the Atlas extension (40)

$$({}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}))^+ = {}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} (Z(\mathfrak{b})^+)$$

and so using induction by stages as in (163) we have

$$({}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}))^+ = {}^u\mathcal{R}^{\mathfrak{gl}_N, K\rtimes\langle\vartheta\rangle}_{\mathfrak{p}, (M\cap K)\rtimes\langle\vartheta\rangle}\, {}^u\mathcal{R}^{\mathfrak{m}, (K\cap M)\rtimes\langle\vartheta\rangle}_{\mathfrak{b}\cap\mathfrak{m}, (H\cap K)\rtimes\langle\vartheta\rangle} Z(\mathfrak{b})^+ = {}^u\mathcal{R}^{\mathfrak{gl}_N, K\rtimes\langle\vartheta\rangle}_{\mathfrak{p}, (M\cap K)\rtimes\langle\vartheta\rangle}\pi_M^+. \tag{164}$$

Therefore, if $\pi_M^+ = \pi_M^\sim$ then it follows that

$$({}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}))^+ = ({}^u\mathcal{R}^{\mathfrak{gl}_N, K}_{\mathfrak{b}, H\cap K} Z(\mathfrak{b}))^\sim,$$

i.e. the Atlas extension and Whittaker extensions are equal.

For the proof of $\pi_M^+ = \pi_M^\sim$ we may assume without loss of generality that $M = \mathrm{GL}_2$ and $\pi_M = D_m$, and appeal to Lemma 7.5. □

The next lemma is a generalization of the previous one to include generic representations with non-integral infinitesimal characters.

Lemma 7.7. *Suppose π_{gen} is a ϑ-stable irreducible generic representation of $\mathrm{GL}_N(\mathbb{R})$ with infinitesimal character satisfying (32). Then π_{gen} embeds into a ϑ-stable standard representation $M(\xi_p)$, $\xi_p \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ such that $M(\xi_p)^\sim = M(\xi_p)^+$.*

Proof. According to [Vog78]*Theorem 6.2, π_{gen} is infinitesimally equivalent to a parabolically induced representation $\mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi' \otimes e^\nu)$. Here, $P(\mathbb{R})$ is a cuspidal standard parabolic subgroup whose Levi subgroup $M(\mathbb{R})$ has Langlands decomposition $M(\mathbb{R}) = M^1 A$ ([Kna86]*Section V.5), π' is a (limit of) discrete series representation of M^1, and ν lies in the complex Lie algebra $\mathfrak{a}$ of A. Since $P(\mathbb{R})$ is standard and cuspidal the Levi subgroup $M(\mathbb{R})$ decomposes diagonally into blocks

$$M(\mathbb{R}) = M_1(\mathbb{R}) \times \cdots \times M_\ell(\mathbb{R})$$

in which each block $M_j(\mathbb{R})$ is isomorphic to either $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_1(\mathbb{R})$. Accordingly,

$$M^1 = M_1^1 \times \cdots \times M_\ell^1, \tag{165}$$

where M_j' is isomorphic to $\mathrm{SL}_2^{\pm}(\mathbb{R})$ or $\{\pm 1\}$; and

$$\pi' = \pi_1' \otimes \cdots \otimes \pi_\ell',$$

where π_j' is equivalent to $D_{m_j}' := (D_{m_j})_{|\mathrm{SL}_2^{\pm}(\mathbb{R})}$ (*cf.* (156)) when $M_j' \cong \mathrm{SL}_2^{\pm}(\mathbb{R})$, and is equivalent to 1 or *sgn* when $M_j^1 \cong \{\pm 1\}$. In addition,

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_\ell \tag{166}$$

and $\nu = \nu_1 + \cdots + \nu_\ell$. One should expect that the ϑ-stability of $\pi_{\mathrm{gen}} \cong \mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi' \otimes e^{\nu})$ would put constraints on the constituent representations $\pi_j' \otimes e^{\nu_j}$ of $M_j(\mathbb{R})$. This is indeed so, and we now prove that the ϑ-stability yields a partition of the set

$$\{\pi_1 \otimes e^{\nu_1}, \ldots, \pi_\ell \otimes e^{\nu_\ell}\} \tag{167}$$

into either pairs of the form $\{D_{m_j}' \otimes e^{\nu_j}, D_{m_j}' \otimes e^{-\nu_j}\}$ or singletons of the form $\{D_{m_j}' \otimes e^{0}\} = \{D_{m_j}\}$.

The ϑ-stability implies that the distribution character of π_{gen} is equal to the distribution character of

$$\pi_{\mathrm{gen}} \circ \vartheta \cong \left(\mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi' \otimes e^{\nu})\right) \circ \vartheta \cong \mathrm{ind}_{\vartheta(P(\mathbb{R}))}^{\mathrm{GL}_N(\mathbb{R})}\left(\pi' \circ \vartheta \otimes e^{\vartheta \cdot \nu}\right).$$

By the Langlands Disjointness Theorem ([Lan89]*pp.* 149-150, *cf.* [Kna86]*Theorem 14.90), there exists $g \in \mathrm{O}(N)$ such that

$$\begin{aligned} \mathrm{Int}(g) \circ \vartheta(M^1) = M^1 \quad &, \quad \mathrm{Int}(g) \circ \vartheta(A) = A, \\ \pi' \circ \vartheta \circ \mathrm{Int}(g^{-1}) \cong \pi' \quad &, \quad \text{and} \quad \vartheta \cdot (\mathrm{Ad}(g^{-1})\nu) = \nu. \end{aligned} \tag{168}$$

Recall that ϑ is the composition of $\mathrm{Int}(\tilde{J})$ and inverse-transpose. The inverse-transpose automorphism stabilizes M^1 and A. Since inverse-transpose acts on $\mathrm{SL}_2(\mathbb{R})$ as the inner automorphism $\mathrm{Int}\left(\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\right)$, and acts trivially on $\{\pm 1\}$, it is easy to see that inverse-transpose stabilizes π'. The value of the differential of inverse-transpose at ν is $-\nu$. Taking these facts into consideration, we may read (168) as

$$\begin{aligned} \mathrm{Int}(g_1)(M^1) = M^1 \quad &, \quad \mathrm{Int}(g_1)(A) = A, \\ \pi' \circ \mathrm{Int}(g_1^{-1}) \cong \pi' \quad &, \quad \text{and} \quad \mathrm{Ad}(g_1^{-1})\nu = -\nu, \end{aligned}$$

where $g_1 = g\tilde{J} \in \mathrm{O}(N)$. After possibly multiplying by an element in $M^1 \cap \mathrm{O}(N)$, we may assume that $\mathrm{Int}(g_1)$ fixes a representative $\lambda' \in {}^\vee\mathfrak{m}$ of the infinitesimal character of π'. The infinitesimal character of π' decomposes as

$$\lambda' = \lambda_1' + \cdots + \lambda_\ell',$$

where $\lambda_j' \in {}^\vee\mathfrak{m}_j$. If $M_j = \mathrm{GL}_2$ then λ_j' determines $\pi_j' = D_{m_j}'$ up to equivalence. Since g_1 normalizes M^1 and fixes λ', $\mathrm{Int}(g_1)(M_j^1) = M_k^1$ for some $1 \leq k \leq \ell$, and $\mathrm{Ad}(g_1)(\lambda_j') = \lambda_k'$. In particular, if $M_j \cong \mathrm{GL}_2$ then $\pi_j' \cong \pi_k' \cong D_{m_j}'$. If $M_j \cong \mathrm{GL}_1$ then $\mathrm{Int}(g)_{|M_j^1}$ is the unique isomorphism of $M_j^1 \cong \{\pm 1\}$ onto $M_k^1 \cong \{\pm 1\}$ and so $\pi_j' = \pi_k'$. If $k = j$ in either of these two cases then $-\nu_j = \mathrm{Ad}(g_1^{-1})(\nu_j) = \nu_j$ and $\nu_j = 0$. In this manner, the element g_1 specifies the singletons for the partition of (167). The pairs in the partition of (167) become evident once we establish that g_1 acts involutively on the factors of (165) and (166). For this, we observe that $\mathrm{Ad}(g_1^2)$ fixes both λ' and ν and so fixes a representative of the infinitesimal character of π_{gen} ([Kna86]*Proposition 8.22). As we are assuming that the infinitesimal character is regular, g_1^2 belongs to the Cartan subgroup determined by λ' and ν, which is a subgroup of $M(\mathbb{R})$. Thus, $\mathrm{Int}(g_1^2)(M_j^1) = M_j^1$, $\mathrm{Ad}(g_1^2)\mathfrak{a}_j = \mathfrak{a}_j$, which proves the desired involutive action of g_1.

