Lifting of Characters and Harish-Chandra's Method of Descent

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Suppose G is a reductive algebraic group defined over a local field F. One of the fundamental difficulties in Langlands' program for understanding representations of G(F) (and automorphic forms over global fields) is L-indistinguishability: irreducible representations are most easily understood not separately but in finite sets (L-packets). The members of a single L-packet are said to be L-indistinguishable. The resolution of this difficulty, which is still far from complete, is the goal of the theory of endoscopic groups and lifting, developed for example in [LS] and [S1]. (Langlands and Shelstad prefer the term transfer to lifting.) This theory seeks to relate representation theory on G(F) to representation theory on a collection of smaller groups, the endoscopic groups of G. Viewed from the perspective of an endoscopic group, the representations in an L-packet look a little different from each other. Taking into account the differences as seen from all possible endoscopic groups, one hopes to understand individual representations .

There are two problems. The first is that it turns out to be very difficult to formulate precisely the theorems that ought to be true. The second is that, once formulated, the theorems are very difficult to prove. The L-packets themselves have been defined completely only if F is $\mathbb{R}$ or $\mathbb{C}$, so one can hope for complete results only in that case. Shelstad has proved such results in [S1]. Our purpose is to reprove her results using an entirely different definition of lifting. We have two motivations. First, our approach leads to cleaner formulations of the results. (We will return to this point later in the introduction.) Second, we believe that the new definition of lifting sheds a great deal of light on Arthur's conjectures ([A1], [A2]), of which we are now able to give a very precise formulation. The proof of at least some of the conjectures (for $\mathbb{R}$ and $\mathbb{C}$) is thereby reduced to a difficult but (we hope) accessible representation-theoretic problem. We hope to return to this in a later paper.

In order to understand why Shelstad's results are at all unsatisfactory, we need to review briefly the philosophy of endoscopy. (It is not very important for the moment to know the definitions of the various objects under consideration; what matters are their formal relationships.) Details for this paragraph may be found in [Bo]. Write $\Pi(G(F))$ for the set of equivalence classes of irreducible admissible repesentations of $G(F)$. Recall the definition of the dual group ${}^\vee G$ (sometimes called ${}^L G^0$), here taken to be a complex reductive algebraic group. One of Langlands' basic ideas is that representations of $G(F)$ ought to be closely related to ${}^\vee G$. More precisely, write ${}^\vee G^\Gamma$ for the (Galois form of the) L-group of G. This is (among other things) an extension

0.1 $\quad 1 \rightarrow {}^\vee G \rightarrow {}^\vee G^\Gamma \rightarrow \Gamma \rightarrow 1$

with Γ the Galois group of F. The extension is determined by the (inner class of the) F-form of G. If G is F-split then ${}^\vee G^\Gamma$ is just the direct product of ${}^\vee G$ with Γ. Now let W'_F be the Weil-Deligne group of F. This group is also endowed with a homomorphism

0.2 $\quad W'_F \rightarrow \Gamma.$

Write $\Phi(G(F))$ for the set of ${}^\vee G$ -conjugacy classes of admissible homomorphisms

0.3 $\quad \varphi: W'_F \rightarrow {}^\vee G^\Gamma.$

(The most important part of the definition of admissible is that φ should respect the maps to Γ in 0.1 and 0.2).

0.4 Conjecture (Langlands-Deligne):

To any φ in $\Phi(G(F))$ there corresponds a finite subset Π_φ of $\Pi(G(F))$. The various subsets Π_φ should partition $\Pi(G(F))$: that is, each irreducible representation should belong to exactly one subset.

Deligne's contribution here is in replacing the Weil group by the Weil-Deligne group. The sets Π_φ are the L-packets of the first paragraph. If F is $\mathbb{R}$ or $\mathbb{C}$, the conjecture is proved in [L].