The distribution character, and therefore the infinitesimal equivalence class, of the irreducible representation $\mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi' \otimes e^{\nu})$ is independent of the choice of parabolic subgroup and is invariant under conjugation by elements in $\mathrm{GL}_N(\mathbb{R})$. This allows us to permute the factors of M, π' and

ν. In view of our partition of (167) into pairs and singletons, we may choose a permutation such that, without loss of generality,

$$\begin{aligned}\pi' \otimes e^{\nu} =&(\varpi'_1 \otimes e^{\nu'_1}) \otimes \cdots \otimes (\varpi'_k \otimes e^{\nu'_k})\\ &\otimes (\varpi'_{k+1} \otimes e^{0}) \otimes \cdots \otimes (\varpi'_h \otimes e^{0})\\ &\otimes (\varpi'_{h+1} \otimes e^{0}) \otimes \cdots \otimes (\varpi'_i \otimes e^{0})\\ &\otimes (\varpi'_k \otimes e^{-\nu'_k}) \otimes \cdots \otimes (\varpi'_1 \otimes e^{-\nu'_1}).\end{aligned} \tag{169}$$

The factors ϖ'_j here have the same form as the π'_j. What is different in this decomposition of $\pi' \otimes e^{\nu}$ is that we are separating the factors into three groups. The first group is comprised of the first and fourth lines of (169). This group corresponds to the pairs in the partition of (167). The second and third groups encompass the singletons, which are essentially (limits of) discrete series. The group in the second line is taken to be those singletons in the partition whose infinitesimal characters in ${}^{\vee}\mathfrak{h}$ all have half-integral entries (elements in $\mathbb{Z} + \frac{1}{2}$). The group in the third line is comprised of the singletons whose infinitesimal characters all have integral entries.

It is not difficult to realize representations in the first and second groups as subquotients of a principal series representation. For example, letting B_1 be the upper-triangular Borel subgroup in GL_2, the representation

$$(D'_m \otimes e^{\nu'_1}) \otimes (D'_m \otimes e^{-\nu'_1})$$

embeds into

$$\mathrm{ind}_{B_1(\mathbb{R})\times B_1(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})\times \mathrm{GL}_2(\mathbb{R})} \left((|\cdot|^{(m-1)/2} \otimes sgn(\cdot)|\cdot|^{-(m-1)/2}) \otimes e^{\nu'_1}\right) \otimes \left((sgn(\cdot)|\cdot|^{(m-1)/2} \otimes |\cdot|^{-(m-1)/2}) \otimes e^{-\nu'_1}\right) \tag{170}$$

if m is odd, and embeds into

$$\mathrm{ind}_{B_1(\mathbb{R})\times B_1(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})\times \mathrm{GL}_2(\mathbb{R})} \left((|\cdot|^{(m-1)/2} \otimes |\cdot|^{-(m-1)/2}) \otimes e^{\nu'_1}\right) \otimes \left(|\cdot|^{(m-1)/2} \otimes |\cdot|^{-(m-1)/2}) \otimes e^{-\nu'_1}\right) \tag{171}$$

if m is even (*cf.* (156)). More generally, one may parabolically induce

$$\left((\varpi'_1 \otimes e^{\nu'_1}) \otimes \cdots \otimes (\varpi'_k \otimes e^{\nu'_k})\right) \otimes \left((\varpi'_k \otimes e^{-\nu'_k}) \otimes \cdots \otimes (\varpi'_1 \otimes e^{-\nu'_1})\right)$$

to a representation π_I of $\mathrm{GL}_{n_1}(\mathbb{R}) \times \mathrm{GL}_{n_1}(\mathbb{R})$, where n_1 is the sum of the block sizes of the first k blocks. By induction in stages, one may show that π_I embeds into a principal series representation of $\mathrm{GL}_{n_1}(\mathbb{R}) \times \mathrm{GL}_{n_1}(\mathbb{R})$.

The representation

$$(\varpi'_{k+1} \otimes e^{0}) \otimes \cdots \otimes (\varpi'_h \otimes e^{0})$$

in the second group of (169) is equal to

$$(D'_{n_{k+1}} \otimes e^{0}) \otimes \cdots \otimes (D'_{n_h} \otimes e^{0})$$

where $n_{k+1}, \ldots, n_h$ are all even integers, and without loss of generality, $n_{k+1} \geq n_{k+2} \geq \cdots \geq n_h$. This places us in the context of Lemma 7.6 ([Kna86]*Theorem 14.91), but it is easy to write things out explicitly here again. By (156), each factor embed into a principal series representation of $\mathrm{GL}_2(\mathbb{R})$. One may parabolically induce $(\varpi'_{k+1} \otimes e^{0}) \otimes \cdots \otimes (\varpi'_h \otimes e^{0})$ to a representation π_{II} of $\mathrm{GL}_{n_2}(\mathbb{R})$, where $n_2 = 2(h-(k+1))$ is the sum of the block sizes from $k+1$ to h. Using induction in stages, one may show that π_{II} embeds into a principal series representation of $\mathrm{GL}_{n_2}(\mathbb{R})$. After conjugating by an element in $\mathrm{GL}_{n_2}(\mathbb{R})$, the principal series representation may be taken to be induced from the upper-triangular Borel subgroup with the quasicharacter

$$\begin{aligned}&|\cdot|^{(n_{k+1}-1)/2} \otimes |\cdot|^{(n_{k+2}-1)/2} \otimes \cdots \otimes |\cdot|^{(n_h-1)/2}\\ &\qquad \otimes |\cdot|^{-(n_h-1)/2} \otimes \cdots \otimes |\cdot|^{-(n_{k+2}-1)/2} \otimes |\cdot|^{-(n_{k+1}-1)/2}.\end{aligned} \tag{172}$$

This brings us to the representation $(\varpi'_{h+1} \otimes e^{0}) \otimes \cdots \otimes (\varpi'_i \otimes e^{0})$ in the third group of (169). It too may be parabolically induced to a representation π_{III} of $\mathrm{GL}_{n_3}(\mathbb{R})$ where n_3 is the sum of the block sizes from $h+1$ to i. This induced representation must be irreducible, as otherwise, by induction in stages, $\mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi'\otimes e^{\nu}) \cong \pi_{\mathrm{gen}}$ would be reducible. It follows from [Kna86]*Theorem 14.91 that π_{III} is tempered. In addition, the ϑ-stability of $\mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi'\otimes e^{\nu})$ combined with the Langlands Disjointness Theorem imply that π_{III} is a ϑ-stable representation of $\mathrm{GL}_{n_3}(\mathbb{R})$. By Lemma 7.6, π_{III} appears as a subquotient of either a ϑ-stable principal series representation of $\mathrm{GL}_{n_3}(\mathbb{R})$ or of a representation parabolically induced from a representation of the form (158).

Taking the tensor product of π_I, π_{II} and π_{III}, we obtain a representation of a block-diagonal Levi subgroup L of $\mathrm{GL}_N(\mathbb{R})$ such that

$$L(\mathbb{R}) \cong \mathrm{GL}_{n_1}(\mathbb{R}) \times \mathrm{GL}_{n_2}(\mathbb{R}) \times \mathrm{GL}_{n_3}(\mathbb{R}) \times \mathrm{GL}_{n_1}(\mathbb{R}).$$

By construction

$$\pi_I \otimes \pi_{II} \otimes \pi_{III} = \mathrm{ind}_{(P\cap L)(\mathbb{R})}^{L(\mathbb{R})}(\pi'\otimes e^{\nu}),$$

and appears as a subquotient of either a principal series representation of $L(\mathbb{R})$, or a representation described by (158) which is nearly in the principal series.

Suppose first that $\pi_I \otimes \pi_{II} \otimes \pi_{III}$ is a subquotient of a principal series representation of $L(\mathbb{R})$ and let Q be a parabolic subgroup of GL_N with L as its Levi subgroup. Then

$$\pi_{\mathrm{gen}} \cong \mathrm{ind}_{P(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi'\otimes e^{\nu}) \cong \mathrm{ind}_{Q(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\mathrm{ind}_{(P\cap L)(\mathbb{R})}^{L(\mathbb{R})}(\pi'\otimes e^{\nu}) \cong \mathrm{ind}_{Q(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}(\pi_I \otimes \pi_{II} \otimes \pi_{III})$$

appears as a subquotient of a principal series representation $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0$ of $\mathrm{GL}_N(\mathbb{R})$. We are free to conjugate this principal series representation by an element of $\mathrm{GL}_N(\mathbb{R})$ in order to permute the $\mathrm{GL}_1(\mathbb{R})$ factors of π_0. The factors of π_0 are given by (170), (171) and (172) (*cf.* Lemma 7.6 for the factors coming from π_{II} and π_{III}). Clearly, the factors of π_0 are paired off so that the tensor product of some permutation of them is fixed under ϑ. Taking π_0 to be this ϑ-fixed tensor product allows us to conclude that π_{gen} is infinitesimally equivalent to a subquotient of a ϑ-stable principal series representation $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0$. The principal series $\mathrm{ind}_{B(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{R})}\pi_0$ is represented by $M(\xi_p)$ where $\xi_p \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ is as in Lemma 7.1 and so $M(\xi_p)^\sim = M(\xi_p)^+$

Finally, suppose that $\pi_I \otimes \pi_{II} \otimes \pi_{III}$ is infinitesimally equivalent to a subquotient of an induced representation which is described by (158). Then we may argue as in the previous paragraph, except that now exactly one of the factors of π_0, stemming from π_{III}, is a relative discrete series representation D_{2j+1} of $\mathrm{GL}_2(\mathbb{R})$ for a positive integer j. After possibly permuting the factors of π_0, the factor D_{2j+1} may be assumed to occupy the middle block of GL_N (N is even by Lemma 7.6). The remaining factors of π_0 are quasicharacters of $\mathrm{GL}_1(\mathbb{R})$ which are paired off as before, so that we may assume that π_0 is ϑ-stable (Lemma 7.5). The representation of $\mathrm{GL}_N(\mathbb{R})$ which is parabolically induced from π_0 using a standard parabolic subgroup, is then ϑ-stable. This representation may be represented by $M(\xi_p)$ for some $\xi_p \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ or as $M(x,y)$ for the equivalent Atlas parameter (Lemma 2.2). The representation $M(x,y)$ has the same form as (163), with $x = x'$, π_0 replacing π_M, y replacing y', and the infinitesimal character of π_0 replacing λ. The arguments following (163) apply equally well to $M(x,y)$ and so we conclude that

$$M(\xi_p)^\sim = M(x,y)^\sim = M(x,y)^+ = M(\xi_p)^+.$$

□

Proposition 7.8. *Suppose* $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$. *Then*

$$M(\xi)^\sim(\vartheta) = (-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\; M(\xi)^+(\vartheta)$$

and

$$\pi(\xi)^\sim(\vartheta) = (-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\; \pi(\xi)^+(\vartheta).$$

Proof. Let $\pi_{\mathrm{gen}} = \pi(\xi_0)$ be the generic representation of Lemma 7.2 and let ξ_p be as in Lemma 7.7. Then the proof follows the proof of Proposition 7.4 exactly, although it is not as straightforward to show that $l^I(\xi_p) = l^I_\vartheta(\xi_p)$ when $M(\xi_p)$ is not a principal series representation. Let (x,y) be the Atlas parameter equivalent to ξ_p. When $M(\xi_p)$ is not a principal series representation it was noted in the proof of Lemma 7.7 that N is even and that the Cartan involution corresponding to $x = x'$ acts on roots as $-s_{\alpha_{N/2}}$ (see (162)). From (60), (61) and the fact that $\alpha_{N/2}$ is also simple root of type 1 in $R_\vartheta(\mathrm{GL}_N, H)$ (see (84)), we compute

$$\begin{aligned} l^I(\xi_p) &= -\frac{1}{2}\left(\left|\left\{\alpha \in R^+(\lambda) : -s_{\alpha_{N/2}}(\alpha) \in R^+(\lambda)\right\}\right| + \dim\left(H^{-s_{\alpha_{N/2}}}\right)\right) \\ &= -\frac{1}{2}(1+1) \\ &= -\frac{1}{2}\left(\left|\left\{\alpha \in R^+_\vartheta(\lambda) : -s_{\alpha_{N/2}}(\alpha) \in R^+_\vartheta(\lambda)\right\}\right| + \dim\left(\left(H^\vartheta\right)^{-s_{\alpha_{N/2}}}\right)\right) \\ &= l^I_\vartheta(\xi_p). \end{aligned}$$

□

It is worth observing that Theorem 3.5 now has the following simple form, reminiscent of Theorem 3.1.

Corollary 7.9. *The pairing* (62)*, defined by* (63)*, satisfies*

$$\langle M(\xi)^\sim, \mu(\xi')^+\rangle = \delta_{\xi,\xi'}$$

and

$$\langle \pi(\xi)^\sim, P(\xi')^+\rangle = (-1)^{d(\xi)}\,\delta_{\xi,\xi'}$$

for $\xi, \xi' \in \Xi({}^\vee\mathcal{O}{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$*. Equivalently,*

$$m_r^\sim(\xi', \xi) = (-1)^{d(\xi)-d(\xi')} c_g^\vartheta(\xi, \xi'), \tag{173}$$

where $m_r^\sim(\xi', \xi)$ *is defined by the decomposition*

$$M(\xi)^\sim = \sum_{\xi' \in (\Xi(\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^1)^\vartheta} m_r^\sim(\xi', \xi)\, \pi(\xi')^\sim. \tag{174}$$

in $K\Pi(\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$*.*

Proof. The first assertion is an immediate consequence of Proposition 7.8. For the second assertion, we return to decomposition (65). Substituting the Whittaker extensions of Proposition 7.8 into this decomposition and comparing with (174), we deduce that

$$m_r^\vartheta(\xi', \xi) = (-1)^{l^I(\xi') - l^I_\vartheta(\xi') - (l^I(\xi) - l^I_\vartheta(\xi))}\, m_r^\sim(\xi', \xi).$$

Substituting this expression into the identity of Proposition 3.7 we see that the first assertion is equivalent to

$$m_r^\sim(\xi', \xi) = (-1)^{l^I(\xi) - l^I(\xi')}\, c_g^\vartheta(\xi, \xi').$$

This identity is equivalent to (173), as

$$l^I(\xi) - d(\xi) = l^I(\xi') - d(\xi')$$

is a constant independent of $\xi, \xi' \in \Xi(\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ (Proposition B.1 [AMR17]). □

Another consequence of Proposition 7.8 is that the endoscopic lifting map Lift_0 (132) is equal to the endoscopic transfer map $\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}$ used in Arthur's definition (5) of $\eta^{\mathrm{Ar}}_{\psi_G}$. This is a crucial step in the comparison of $\eta^{\mathrm{Ar}}_{\psi_G}$ and $\eta^{\mathrm{ABV}}_{\psi_G}$.

Corollary 7.10. *Suppose* G *is a simple twisted endoscopic group as in Section 5.2. Suppose further that* $S_G \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ *is a* ${}^\vee G$*-orbit and let* $\epsilon(S_G) \subset X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ *be the* ${}^\vee\mathrm{GL}_N$*-orbit of the image of* S_G *under* ϵ *(113). Then*

(a)

$$\mathrm{Lift}_0(\eta^{\mathrm{loc}}_{S_G}(\sigma)(\delta_q)) = M(\epsilon(S_G), 1)^\sim,$$

(b)

$$\mathrm{Lift}_0 = \mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}$$

on $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$.

Proof. The first assertion is an immediate consequence of Propositions 5.3 and 7.8. The second assertion follows from the identity

$$\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}(\eta_{S_G}^{\mathrm{loc}}(\delta_q)) = M(\epsilon(S_G), 1)^{\sim}$$

([AMR18]*(1.0.3), [Mez16]) and the fact that the stable virtual characters $\eta_{S_G}^{\mathrm{loc}}(\delta_q)$ form a basis for $K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$ as S_G runs over the ${}^\vee G$-orbits in $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$. □

8 The comparison of Π_{ψ_G} and $\Pi_{\psi_G}^{\mathrm{ABV}}$ for regular infinitesimal character

In this section we prove the main results comparing $\eta_{\psi_G}^{\mathrm{Ar}}$ with $\eta_{\psi_G}^{\mathrm{ABV}}$ ((19), (20)). We shall work under the assumptions of Section 5.2. In particular, ψ_G and $\psi = \epsilon \circ \psi_G$ are A-parameters with respective infinitesimal characters ${}^\vee\mathcal{O}_G$ and ${}^\vee\mathcal{O}$. The assumption on the infinitesimal characters is that they are regular with respect to GL_N. This assumption shall be removed in the next section.

The definition of $\eta_{\psi_G}^{\mathrm{Ar}}$ was outlined in (5). Let us provide a few more details from [Art13]. The key lemma is

Lemma 8.1. *Let $S_\psi \subset X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ be the ${}^\vee\mathrm{GL}_N$-orbit corresponding to ϕ_ψ ([ABV92]*Proposition 6.17, (26)).*

(a) There exist integers n_S such that

$$\pi(S_\psi, 1)^{\sim} = \sum_{(S,1)\in\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N)^\vartheta} n_S\, M(S,1)^{\sim} \tag{175}$$

in $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$.

(b) Moreover, for every S such that $n_S \neq 0$ in (175) there exists a unique ${}^\vee G$-orbit $S_G \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ which is carried to S under ϵ.

(c) Writing

$$S = \epsilon(S_G)$$

for the orbits in part (b), we have

$$\begin{aligned} \pi(S_\psi, 1)^{\sim} &= \mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}\left(\sum_{S_G} n_{\epsilon(S_G)}\, \eta_{S_G}^{\mathrm{loc}}(\delta_q)\right) \\ &= \mathrm{Lift}_0\left(\sum_{S_G} n_{\epsilon(S_G)}\, \eta_{S_G}^{\mathrm{loc}}(\delta_q)\right). \end{aligned} \tag{176}$$

Proof. By virtue of Proposition 7.8, (175) is equivalent to a decomposition

$$\pi(S_\psi, 1)^{+} = \sum_{(S,1)\in\Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N)^\vartheta} n'_S\, M(S,1)^{+}$$

of Atlas extensions. The latter decomposition follows from (65) and Lemma 3.6. The existence of the orbit S_G in part (b) is established on the first page of the proof of [Art13]*Lemma 2.2.2. The uniqueness of the orbit follows from Proposition 5.2. Part (c) is a consequence of Corollary 7.10. □

Arthur's definition of $\eta^{\mathrm{Ar}}_{\psi_G}$ is easiest to state when the endoscopic group G is not equal to SO_N for even N. In this case, the group G has no outer automorphisms and the defining equation (5) is the same as (176). It follows that

$$\eta^{\mathrm{Ar}}_{\psi_G} = \sum_{S_G} n_{\epsilon(S_G)}\, \eta^{\mathrm{loc}}_{S_G}(\delta_q) \in K_{\mathbb{C}}\Pi({}^{\vee}\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}} \tag{177}$$

(*cf.* [Art13]*(2.2.12)). By definition, the A-packet $\Pi^{\mathrm{Ar}}_{\psi_G}$ consists of those irreducible characters in $\Pi({}^{\vee}\mathcal{O}_G, G(\mathbb{R},\delta_q))$ which occur with non-zero multiplicity when (177) is expressed as a linear combination in the basis of irreducible characters.