Having defined L-packets, we can now say in what sense the whole packet is easier to understand than individual representations in it. Recall that an irreducible admissible representation π of G(F) has a distribution character Θ_π, which may be regarded as a smooth function on the (open dense) set $G(F)_{sr}$ of strongly regular semisimple elements of G(F). (Recall that the character is actually a generalized function rather than a distribution; no choice of Haar measure is involved in associating a function to it.) This function is automatically constant on G(F) conjugacy classes. Write $\bar{F}$ for the algebraic closure of F. A sum of characters of G(F) is called stable if the corresponding function is constant on the intersection with $G(F)_{sr}$ of conjugacy classes in $G(\bar{F})$. Now conjugacy classes behave a little more simply over algebraically closed fields, so stable sums of characters are a little less complicated than individual characters. Stability is related to L-packets by

0.5 <u>Conjecture</u> (Langlands):

Fix an L-packet Π_φ, and assume that one of the representations in Π_φ is tempered. Then all of them are, and there are integers n_π (unique up to a common multiple) so that

$$\Theta(\varphi)=\Sigma_{\pi\in\Pi_\varphi} n_\pi \Theta_\pi$$

is stable.

This has been proved over $\mathbb{R}$ and $\mathbb{C}$ [S3].

We turn now to the problem of parametrizing the representations in a single L-packet. Fix a map φ from W'_F to ${}^\vee G^\Gamma$ (representing an equivalence class) in $\Phi(G(F))$. Define

0.6 (a) S_φ = centralizer in ${}^\vee G$ of the image of φ.

The group S_φ is an algebraic group, often disconnected. We mention two examples. If φ corresponds to a generic spherical representation of a split group, then S_φ is a maximal torus in ${}^\vee G$. On the other hand, if S_φ corresponds to an L-packet of discrete series representations over $\mathbb{R}$, then S_φ is the group of elements of order two in a maximal torus in ${}^\vee G$. Finally, define

0.6 (b) $\mathbb{S}_\varphi$ = group of connected components of S_φ.

0.7 <u>Naive Conjecture</u>:

There is an injective correspondence $\pi \to a_\pi$ from Π_φ to the set $\mathbb{S}_\varphi\hat{}$ of irreducible characters of $\mathbb{S}_\varphi\hat{}$. The integers n_π in Conjecture 0.5 may be taken to be $a_\pi(1)$ (the dimensions of the corresponding irreducible representations of $\mathbb{S}_\varphi$).

As stated, Conjecture 0.7 may actually be true. To see why it is naive, consider an L-packet of discrete series in $SL(2,\mathbb{R})$. This consists of two representations differing by an outer automorphism of $SL(2,\mathbb{R})$ (and therefore difficult to distinguish intrinsically). The group $\mathbb{S}_\varphi$ has two elements, and therefore two irreducible characters, so the conjecture is true. Unfortunately the two characters of $\mathbb{S}_\varphi$ look quite different, so any bijection from Π_φ to $\mathbb{S}_\varphi\hat{}$ must be to some extent unnatural.

Setting such difficulties aside, suppose that Conjecture 0.7 holds. For any element s in S_φ, we can then form the invariant distribution

0.8 $\quad \Theta(\varphi)(s) = \Sigma_{\pi\in\Pi_\varphi} a_\pi(s)\Theta_\pi.$

Of course this will depend only on the image of s in $\mathbb{S}_\varphi$. Elementary character theory for finite groups allows one to express each irreducible character Θ_π as a linear combination of various $\Theta(\varphi)(s)$. (This uses the injectivity of the correspondence of Conjecture 0.7.) The theory of endoscopy seeks (among other things) to describe the distributions $\Theta(\varphi)(s)$ in terms of stable characters on smaller groups. Notice that $\Theta(\varphi)(1)$ is just the stable character $\Theta(\varphi)$ of Conjecture 0.5.

To define endoscopic groups, one must first introduce a class of (closed) subgroups of ${}^\vee G^\Gamma$. Let ${}^\vee\mathcal{H}$ be such a subgroup. Assume

0.9 (1) the natural projection from ${}^\vee\mathcal{H}$ to Γ (coming from (0.1)) is surjective;

(2) its kernel is the identity component of the centralizer in $^\vee G$ of a semisimple element s of $^\vee G$,

(3) s centralizes all of $^\vee\mathcal{H}$.

These conditions appear at first to be unwieldy and unnatural. They arise in practice in the following way. Fix an admissible map φ from W'_F to $^\vee G^\Gamma$. If s is any semisimple element of S_φ (cf. (0.6)(a)) define $^\vee H$ to be the identity component of the centralizer of s in $^\vee G$. Then the closed subgroup

$$0.10 \qquad {}^\vee\mathcal{H} = \overline{<{}^\vee H, \varphi(W'_F)>}$$

generated by $^\vee H$ and the image of φ satisfies (0.9). An endoscopic group for G is (roughly speaking) a quasisplit group H over F of which the L-group $^\vee H^\Gamma$ is isomorphic to a subgroup $^\vee\mathcal{H}$ satisfying (0.6). We must say "roughly" for two reasons. First, one wants to allow all the subgroups $^\vee\mathcal{H}$ of 0.10, and these need not be L-groups of anything. Second, one wants to keep track of substantially more than the isomorphism class of H to specify an endoscopic group; most importantly the $^\vee G$ conjugacy class of $^\vee H$, but also something about the element s.

0.11 <u>Naive Conjecture</u>:

Suppose H is an endoscopic group for G. Then there is a natural map <u>Lift</u> from stable linear combinations of characters on H to linear combinations of characters on G. Suppose φ is an admissible map from W'_F to the L-group of H. Since we are regarding this L-group as a subgroup of $^\vee G^\Gamma$, we may also regard φ as a map to the L-group of G. (It may or may not be admissible.) Assume that the representations in $\Pi_\varphi(H(F))$ are tempered. Then those in $\Pi_\varphi(G(F))$ are as well, and

$$\mathrm{Lift}(\Theta_H(\varphi)) = \Theta_G(\varphi)(s).$$

Here the character on the right is taken to be zero if φ is not admissible for G.

This conjecture provides the desired description of the distributions 0.8 in terms of stable characters on endoscopic groups (assuming that Lift is well understood).

In case F = $\mathbb{R}$ Shelstad has proved versions of Conjectures 0.5, 0.7, and 0.11 (cf. [S1]). The most obviously unsatisfactory aspect of her results is the large number of choices required, particularly in the correspondence of Conjecture 0.7. For us the most compelling evidence that there was more to be said was the following example. Suppose $G(\mathbb{R})$ is $U(p,q)$, the group of linear transformations of $\mathbb{C}^n$ preserving a Hermitian form of signature (p,q). (Here n is p+q.) An L-packet of discrete series for G has $\binom{n}{p}$ elements. The corresponding group $\mathbb{S}_\varphi$ has order 2^n. Shelstad's correspondence from Π_φ to $\mathbb{S}_\varphi\hat{}$ requires the choice of one element of Π_φ arbitrarily; it is declared to correspond to the trivial character of $\mathbb{S}_\varphi$. The rest of the correspondence is natural. To get nice formulas for irreducible characters, Shelstad declares that the characters of $\mathbb{S}_\varphi$ not arising in her correspondence are associated to the zero character on $G(\mathbb{R})$. Suppose instead that we look at all the groups $U(p,q)$ (for fixed p+q) at once. There are n+1 of these groups (we ignore for the moment the isomorphism $U(p,q) \approx U(q,p)$) all having the same L-group ${}^\vee G^\Gamma$. A single map φ of $W_{\mathbb{R}}$ into ${}^\vee G^\Gamma$ parametrizes one L-packet of discrete series in each of the n+1 groups: a total of $\Sigma_p \binom{n}{p}$, or 2^n representations. It is natural to hope that these might be associated in some way with all 2^n characters of $\mathcal{S}_\varphi$. One of the main results of this paper (Theorem 5.1) provides (for real groups) at least a partial realization of this hope. Before describing the result, we want to restate the basic conjectures above to take this point of view into account.

We still fix a local field F and a group ${}^\vee G^\Gamma$ as in 0.1. Now, however, we think of G as defined over the separable algebraic closure $\bar{F}^s$ of F, and consider all the F-forms of G having L-group ${}^\vee G^\Gamma$. (These will be exactly the inner forms of a single quasisplit F-form of G.) It is convenient for the formulation of the ideas to fix representatives $\{ G_i \}$ for the equivalence classes of F-forms. If G has a center, one needs a more stringent notion of equivalence than the usual one. We call this conjectural notion strong equivalence. We do not know how to formulate it except over $\mathbb{R}$ (cf. §1). The main requirement is that strongly equivalent F-forms G' and G" should come equipped with an F-isomorphism, unique up to an inner automorphism from G'(F). Such an isomorphism gives rise to a canonical bijection between the representations of G'(F) and those of G"(F). A small technical penalty of this requirement is there may be infinitely many strong equivalence classes of F-forms (if the center of G is infinite.) Each G_i is simply G endowed with a certain action of the Galois group Γ on $G(\bar{F}^s)$. A super L-packet will be a collection of representations of each $G_i(F)$.