In the case that N is even and $G = \mathrm{SO}_N$, the stable virtual character $\eta^{\mathrm{Ar}}_{\psi_G}$ is defined to be invariant under the action of the outer automorphisms induced by the orthogonal group O_N ([Art13]**pp.* 12, 41). Fix

$$\tilde{w} \in \mathrm{O}_N - \mathrm{SO}_N. \tag{178}$$

The orthogonal group acts on geometric parameters in $X({}^{\vee}\mathcal{O}_G, {}^{\vee}G^{\Gamma})$ in a straightforward manner, sending them to geometric parameters in $X(\tilde{w}\cdot {}^{\vee}\mathcal{O}_G, {}^{\vee}G^{\Gamma})$. The stable virtual character

$$\frac{1}{2}(\eta^{\mathrm{loc}}_{S_G}(\delta_q) + \eta^{\mathrm{loc}}_{\tilde{w}\cdot S_G}(\delta_q)) \in K_{\mathbb{C}}\Pi({}^{\vee}\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}} \oplus K_{\mathbb{C}}\Pi(\tilde{w}\cdot{}^{\vee}\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}}$$

is O_N-invariant by design. Extending the domain of $\mathrm{Trans}_G^{\mathrm{GL}_N \rtimes \vartheta}$ to the space on the right, equations (5) and (176) imply

$$\eta^{\mathrm{Ar}}_{\psi_G} = \sum_{S_G} \frac{n_{\epsilon(S_G)}}{2}\,(\eta^{\mathrm{loc}}_{S_G}(\delta_q) + \eta^{\mathrm{loc}}_{\tilde{w}\cdot S_G}(\delta_q)).$$

This is a virtual character in $K_{\mathbb{C}}\Pi({}^{\vee}\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}} \oplus K_{\mathbb{C}}\Pi(\tilde{w}\cdot{}^{\vee}\mathcal{O}_G, G(\mathbb{R},\delta_q))^{\mathrm{st}}$ and the A-packet Π_{ψ_G} consists of the irreducible characters in its support.

Theorem 8.2. *(a) If G is not isomorphic to SO_N for even N then*

$$\eta^{\mathrm{Ar}}_{\psi_G} = \eta^{\mathrm{mic}}_{\psi_G}(\delta_q) = \eta^{\mathrm{ABV}}_{\psi_G} \quad \textit{and} \quad \Pi^{\mathrm{Ar}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G}.$$

(b) If N is even and $G \cong \mathrm{SO}_N$ then

$$\eta_{\psi_G} = \frac{1}{2}\left(\eta^{\mathrm{mic}}_{\psi_G}(\delta_q) + \eta^{\mathrm{mic}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}(\delta_q)\right) = \frac{1}{2}\left(\eta^{\mathrm{ABV}}_{\psi_G} + \eta^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}\right)$$

and

$$\Pi^{\mathrm{Ar}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G} \cup \Pi^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}$$

where the union is disjoint.

Proof. This just involves putting together the pieces. Let $\xi = (S_\psi, 1)$ as in Corollary 6.4.

$$\begin{aligned}
\mathrm{Lift}_0\,(\eta^{\mathrm{mic}}_{\psi_G}(\delta_q)) &= \mathrm{Lift}_0\,(\eta^{\mathrm{mic}}_{\psi_G}(\sigma)(\delta_q)) \quad \text{(by (138))}\\
&= (-1)^{l^I(\xi)-l^I_\vartheta(\xi)}\pi(\xi)^+ \quad \text{(Corollary 6.4)}\\
&= \pi(\xi)^\sim \quad \text{(Proposition 7.8)}\\
&= \pi(S_\psi, 1)^\sim\\
&= \mathrm{Trans}_G^{\mathrm{GL}_N\rtimes\vartheta}\left(\sum_{S_G} n_{\epsilon(S_G)}\,\eta^{\mathrm{loc}}_{S_G}(\delta_q)\right) \quad \text{(Lemma 8.1)}\\
&= \mathrm{Lift}_0\left(\eta^{\mathrm{Ar}}_{\psi_G}\right) \quad \text{(Corollary 7.10}(b)\text{)}.
\end{aligned}$$

The equality of the stable virtual characters follows from the injectivity of Lift_0 (Proposition 5.4). The equality of packets follows immediately.

For part (b), identical reasoning leads to the identity

$$\mathrm{Lift}_0(\eta^{\mathrm{mic}}_{\psi_G}(\delta_q)) = \mathrm{Trans}_G^{\mathrm{GL}_N\rtimes\vartheta}\left(\sum_{S_G} n_{\epsilon(S_G)}(\eta^{\mathrm{loc}}_{S_G}(\delta_q) + \eta^{\mathrm{loc}}_{\tilde{w}\cdot S_G}(\delta_q))\right) = \mathrm{Lift}_0(\eta^{\mathrm{Ar}}_{\psi_G}).$$

The first assertion of part (b) follows from the injectivity of Lift_0 as before. The irreducible characters in the support of $\sum_{S_G} n_{\epsilon(S_G)}\eta^{\mathrm{loc}}_{S_G}(\delta_q)$ are those in the packet $\Pi^{\mathrm{ABV}}_{\psi_G}$. These irreducible characters lie in $\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))$. Similarly the irreducible characters in the support of $\sum_{S_G} n_{\epsilon(S_G)}\eta^{\mathrm{loc}}_{\tilde{w}\cdot S_G}(\delta_q)$ are those in $\Pi^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}$. These irreducible characters lie in $\Pi(\tilde{w}\cdot{}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q))$. The regularity of the infinitesimal character ${}^\vee\mathcal{O}_G$ implies ${}^\vee\mathcal{O}_G \cap (\tilde{w}\cdot{}^\vee\mathcal{O}_G) = \emptyset$, otherwise there exists $g \in {}^\vee\mathrm{SO}_N$ and $\lambda \in {}^\vee\mathcal{O}_G \cap {}^\vee\mathfrak{h}^\vartheta$ such that $\mathrm{Ad}(\tilde{w}g)\lambda = \lambda$. This would imply $\tilde{w}g \in {}^\vee H^\vartheta \subset {}^\vee\mathrm{SO}_N$, contradicting the definition of $\tilde{w}$. In consequence,

$$\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q)) \cap \Pi(\tilde{w}\cdot{}^\vee\mathcal{O}_G, G(\mathbb{R},\delta_q)) = \emptyset$$

and

$$\Pi^{\mathrm{ABV}}_{\psi_G} \cap \Pi^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\cdot\psi_G} = \emptyset.$$

This proves the final assertion. □

9 The comparison of Π_{ψ_G} and $\Pi^{\mathrm{ABV}}_{\psi_G}$ for singular infinitesimal character

To conclude our comparison of stable virtual characters, we retain the setup of the previous section, but without the hypothesis of regularity on the infinitesimal character. In other words, the orbits ${}^\vee\mathcal{O}$ and ${}^\vee\mathcal{O}_G$ are now allowed to be orbits of singular infinitesimal characters and the reader should think of them as such. In order to prove something like Theorem 8.2 for singular ${}^\vee\mathcal{O}$, we must extend the pairing of Theorem 3.5 and extend the twisted endoscopic lifting (132) to include representations with singular infinitesimal character. The main tool for this extension is the *Jantzen-Zuckerman translation principle*, which we refer to simply as *t*ranslation. In essence the Jantzen-Zuckerman translation principle allows one to transfer results for regular infinitesimal character to results for singular infinitesimal character. Applying this principle to the results of the previous section will allow us to compare Π_{ψ_G} with $\Pi^{\mathrm{ABV}}_{\psi_G}$ with no restriction on the infinitesimal character.

The reader is assumed to have some familiarity with then Jantzen-Zuckerman translation principle, which for us begins with the existence of a regular orbit ${}^\vee\mathcal{O}' \subset {}^\vee\mathfrak{gl}_N$ and a *translation datum* $\mathcal{T}$ from ${}^\vee\mathcal{O}$ to ${}^\vee\mathcal{O}'$ ([ABV92]*Definition 8.6, Lemma 8.7). A key feature of the translation datum is that if ${}^\vee\mathcal{O}$ is the ${}^\vee\mathrm{GL}_N$-orbit of $\lambda \in {}^\vee\mathfrak{h}$ then ${}^\vee\mathcal{O}'$ is the ${}^\vee\mathrm{GL}_N$-orbit of

$$\lambda' = \lambda + \lambda_1 \in {}^\vee\mathfrak{h} \tag{179}$$

where $\lambda_1 \in X_*(H)$ is regular and dominant with respect to the positive system of $R^+(\mathrm{GL}_N, H)$. We may and shall take λ_1 to be the sum of the positive roots. In this way, each of λ, λ_1 and λ' are fixed by ϑ. The translation datum $\mathcal{T}$ induces a ${}^\vee\mathrm{GL}_N$-equivariant morphism

$$f_{\mathcal{T}} : X({}^\vee\mathcal{O}', {}^\vee\mathrm{GL}_N^\Gamma) \to X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma) \tag{180}$$

of geometric parameters ([ABV92]*Proposition 8.8). The morphism has connected fibres of fixed dimension, a fact we shall use when comparing orbit dimensions. The ${}^\vee\mathrm{GL}_N$-equivariance of (180) is tantamount to a coset map commuting with left-multiplication by ${}^\vee\mathrm{GL}_N$ ([ABV92]*(6.10)(b)). Since both λ and λ' are fixed by ϑ, it is just as easy to see that the action of ϑ commutes with the same coset map. We leave this exercise to the reader, taking for granted the resulting $({}^\vee\mathrm{GL}_N \rtimes \langle\vartheta\rangle)$-equivariance of (180).