Write $\Pi(G)$ for the union of the various $\Pi(G_i(F))$. Using the bijections discussed in the parenthetical remark above, $\Pi(G)$ may be regarded as containing the representations of any one of our F-forms of G. A homomorphism φ of W'_F into ${}^\vee G^\Gamma$ is called quasiadmissible if it is admissible for at least one G_i. (It is equivalent to require φ to be admissible for the quasisplit inner form, or simply to drop the "relevancy" requirement ([Bo], 8.2(ii)).) Write $\Phi(G)$ for the set of $^\vee G$ conjugacy classes of such quasiadmissible homomorphisms. We can now state

0.12 Super-Conjecture:

To any φ in $\Phi(G)$ there corresponds a subset Π_φ of $\Pi(G)$. The various subsets Π_φ should partition $\Pi(G)$: that is, each irreducible representation of each $G_i(F)$ should belong to exactly one subset.

Modulo the problem of formulating the notion of strong equivalence, this is not really different from Conjecture 0.4. Of course Π_φ will be a super L-packet.

We turn now to stability. Suppose we are given a virtual character Θ_i of each group $G_i(F)$. We say that the (formal symbol) $\Sigma_i \Theta_i$ is a super character. It is called super-stable (cf. §8) if $\Theta_i(g)$ is equal to $\Theta_i(g')$ whenever both terms are defined and g and g' are G-conjugate strongly regular elements. (Here Θ_i is the function representing the character on the strongly regular semisimple elements.) Thus a super-stable character corresponds to a class function on the set of strongly regular elements of $G(\bar{F}^s)$ that belong to one of our inner forms of G. We can now formulate

0.13 Super-Conjecture:

Fix a super L-packet Π_φ , and assume that one of the representations in Π_φ is tempered. Then all of them are, and there are integers n_π (unique up to a common multiple) so that

$$\Theta^*(\varphi)=\Sigma_{\pi\in\Pi_\varphi} n_\pi \, \Theta_\pi \text{ is super-stable.}$$

This is substantially stronger than Conjecture 0.5, but it does little more than incorporate into that conjecture the case s=1 of Conjecture 0.11. (What has been added is a specification of Lift in that case.)

Before discussing the super version of conjecture 0.5, we consider an example. Consider the case $G=SL(2)$ and $F=\mathbb{R}$. There are two real forms of G, SU(1,1) and SU(2). In fact there are three strong real forms, which may be thought of as SU(2,0), SU(1,1) and SU(0,2). A super L-packet of discrete series therefore contains four elements: two discrete series for SU(1,1) and one each for SU(2,0) and SU(0,2). The group $\mathbb{S}_\varphi$ has order two, so $\mathbb{S}_\varphi\hat{}$ has just two elements. Thus we cannot hope to find an injection $\Pi_\varphi \hookrightarrow \mathbb{S}_\varphi\hat{}$.

This problem does not arise if the center of G is trivial. This suggests passing to the simply connected cover ${}^\vee G^{sc}$ of ${}^\vee G$. In fact (for F equal to $\mathbb{R}$) we introduce a covering ${}^\vee G^{can}$ of the dual group intermediate between ${}^\vee G$ and ${}^\vee G^{sc}$. The inverse image $\tilde{S}_\varphi$ of S_φ is a central extension

0.14 $\quad 1 \to Z \to \tilde{S}_\varphi \to S_\varphi \to 1$

of S_φ. Each strong real form G_i of G defines a character τ_i of Z. Let $\tilde{\mathbb{S}}_\varphi$ be the component group of $\tilde{S}_\varphi$.

0.15 <u>Super-Conjecture</u>:

There is an injective correspondence $\pi \to a_\pi$ from Π_φ to the set $\tilde{\mathbb{S}}_\varphi\hat{}$ of irreducible characters of $\tilde{\mathbb{S}}_\varphi$. To each of the F-forms G_i there is attached a sign ε_i so that the integers n_π in Super-Conjecture 0.5 may be taken to be $\varepsilon_i a_\pi(1)$. If π is a representation of G_i, then a_π restricted to (the image of) Z is τ_i.

In analogy with 0.8, Super-Conjecture 0.15 allows us to define for any s in $\tilde{S}_\varphi$ a super character

0.16 $\Theta^*(\varphi)(s) = \Sigma_{\pi\in\Pi_\varphi} \varepsilon_i a_\pi(s)\Theta_\pi$.

Here the sign ε_i is the one from Super-Conjecture 0.15 above. As before, character theory for the group $\tilde{S}_\varphi$ will permit the recovery of individual irreducible characters from a knowledge of all these super characters.

Finally, we consider lifting.

0.17 Super-Conjecture:

Suppose H is an endoscopic group for G. Then there is a natural map Lift* from stable characters on H to super-characters on G. Suppose φ_H is in $\Phi(H)$. Since we are regarding the L-group of H as a subgroup of $^\vee G^\Gamma$ (well-defined up to $^\vee G$ conjugacy) φ_H also defines an element φ_G of $\Phi(G)$. (The condition of quasi-admissibility is automatically inherited by φ_G.) Assume that the representations in Π_{φ_H} are tempered. Then those in $\Pi_\varphi(G(F))$ are as well, and

$$\mathrm{Lift}^*(\Theta_H(\varphi)) = \Theta^*(\varphi)(s) \text{ (notation (0.16).}$$

Super-Conjecture 0.14 guarantees that a stable character on a quasisplit inner form is more or less the same thing as a super-stable character. We could therefore consider the domain of super-lifting to be super-stable characters, and regard a super-endoscopic group as a group H over $\bar{F}^s$, endowed with an inner class of F-forms having L-group $^\vee\mathcal{H}$, where $^\vee\mathcal{H}$ is as in 0.6.

Super-Conjecture 0.17 forces on us the characters a_π of Super-Conjecture 0.15. The introduction of $\tilde{S}_\varphi$ also resolves another problem, that of the arbitrary choices necessary to define lifting.

We can now state a little more precisely what we do in this paper: we formulate and prove precise versions of the "Super-Conjectures" 0.13, 0.15, and 0.17 for real groups. For reasons explained below it is best to work first with regular infinitesimal character, and then pass to the singular case via the translation principle [Z]. The main results are Theorems 5.1 and 9.7 (in the regular infinitesimal character case), and we extend them to singular infinitesimal character in Theorems 10.19 and 10.42.

To say what this requires, we first list four problems alluded to in the discussion so far.

Problem 1. Suppose $^{\vee}\mathcal{H}$ is a subgroup of $^{\vee}G^{\Gamma}$ satisfying 0.9, and that H is the unique reductive group over $\bar{F}^s$ with dual group $^{\vee}H$. Then $^{\vee}\mathcal{H}$ need not be isomorphic to an L-group of H.

Problem 2. Suppose G_1 and G_2 are equivalent F-forms of G; that is, conjugation by some element of $G(\bar{F}^s)$ induces an F-isomorphism beween them. Then the isomorphisms induced by different elements of G may not differ by inner automorphisms of $G_i(F)$. In particular, there is no canonical bijection between representations of $G_1(F)$ and those of $G_2(F)$.

Problem 3. The set Π_{φ} can have more elements than there are representations of $\mathbb{S}_{\varphi}$. That is, the group $\mathbb{S}_{\varphi}$ is too small.

Problem 4. The group S_{φ} and the element s of 0.9 do not suffice to specify a canonical lifting map.

We can overcome these problems only when $F=\mathbb{R}$, as we assume for the rest of the paper. In that case Problem 1 was resolved rather completely in [AV2]: the group ${}^\vee\mathcal{H}$ is what we called an E-group of H. We showed that maps of $W_{\mathbb{R}}$ into an E-group of H parametrize certain representations of a covering group of $H(\mathbb{R})$; more precisely, of the preimage of $H(\mathbb{R})$ in a certain (connected but not necessarily algebraic) cover H^{can} of the complex group H. The first problem therefore requires only that we consider these projective representations as the appropriate domain of lifting. (The history of this idea is a little unclear; we will say more about it at the end of the introduction.)

The second problem is resolved by fixing as a base point a pair consisting of a quasisplit real form G_1 of G and a discrete series representation of $G_1(\mathbb{R})$ having a Whittaker model. The point of fixing the pair is that any element of $G(\mathbb{C})$ normalizing the pair acts by inner automorphisms on $G_1(\mathbb{R})$. Other real forms G' are described relative to this base point, in a slightly convoluted way that makes use of ideas peculiar to the real case (notably the Cartan involution). Details appear in Section 1.