According to [ABV92]*Proposition 7.15, the morphism $f_{\mathcal{T}}$ induces an inclusion

$$f^*_{\mathcal{T}} : \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma) \hookrightarrow \Xi({}^\vee\mathcal{O}', {}^\vee\mathrm{GL}_N^\Gamma) \tag{181}$$

of complete geometric parameters. The ϑ-equivariance of (180) implies that this inclusion restricts to an inclusion (denoted by the same symbol)

$$f_{\mathcal{T}}^{*} : \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta} \hookrightarrow \Xi({}^{\vee}\mathcal{O}', {}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}.$$

The (Jantzen-Zuckerman) translation functor ([AvLTV20]*(17.8j))

$$T_{\lambda'}^{\lambda} = T_{\lambda+\lambda_1}^{\lambda}$$

is an exact functor on a category of Harish-Chandra modules, which we shall often regard as a homomorphism

$$T_{\lambda+\lambda_1}^{\lambda} : K\Pi({}^{\vee}\mathcal{O}', \mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle) \to K\Pi({}^{\vee}\mathcal{O}, \mathrm{GL}_N(\mathbb{R}) \rtimes \langle\vartheta\rangle) \tag{182}$$

of Grothendieck groups. It is surjective ([AvLTV20]*Corollary 17.9.8). This translation functor is an extended version of the usual translation functor ([AvLTV20]*(16.8f)), which we also denote by

$$T_{\lambda+\lambda_1}^{\lambda} : K\Pi({}^{\vee}\mathcal{O}', \mathrm{GL}_N(\mathbb{R})) \to K\Pi({}^{\vee}\mathcal{O}, \mathrm{GL}_N(\mathbb{R})). \tag{183}$$

Let us take a moment to make (182) more precise. The sum of the positive roots λ_1 is the infinitesimal character of a finite-dimensional representation of $\mathrm{GL}_N(\mathbb{R})$. Therefore, λ_1 is the differential of a ϑ-fixed quasicharacter Λ_1 of the split real diagonal torus $H(\mathbb{R})$, which matches the weight of this finite-dimensional representation. The quasicharacter Λ_1 may be extended to a quasicharacter Λ_1^+ of the semi-direct product $H(\mathbb{R}) \rtimes \langle\vartheta\rangle$ by setting

$$\Lambda_1^+(\vartheta) = 1. \tag{184}$$

We define translation in the extended setting of (182) using this representation of the extended group. Since the extension is evident here we continue to write $T_{\lambda+\lambda_1}^{\lambda}$ instead of $T_{\lambda+\Lambda_1^+}^{\lambda}$.

In the ordinary setting of (183) we have

$$\pi(\xi) = T_{\lambda+\lambda_1}^{\lambda}\left(\pi(f_{\mathcal{T}}^{*}(\xi))\right),$$
$$M(\xi) = T_{\lambda+\lambda_1}^{\lambda}\left(M(f_{\mathcal{T}}^{*}(\xi))\right), \quad \xi \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_N^{\Gamma})$$

([AvLTV20]*Corollary 16.9.4, 16.9.7 and 16.9.8, or [ABV92]* Theorem 16.4 and Proposition 16.6). We *define* the Atlas extensions of $\pi(\xi)$ and $M(\xi)$, with $\xi \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}$, by

$$\pi(\xi)^+ = T_{\lambda+\lambda_1}^{\lambda}(\pi(f_{\mathcal{T}}^{*}(\xi))^+)$$
$$M(\xi)^+ = T_{\lambda+\lambda_1}^{\lambda}(M(f_{\mathcal{T}}^{*}(\xi))^+).$$

(To be careful, one should verify that this definition does not conflict with Section 2.5 when ${}^{\vee}\mathcal{O}$ is regular. This amounts to the observation that the translate of the $(\mathfrak{h}, (H \cap K_\delta) \rtimes \langle\vartheta\rangle)$-module underlying an Atlas extension remains trivial on ϑ. Justification for this observation is given in proof of Proposition 9.1).

With the definition of Atlas extensions in place, the discussion of Section 2.7 is valid, and we see that $T_{\lambda+\lambda_1}^{\lambda}$ factors to a homomorphism of $K\Pi({}^{\vee}\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta)$ (see (44)). We use the same notation $T_{\lambda+\lambda_1}^{\lambda}$ to denote the functor of Harish-Chandra modules, and either of the earlier homomorphisms. The reader will be reminded of the context when it is important.

The definition of a Whittaker extension does not depend on the regularity of the infinitesimal character. The following proposition shows that translation sends Whittaker extensions to Whittaker extensions.

Proposition 9.1. *Suppose $\xi \in \Xi({}^{\vee}\mathcal{O}, {}^{\vee}\mathrm{GL}_N^{\Gamma})^{\vartheta}$. Then (as Harish-Chandra modules)*

$$T_{\lambda+\lambda_1}^{\lambda}\left(M(f_{\mathcal{T}}^{*}(\xi))^{\sim}\right) = M(\xi)^{\sim},$$

and

$$T_{\lambda+\lambda_1}^{\lambda}\left(\pi(f_{\mathcal{T}}^{*}(\xi))^{\sim}\right) = \pi(\xi)^{\sim}.$$

Proof. Lemma 7.2 does not require ${}^\vee\mathcal{O}$ to be regular, and so we may choose $\xi_0 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ to be the unique parameter such that $\pi(\xi_0)$ is generic and embeds as a subrepresentation of $M(\xi)$. (Here, we are implicitly assuming that we are working with actual admissible representations (or Harish-Chandra modules) rather than equivalence classes.) Since the orbit $S_{\xi_0} \subset X({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)$ of ξ_0 is open (see the proof of Proposition 7.3), it is an immediate consequence of the definition of $f_\mathcal{T}^*$ ([ABV92]*(7.16)(b)) that $f_\mathcal{T}^*(S_{\xi_0})$ is open and therefore that $\pi(f_\mathcal{T}^*(\xi_0))$ is generic.

According to Lemma 7.7, $\pi(f_\mathcal{T}^*(\xi_0))$ embeds into a standard representation $M(\xi_p')$ for some $\xi_p' \in \Xi({}^\vee\mathcal{O}', {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$, which satisfies $M(\xi_p')^\sim = M(\xi_p')^+$. Furthermore, $\pi(f_\mathcal{T}^*(\xi_0))$ occurs as a subrepresentation with multiplicity one, and $\pi(f_\mathcal{T}^*(\xi_0))^\sim$ is a subrepresentation of $M(\xi_p')^\sim$ (Lemma 7.2). Applying the exact functor $T_{\lambda+\lambda_1}^\lambda$ (of Harish-Chandra modules), we see that $T_{\lambda+\lambda_1}^\lambda(\pi(f_\mathcal{T}^*(\xi_0))^\sim)$ is a subrepresentation of $T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)$.

Suppose first that $M(\xi_p')$ is a principal series representation. By [AvLTV20]*Corollary 17.9.7 $T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)$ is an extension of a principal series representation. Indeed, $M(\xi_p')^\sim = M(\xi_p')^+$ (Lemma 7.1) and is parabolically induced from a quasicharacter of a split Cartan subgroup extended by $\langle\vartheta\rangle$–the value of this quasicharacter on ϑ being one (Section 2.5). [AvLTV20]*Corollary 17.9.7 tells us that $T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)$ is parabolically induced from the tensor product of the aforementioned quasicharacter with the inverse of Λ_1^+ as in (184) ([AvLTV20]*Theorem 17.7.5). This justifies $T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)$ being an extended principal series representation, but more can be said. In view of (184), translation by $(\Lambda_1^+)^{-1}$ does not affect the value of the quasicharacter on ϑ. Consequently its value on ϑ is still one. The arguments of Lemma 7.1 therefore apply to $T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)$ as they do for $M(\xi_p)^\sim$ and we deduce

$$\omega = \omega \circ T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)(\vartheta) \tag{185}$$

for the Whittaker functional ω defined by (151).

If $M(\xi_p')$ is not a principal series representation then it is of the form (164), which is a parabolically induced representation, essentially from a relative discrete series representation on $\mathrm{GL}_2(\mathbb{R})$. Such a representation may still be regarded as being induced, albeit not parabolically induced, from a quasicharacter of a non-split Cartan subgroup. The earlier arguments from [AvLTV20]* Corollary 17.9.7 apply. We leave it to the reader, to verify that (185) holds in any case. In consequence of (185) and the exactness of $T_{\lambda+\lambda_1}^\lambda$,

$$\omega \circ T_{\lambda+\lambda_1}^\lambda(\pi(f_\mathcal{T}^*(\xi_0))^\sim)(\vartheta) = \omega \circ T_{\lambda+\lambda_1}^\lambda\left(M(\xi_p')^\sim\right)(\vartheta)_{|\pi(\xi_0)} = \omega.$$

This proves that $T_{\lambda+\lambda_1}^\lambda(\pi(f_\mathcal{T}^*(\xi_0))^\sim)$ is the Whittaker extension of

$$T_{\lambda+\lambda_1}^\lambda(\pi(f_\mathcal{T}^*(\xi_0)) = \pi(\xi_0),$$

that is

$$T_{\lambda+\lambda_1}^\lambda(\pi(f_\mathcal{T}^*(\xi_0))^\sim) = \pi(\xi_0)^\sim. \tag{186}$$

To complete the proposition we embed $\pi(f_\mathcal{T}^*(\xi_0))^\sim$ as a subrepresentation of $M(f_\mathcal{T}^*(\xi))^\sim$ using Lemma 7.2. Applying the exact functor $T_{\lambda+\lambda_1}^\lambda$ (of Harish-Chandra modules), we see that (186) is a subrepresentation of $T_{\lambda+\lambda_1}^\lambda(M(f_\mathcal{T}^*(\xi))^\sim)$. Since a Whittaker functional ω of $T_{\lambda+\lambda_1}^\lambda(M(f_\mathcal{T}^*(\xi))) = M(\xi)$ restricts to a non-zero Whittaker functional $\omega_{|\pi(\xi_0)}$ on $\pi(\xi_0)$ and

$$c\,\omega = \omega \circ T_{\lambda+\lambda_1}^\lambda(M(f_\mathcal{T}^*(\xi))^\sim)(\vartheta)$$

for $c = \pm 1$, we deduce in succession that

$$c\,\omega_{|\pi(\xi_0)} = \omega \circ T_{\lambda+\lambda_1}^\lambda(M(f_\mathcal{T}^*(\xi))^\sim)(\vartheta)_{|\pi(\xi_0)} = \omega \circ \pi(\xi_0)^\sim(\vartheta) = \omega_{|\pi(\xi_0)},$$

$c = 1$ and

$$\omega = \omega \circ T_{\lambda+\lambda_1}^\lambda(M(f_\mathcal{T}^*(\xi))^\sim)(\vartheta).$$

The final equation implies

$$T_{\lambda+\lambda_1}^\lambda(M(f_\mathcal{T}^*(\xi))^\sim) = M(\xi)^\sim.$$

Since $\pi(f_\mathcal{T}^*(\xi))^\sim$ and $\pi(\xi)^\sim$ are the unique irreducible quotients of $M(f_\mathcal{T}^*(\xi))^\sim$ and $M(\xi)^\sim$ respectively, and $T_{\lambda+\lambda_1}^\lambda$ is exact (on Harish-Chandra modules), the proposition follows. ☐

Our translation datum $\mathcal{T}$ for GL_N is defined by (179), in which both λ and λ' are fixed by the endoscopic datum $\mathrm{Int}(s) \circ \vartheta$. For this reason (179) also determines a translation datum $\mathcal{T}_G$ from ${}^\vee G$-orbits ${}^\vee\mathcal{O}_G$ to ${}^\vee\mathcal{O}'_G$ for the twisted endoscopic group G ([ABV92]*Definition 8.6 (e)). Just as for GL_N, we have maps

$$f_{\mathcal{T}_G} : X({}^\vee\mathcal{O}', {}^\vee G^\Gamma) \to X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$$
$$f^*_{\mathcal{T}_G} : \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \hookrightarrow \Xi({}^\vee\mathcal{O}', {}^\vee G^\Gamma)$$

and the translation functor $T^\lambda_{\lambda+\lambda_1}$ which satisfies

$$\pi(\xi) = T^\lambda_{\lambda+\lambda_1}\left(\pi(f^*_{\mathcal{T}_G}(\xi))\right), \quad \xi \in \Xi({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$$

([ABV92]*Proposition 16.6, [AvLTV20]*Section 16).

The translation data $\mathcal{T}$ and $\mathcal{T}_G$ allow us to transport properties of our pairings at regular infinitesimal character (Proposition 7.9) to the same properties for pairings at singular infinitesimal character.

Proposition 9.2. *Define the pairing*

$$\langle \cdot, \cdot \rangle : K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{R}), \vartheta) \times KX({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma, \sigma) \to \mathbb{Z} \tag{187}$$

by

$$\langle M(\xi)^\sim, \mu(\xi')^+ \rangle = \delta_{\xi,\xi'}.$$

Then

$$\langle \pi(\xi)^\sim, P(\xi')^+ \rangle = (-1)^{d(\xi)}\, \delta_{\xi,\xi'}$$

where $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$.

Proof. We first sketch the proof for the ordinary pairing of Theorem 3.1 in [ABV92], which does not involve twisting by ϑ. This will allow us to point out the portions of the proof that must be modified in the twisted setting. In the ordinary case there are no Whittaker or Atlas extensions, and the identity to be proven is (53)

$$m_r(\xi_1, \xi_2) = (-1)^{d(\xi_1)-d(\xi_2)}\, c_g(\xi_2, \xi_1), \quad \xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$$

for the possibly singular orbit ${}^\vee\mathcal{O}$. The idea of the proof is to show that both sides of (53) are invariant under translation. Starting with the right-hand side of (53), we use [ABV92]*Proposition 8.8 (b), which provides an exact functor from $\mathcal{P}(X({}^\vee\mathcal{O}, {}^\vee G^\Gamma))$ to $\mathcal{P}(X({}^\vee\mathcal{O}', {}^\vee G^\Gamma))$ satisfying

$$P(\xi) \mapsto P(f^*_{\mathcal{T}_G}(\xi))$$

and

$$c_g(f^*_{\mathcal{T}_G}(\xi_1), f^*_{\mathcal{T}_G}(\xi_2)) = c_g(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma). \tag{188}$$

The invariance of the left-hand side of (53)

$$m_r(f^*_{\mathcal{T}_G}(\xi_1), f^*_{\mathcal{T}_G}(\xi_2)) = m_r(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma),$$

is given by [ABV92]*Proposition 16.6 and (16.5)(d), which rely on the translation functor ([ABV92]*(16.3)). All that is now needed to prove (53) for the possibly singular orbit ${}^\vee\mathcal{O}$ is to line up the equations

$$\begin{aligned} m_r(\xi_1, \xi_2) &= m_r(f^*_{\mathcal{T}_G}(\xi_1), f^*_{\mathcal{T}_G}(\xi_2)) \\ &= (-1)^{d(f^*_{\mathcal{T}_G}(\xi_1))-d(f^*_{\mathcal{T}_G}(\xi_2))}\, c_g(f^*_{\mathcal{T}_G}(\xi_2), f^*_{\mathcal{T}_G}(\xi_1)) \\ &= (-1)^{(d(\xi_1)-d)-(d(\xi_2)-d)}\, c_g(\xi_2, \xi_1) \\ &= (-1)^{d(\xi_1)-d(\xi_2)}\, c_g(\xi_2, \xi_1). \end{aligned} \tag{189}$$

In the third equation, we have used [ABV92]* (7.16)(b) and the dimension d of the connected fibres of $f^*_{\mathcal{T}_G}$ to describe the orbit dimensions.

Let us repeat the preceding proof in the twisted setting. The desired analogue of (188) is

$$c_g(f_{\mathcal{T}}^*(\xi_1)_\pm, f_{\mathcal{T}}^*(\xi_2)_\pm) = c_g(\xi_{1\pm}, \xi_{2\pm}), \quad \xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta \tag{190}$$

(see (66)). As before, [ABV92]*Proposition 8.8 (b) ensures this as long as

$$P(\xi)^+ \mapsto P(f_{\mathcal{T}}^*(\xi))^+, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta \tag{191}$$

([ABV92]*Proposition 7.15 (b)). This may be seen as follows. The complete geometric parameter $f_{\mathcal{T}}^*(\xi)$ determines a $({}^\vee\mathrm{GL}_N \rtimes \langle\vartheta\rangle)$-equivariant local system, and the perverse sheaf $P(\xi)^+$ is mapped to the intermediate extension of this local system (to its ${}^\vee\mathrm{GL}_N$-orbit closure) ([ABV92]*(7.10)(d), [BBD82]**p.* 110). Let us call the resulting perverse sheaf P^+. We would like $P^+ = P(f_{\mathcal{T}}^*(\xi))^+$ and this holds when the $({}^\vee\mathrm{GL}_N \rtimes \langle\vartheta\rangle)$-equivariant irreducible constructible sheaf $\mu(f_{\mathcal{T}}^*(\xi))^+$ occurs in the decomposition of P^+ (*cf.* (66)). The latter property is true, for P^+ and $\mu(f_{\mathcal{T}}^*(\xi))^+$ are obtained from the same $({}^\vee\mathrm{GL}_N \rtimes \langle\vartheta\rangle)$-equivariant local system by intermediate extension and extension by zero respectively (*cf.* [ABV92]*(7.11)(b)). This justifies (191) and therefore also (190). By definition (67) and (190)

$$c_g^\vartheta(f_{\mathcal{T}}^*(\xi_1), f_{\mathcal{T}}^*(\xi_2)) = c_g^\vartheta(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta. \tag{192}$$