We have already discussed the third problem above, and the fourth problem is solved almost automatically along with the third. For our definition of super-lifting, we need to choose (in the setting of 0.9) a preimage $\tilde{s}$ of s in ${}^\vee G^{can}$. With this choice, super-lifting is canonically defined.

Having touted the lack of arbitrary choice in our results, we should draw attention to the one place where some choice is still required. Given S and $\tilde{s}$ as in 0.9 and the preceding paragraph, we can get in a natural way an endoscopic group H. We need to choose an identification of ${}^\vee\mathcal{H}$ with an E-group of H, however. Different choices lead to different identifications of maps of $W_{\mathbb{R}}$ into ${}^\vee\mathcal{H}$ with representations of (real forms of) H. Given a map φ of $W_{\mathbb{R}}$ into ${}^\vee\mathcal{H}$, the super-character Lift*($\Theta^*(\varphi)$) is independent of choices; what varies is the super-character of H that we attach to φ. Even this cannot vary much; the situation is described precisely in Lemma 1.5 and the remarks after it.

Our methods are quite different from those of [S1], and we briefly describe them. Our main point of view is that lifting is dual to Harish-Chandra's method of descent. The duality in question is that of [V2], between characters of real forms of G and characters of real forms of ${}^\vee G$. The introduction of real forms of ${}^\vee G$ and the resulting symmetry between G and ${}^\vee G$ is a significant departure from the usual techniques. We obtain lifting of characters directly, without the use of orbital integrals. A consequence is a defintion of stable and super-stable characters in terms of duality. Both the definitions of lifting and of stability are equivalent to the usual ones (up to a constant in the former case) (cf. Theorems 8.3 and 9.17).

This formulation of stability is natural, in particular with respect to the notion of super-lifting. Furthermore it is well suited to unipotent representations, which we leave to another paper. In some sense this definition of lifting realizes a suggestion of Duflo that the constants arising in the definition of lifting might be related to those coming from the method of descent (in our case, on the dual group).

The duality methods of [V2] are best suited to regular infinitesimal character. Thus we work until section 9 entirely with this restriction, and the passage to singular infinitesimal character is carried out in section 10.

Here is an outline of the contents of the paper. Section 1 introduces some structure theory, including the notion of a strong real form. Section 2 describes the translation principle in this context. Section 3 recalls and reformulates the relationship established in [V2] between representations of real forms of G and real forms of $^\vee G$; this will be our main technical tool. This section may be considered a realization of the idea expressed in the mysterious Section 15 of [V2]. Section 4 extends the results of this section to allow for covering groups and non-integral infinitesimal character. Section 5 discusses the pairing $\Pi \times \tilde{\mathbb{S}}_\varphi \to \mathbb{C}^*$, and some calculations needed to identify our lifting with Shelstad's. Section 6 introduces endoscopic groups in a convenient form. Section 7 recalls the reformulation in [AV1] of results of [DHV] and [B] on Harish-Chandra's method of descent. Section 8 gives a convenient characterization of stability and super-stability (Definition 8.1 and Theorem 8.3). Section 9 defines lifting from H to G in terms of descent from $^\vee G$ to $^\vee H$ (using the relationships from Sections 2-4). The new definition is compared with Shelstad's, and an appropriate version of Conjecture 0.11 is established. The extension to singular infinitesimal character is made in Section 10. We discuss some examples in Section 11, and we strongly recommend the reader read the paper with these examples in mind.

We conclude this introduction with a remark about the problem of embeddings of L-groups (Problem 1 above). When we first began to circulate these ideas a few years ago, we believed that the idea of introducing coverings of H to resolve it was entirely original. Now it appears to be a minor variation on the approach in [LS], which in turn originates in [L2]. What our formulation has to recommend it is perhaps a minimization of arbitrary choices.

§ 1

Structure Theory

In the first three sections we set up the machinery for the rest of the paper, starting with some structure theory. Most of this is from [AV2].

Let G be a connected reductive algebraic group. Let ${}^LG^0$ be the dual group for G, i.e. with datum dual to that for G [Bo]. For G defined over $\mathbb{R}$ let ${}^LG$ be the L-group of G [Bo]. In order to exploit fully the symmetry between the roles of G and ${}^LG^0$, we use different notation.