Moving to the representation-theoretic multiplicities, we appeal to the translation functor $T_{\lambda+\lambda_1}^{\lambda}$ for $\mathrm{GL}_N \rtimes \langle\vartheta\rangle$. In $K\Pi({}^\vee\mathcal{O}', \mathrm{GL}_N(\mathbb{R}), \vartheta)$ we have

$$M(f_{\mathcal{T}}^*(\xi_2))^\sim = \sum_{\xi_1 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} m_r^\sim(f_{\mathcal{T}}^*(\xi_1), f_{\mathcal{T}}^*(\xi_2))\, \pi(f_{\mathcal{T}}^*(\xi_1))^\sim \tag{193}$$
$$+ \sum_{\xi'} m_r^\sim(\xi', f_{\mathcal{T}}^*(\xi_2))\, \pi(\xi')^\sim$$

where ξ' are those parameters in $\Xi({}^\vee\mathcal{O}', {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta$ which do not lie in the image of (181). Applying $T_{\lambda+\lambda_1}^{\lambda}$ to (193) has the effect of annihilating the second sum on the right ([AvLTV20]*Corollary 17.9.4 and 17.9.8). By Proposition 9.1, the remaining terms are

$$M(\xi_2)^\sim = \sum_{\xi_1 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} m_r^\sim(f_{\mathcal{T}}^*(\xi_1), f_{\mathcal{T}}^*(\xi_2))\, \pi(\xi_1)^\sim$$

and this equation implies

$$m_r^\sim(f_{\mathcal{T}}^*(\xi_1), f_{\mathcal{T}}^*(\xi_2)) = m_r^\sim(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta. \tag{194}$$

Using equations (192) and (194), and replacing m_r and c_g with $m_r^\sim$ and c_g^ϑ respectively in (189), we deduce that (173) holds for the possibly singular orbit ${}^\vee\mathcal{O}$. ☐

Proposition 9.2 is the final version of the twisted pairing, and we use it to extend the definition of endoscopic lifting Lift_0 to include singular infinitesimal characters ((131), (132)). In fact, all of the remaining results used in Section 8 easily carry over to the more general setting, except for the injectivity of Lift_0 (Proposition 5.4). In particular, using the pairing (187) in the proof of Proposition 5.3, we see that for any ${}^\vee G$-orbit $S_G \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ we still have

$$\mathrm{Lift}_0\left(\eta_{S_G}^{\mathrm{loc}}(\sigma)(\delta_q)\right) = M(\epsilon(S_G), 1)^\sim.$$

It is explained in [Art13]**p.* 31 that Lift_0 is injective when G is not isomorphic to SO_N for even N. However, when $G \cong \mathrm{SO}_N$ for even N, the endoscopic lifting map is only injective on O_N-orbits [Art13]**pp.* 12, 31). That is to say,

$$\mathrm{Lift}_0\left(\eta_{S_1}^{\mathrm{loc}}(\delta_q)\right) = \mathrm{Lift}_0\left(\eta_{S_2}^{\mathrm{loc}}(\delta_q)\right)$$

for ${}^\vee G$-orbits of complete geometric parameters if and only if $S_2 = \tilde{w} \cdot S_1$ for $\tilde{w}$ as in (178). One might hope to retain injectivity by restricting the infinitesimal character of S_1 to lie in ${}^\vee\mathcal{O}_G$, but this too fails as it is not difficult to construct singular examples in which ${}^\vee\mathcal{O}_G = \tilde{w} \cdot {}^\vee\mathcal{O}_G$.

Recall decomposition (175)

$$\pi(S_\psi,1)^\sim = \sum_{(S,1)\in\Xi({}^\vee\mathcal{O},{}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} n_S\, M(S,1)^\sim.$$

As in the previous section, for each ${}^\vee\mathrm{GL}_N$-orbit S with $n_S \neq 0$ there exists a ${}^\vee G$-orbit $S_G \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ such that $\epsilon(S_G) = S$. The difference now is that when G is an even special orthogonal group the orbit S_G may not be uniquely determined in $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$. The lack of uniqueness forces us to weaken Theorem 8.2 in this context in that the ABV-packets in part (b) below are no longer necessarily disjoint.

Theorem 9.3. *(a) If G is not isomorphic to SO_N for even N then*

$$\eta^{\mathrm{Ar}}_{\psi_G} = \eta^{\mathrm{mic}}_{\psi_G}(\delta_q) = \eta^{\mathrm{ABV}}_{\psi_G} \quad \text{and} \quad \Pi^{\mathrm{Ar}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G}.$$

(b) If N is even and $G \cong \mathrm{SO}_N$ then

$$\eta_{\psi_G} = \frac{1}{2}\left(\eta^{\mathrm{mic}}_{\psi_G}(\delta_q) + \eta^{\mathrm{mic}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}(\delta_q)\right) = \frac{1}{2}\left(\eta^{\mathrm{ABV}}_{\psi_G} + \eta^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}\right)$$

and

$$\Pi^{\mathrm{Ar}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G} \cup \Pi^{\mathrm{ABV}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}.$$

Proof. The proof of the first assertion is completely the same as the proof of Theorem 8.2(a), since the injectivity of Lift_0 holds. Suppose therefore that N is even and $G \cong \mathrm{SO}_N$. As in the proof of Theorem 8.2 we have

$$\begin{aligned}
\mathrm{Lift}_0\left(\eta^{\mathrm{mic}}_{\psi_G}(\delta_q) + \eta^{\mathrm{mic}}_{\mathrm{Int}(\tilde{w})\circ\psi_G}(\delta_q)\right) &= 2\pi(S_\psi,1)^\sim \\
&= \sum_j 2n_S\, M(S,1)^\sim \\
&= \mathrm{Lift}_0\left(\sum_{S_G} n_{\epsilon(S_G)}(\eta^{\mathrm{loc}}_{S_G}(\delta_q) + \eta^{\mathrm{loc}}_{\tilde{w}\cdot S_G}(\delta_q))\right).
\end{aligned}$$

Since Lift_0 is injective on O_N-orbits of stable virtual characters, the second assertion follows. □

10 The comparison of Problems B-E

Theorem 9.3 is a comparison of the solutions to Problem A of Arthur and Adams-Barbasch-Vogan. Let us compare the remaining problems of the introduction.

Problem E, concerning the unitarity of the representations in the A-packets, stands apart from Problems B-D. It is also easy to dispense with. Arthur proves that Π_{ψ_G} consists of unitary representations ([Art13]*Theorem 2.2.1 (b)), and so by Theorem 9.3, every packet $\Pi^{\mathrm{ABV}}_{\psi_G}$ also consists of unitary representations.

For problems B-D, we review Arthur's approach first. The stable virtual character $\eta^{\mathrm{Ar}}_{\psi_G}$ is written

$$\eta^{\mathrm{Ar}}_{\psi_G} = \sum_{\sigma\in\tilde{\Sigma}_{\psi_G}} < s_{\psi_G}, \sigma > \sigma \tag{195}$$

as in [Art13]*(7.1.2). Here, $\tilde{\Sigma}_{\psi_G}$ is a finite set of non-negative integral linear combinations

$$\sigma = \sum_{\pi\in\Pi_{\mathrm{unit}}(G(\mathbb{R}))} m(\sigma,\pi)\,\pi$$

of irreducible unitary characters of $G(\mathbb{R}) = G(\mathbb{R},\delta_q)$. Furthermore, there is an injective map from $\tilde{\Sigma}_{\psi_G}$ into the set of those quasicharacters of

$$A_{\psi_G} = {}^\vee G_{\psi_G}/({}^\vee G_{\psi_G})^0$$

which are trivial on the centre of ${}^\vee G$. The injection is denoted by

$$\sigma \mapsto < \cdot, \sigma > .$$

The element s_{ψ_G} is the image of

$$\psi_G\left(1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

in A_{ψ_G}. The element s_{ψ_G} is clearly of order two. It is easy to rewrite (195) as

$$\eta^{\mathrm{Ar}}_{\psi_G} = \sum_{\pi \in \Pi_{\psi_G}} \left(\sum_{\sigma \in \tilde{\Sigma}_{\psi_G}} m(\sigma, \pi) \ < s_{\psi_G}, \sigma > \right) \pi \tag{196}$$

(*cf.* [Art13]*Proposition 7.4.3 and (7.4.1)). By defining a finite-dimensional representation

$$\tau_{\psi_G}(\pi) = \bigoplus_{\sigma \in \tilde{\Sigma}_{\psi_G}} m(\sigma, \pi) \ < \cdot, \sigma >, \tag{197}$$

equation (196) becomes

$$\eta^{\mathrm{Ar}}_{\psi_G} = \sum_{\pi \in \Pi_{\psi_G}} \mathrm{Tr}\left(\tau_{\psi_G}(\pi)(s_{\psi_G})\right) \pi. \tag{198}$$

The finite-dimensional representations defined in (197) provide a solution to Problem B. Equation (198) is close to a complete resolution of Problem C. We also need to show that the quasicharacters $< \cdot, \sigma >$ occurring in a given $\tau_{\psi_G}(\pi)$ have the same value ε_π at s_{ψ_G}. In this way the trace $\mathrm{Tr}\left(\tau_{\psi_G}(\pi)(s_{\psi_G})\right)$ would reduce to $\varepsilon_\pi \cdot \dim(\tau_{\psi_G}(\pi))$ as expected. We return to this point when we compare with $\eta^{\mathrm{ABV}}_{\psi_G} = \eta^{\mathrm{mic}}_{S_{\psi_G}}(\delta_q)$ (49) below.

Problem D concerns endoscopic lifting from an endoscopic group G' of G. The endoscopic group G' is defined to be a quasisplit form of a complex reductive group whose dual ${}^\vee G'$ is the identity component of the centralizer in ${}^\vee G$ of a semisimple element $s \in {}^\vee G$ (*cf.* Section 5 and [Art13]*Theorem 2.2.1(b)). Furthermore, the element s is taken to centralize the image of ψ_G, and there is a natural embedding $\epsilon' : {}^\vee(G')^\Gamma \hookrightarrow {}^\vee G^\Gamma$. Arthur's solution to Problem D tells us that if

$$\psi_G = \epsilon' \circ \psi_{G'}$$

for an A-parameter $\psi_{G'}$ then there exists a stable virtual character $\eta_{\psi_{G'}}$ on $G'(\mathbb{R})$ such that

$$\mathrm{Trans}^G_{G'}(\eta_{\psi_{G'}}) = \sum_{\pi \in \Pi^{\mathrm{Ar}}_{\psi_G}} \mathrm{Tr}\left(\tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s})\right) \pi \tag{199}$$

([Art13]*Theorem 2.2.1). Here, $\bar{s} \in A_{\psi_G}$ is the coset of s, and $\mathrm{Trans}^G_{G'}$ denotes the standard endoscopic lifting of Shelstad ([She83]). Observe that (198) is obtained from (199) by taking $s = 1$.

Now let us look at Problems B-D from the perspective of [ABV92]. Each $\pi \in \Pi^{\mathrm{ABV}}_{\psi_G}$ is of the form $\pi(\xi)$ for a unique complete geometric parameter $\xi = (S_\xi, \tau_\xi)$. Using this, we set

$$\tau^{\mathrm{ABV}}_{\psi_G}(\pi) = \tau^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi))$$

as in (133). This is a solution to Problem B. The solution to Problem C is then given by (139), which we may write as

$$\eta^{\mathrm{ABV}}_{\psi_G} = \sum_{\pi \in \Pi^{\mathrm{ABV}}_{\psi_G}} (-1)^{d(\pi) - d(S_{\psi_G})} \ \dim\left(\tau^{\mathrm{ABV}}_{\psi_G}(\pi)\right) \pi,$$

where $d(\pi) = d(S_\xi)$ for $\pi = \pi(\xi)$ as above. The solution to Problem D is given by [ABV92]*Theorem 26.25. Translated into the setting of (199), it reads as

$$\mathrm{Lift}^G_0\left(\eta^{\mathrm{ABV}}_{\psi_{G'}}\right) = \sum_{\pi \in \Pi^{\mathrm{ABV}}_{\psi_G}} (-1)^{d(\pi) - d(S_{\psi_G})} \ \mathrm{Tr}\left(\tau^{\mathrm{ABV}}_{\psi_G}(\pi)(\bar{s})\right) \pi \tag{200}$$

[ABV92]*Definition 24.15 and (26.17)(f). Here, Lift_0^G is the standard endoscopic lifting map of [ABV92]*Definition 26.18, which is defined on the stable virtual characters of $G'(\mathbb{R})$ and take values in the virtual characters of $G(\mathbb{R})$.

We wish to compare Arthur's solutions to Problems B-D with those of [ABV92]. This amounts to comparing (199) with (200). For this comparison, we shall, for the sake of simplicity, assume that

$$G = \mathrm{SO}_N, \quad N \text{ odd}$$

from now on. This assumption avoids the irksome complications arising from even special orthogonal groups in Theorem 9.3 (b). Under our assumption Theorem 9.3 tells us that the solutions of Arthur and Adams-Barbasch-Vogan to Problem A are identical.

In comparing (199) with (200), we must choose our endoscopic groups judiciously. Recall that A_{ψ_G} is the component group of the centralizer in ${}^\vee G$ of the image of ψ_G. The explicit description of this centralizer in [Art13]*(1.4.8) makes it clear that every element $\bar{s} \in A_{\psi_G}$ has a diagonal representative $\dot{s}$ in the centralizer with eigenvalues ± 1. The endoscopic group $G'(\dot{s})$ determined by $\dot{s}$ is a direct product $G_1'(\dot{s}) \times G_2'(\dot{s})$ in which each of the two factors is a special orthogonal group of odd rank ([Art13]**pp.* 13-14). The A-parameter $\psi_{G'(\dot{s})}$ decomposes accordingly as a product $\psi_{G_1'(\dot{s})} \times \psi_{G_2'(\dot{s})}$ of A-parameters ([Art13]*pp. 31, 36). Similarly, Arthur's stable virtual character $\eta_{\psi_{G'(\dot{s})}}$ is defined as the tensor product $\eta_{\psi_{G_1'(\dot{s})}} \otimes \eta_{\psi_{G_2'(\dot{s})}}$ ([Art13]*Remark 2 of Theorem 2.2.1). Hence, a particular instance of (199) reads as

$$\mathrm{Trans}_{G'}^G(\eta_{\psi_{G_1'(\dot{s})}} \otimes \eta_{\psi_{G_2'(\dot{s})}}) = \sum_{\pi \in \Pi_{\psi_G}} \mathrm{Tr}\,(\tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s}))\,\pi. \tag{201}$$

We now turn to rewriting the left-hand side of (201) so as to match it with the left-hand side of (200). First, it is noted on [ABV92]**p.* 289 that $\mathrm{Trans}_{G'}^G = \mathrm{Lift}_0^G$. Second, using the arguments in the proof of Corollary 6.2, we see that

$$\eta^{\mathrm{ABV}}_{\psi_{G'(\dot{s})}} = \eta^{\mathrm{ABV}}_{\psi_{G_1'(\dot{s})}} \otimes \eta^{\mathrm{ABV}}_{\psi_{G_2'(\dot{s})}}.$$

Third, since $G_1'(\dot{s})$ and $G_2'(\dot{s})$ are both odd rank special orthogonal groups, Theorem 9.3 (a) tells us that

$$\eta^{\mathrm{Ar}}_{\psi_{G_j'(\dot{s})}} = \eta^{\mathrm{ABV}}_{\psi_{G_j'(\dot{s})}}, \quad j = 1, 2.$$

Taking these three observations together we conclude

$$\begin{aligned}
\mathrm{Trans}_{G'}^G\left(\eta^{\mathrm{Ar}}_{\psi_{G'(\dot{s})}}\right) &= \mathrm{Lift}_0^G\left(\eta^{\mathrm{Ar}}_{\psi_{G_1'(\dot{s})}} \otimes \eta^{\mathrm{Ar}}_{\psi_{G_2'(\dot{s})}}\right) \\
&= \mathrm{Lift}_0^G\left(\eta^{\mathrm{ABV}}_{\psi_{G_1'(\dot{s})}} \otimes \eta^{\mathrm{ABV}}_{\psi_{G_2'(\dot{s})}}\right) \\
&= \mathrm{Lift}_0^G\left(\eta^{\mathrm{ABV}}_{\psi_{G'(\dot{s})}}\right).
\end{aligned}$$

It is now immediate from (199) and (200) that

$$\sum_{\pi \in \Pi_{\psi_G}} \mathrm{Tr}\,(\tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s}))\,\pi = \sum_{\pi \in \Pi_{\psi_G}} (-1)^{d(\pi) - d(S_{\psi_G})}\ \mathrm{Tr}\left(\tau^{\mathrm{ABV}}_{\psi_G}(\pi)(\bar{s})\right)\,\pi$$

for any $\bar{s} \in A_\psi$. By the linear independence of characters on $G(\mathbb{R})$

$$\mathrm{Tr}\,(\tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s})) = (-1)^{d(\pi) - d(S_{\psi_G})}\ \mathrm{Tr}\left(\tau^{\mathrm{ABV}}_{\psi_G}(\pi)(\bar{s})\right)$$

for any $\bar{s} \in A_\psi$. This may be regarded as an equality between virtual (quasi)characters on A_ψ (*cf.* (197)). By appealing to the linear independence of these (quasi)characters we conclude that

$$\tau_{\psi_G}(\pi)(s_{\psi_G}) = (-1)^{d(\pi) - d(S_{\psi_G})}$$

and

$$\tau_{\psi_G}(\pi) = \tau^{\mathrm{ABV}}_{\psi_G}(\pi).$$

The former equation gives a complete solution to Arthur's approach to Problem C.

This completes our solution of Problems B-D for odd rank special orthogonal groups. A similar argument holds for symplectic and even orthogonal groups, keeping in mind the element $\tilde{w}$ of Theorem 9.3 (b) when comparing virtual characters on $G(\mathbb{R})$ or $G'(\mathbb{R})$. We leave the details to the interested reader.

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