We first introduce the category of L-groups with which we will be working.

1.1 Definition:

An extended group containing G is a pair $(G^\Gamma, \mathfrak{D})$ where:

(1) G^Γ is an algebraic group containing G as a subgroup of index 2.

(2) $\mathfrak{D}$ is a conjugacy class of pairs (δ, B), where:

(a) δ is an element of G^Γ not contained in G,

(b) B is a Borel subgroup of G.

(3) Fix $(\delta, B) \in \mathfrak{D}$, and let $\theta(g) = \delta g \delta^{-1}$ $(g \in G)$. Then

(a) θ is a principal involution (cf. [AV2], Definition 6.13),

(b) θ normalizes B.

(4) Let T be a θ-stable maximal torus in B. Then every simple root of T is either complex or non-compact imaginary with respect to θ.

A Borel subgroup satisfying condition 4 is said to be large, cf. [V2] for motivation. Conditions 3 and 4 are independent of the choice of (δ,B). We write $\delta\in\mathcal{D}$ to indicate $(\delta,B)\in\mathcal{D}$ for some B. The condition that θ be a principal involution says that it is the Cartan involution of a quasi-split group ([AV2], Definition 6.13).

We will often drop the extra data from the notation and call G^Γ an extended group. A morphism between extended groups $(G^\Gamma,\mathcal{D})$ and $(G'^\Gamma,\mathcal{D}')$ is by definition a group morphism $G^\Gamma\rightarrow G'^\Gamma$,which takes G to G' and $G^\Gamma-G$ to $G'^\Gamma-G'$. By 3,

1.2 $\qquad \delta^2\in Z(G)^\theta,$

the θ-invariants of the center of G. (Note that conjugation by an element $x\in G^\Gamma-G$ gives an automorphism of Z(G), independent of the choice of x). It follows that an extended group is defined up to isomorphism once a conjugacy class of involutions $\{\theta\}$ and an element $z\in Z(G)^\theta$ have been chosen. For let δ be a formal element, and define G^Γ as a set by $G^\Gamma=G\cup\delta G$. Define multiplication by multiplication in G, and the relations $\delta g\delta^{-1}=\theta(g)$ and $\delta^2=z$. This uniquely determines the isomorphism class of G^Γ. We may pick any Borel subgroup satisfying condition 4, and choose $\mathcal{D}=G(\delta,B)$ to complete the definition of $(G^\Gamma,\mathcal{D})$. Given z, any two such groups are isomorphic by an isomorphism which is canonical up to conjugation by G.

The choice of conjugacy class $\mathcal{D}$ corresponds precisely to the choice of splitting in the definition of an L-group ([L], see also ([AV2], §9).

Given G (defined over $\mathbb{R}$) let ${}^\vee G$ be a connected reductive algebraic group (called ${}^LG^0$ in the previous notation) with root datum dual to that for G. We obtain an extended group containing ${}^\vee G$ as follows ([AV2], Definition 9.3). Let γ_0 denote the anti-holomorphic involution of G defining the real form, i.e. $G^{\gamma_0}=G(\mathbb{R})$. Let γ_0^s denote the anti-holomorphic involution defining a split real form of G. Then $(\gamma_0)^{-1}\gamma_0^s$ is a holomorphic automorphism. Let γ denote the image of $(\gamma_0)^{-1}\gamma_0^s$ in the outer automorphism group $Out(G)\approx Aut(G)/Int(G)$ of G; γ only depends on the conjugacy class of γ_0. Let ${}^\vee\gamma$ denote the element of $Out({}^\vee G)$ corresponding to γ via the natural isomorphism $Out(G)\approx Out({}^\vee G)$. Then ${}^\vee\gamma$ is independent of all choices.

We say two elements of order two in Aut(G) are inner to each other if they have the same image in Out(G). If $\gamma\in Out(G)$ is an element of order two, we refer to the set $\mathcal{C}(\gamma)$ of real forms of G giving rise to γ by the above construction as an inner class.

For T a Cartan subgroup of G, we let $Q^*(G,T)\subset X^*(T)\subset P^*(G,T)$ denote the root lattice, the character lattice, and the weight lattice respectively. We let $Q_*(G,T)\subset X_*(T)\subset P_*(G,T)$ denote the coroot lattice, the lattice of one-parameter subgroups, and the coweight lattice respectively. We let $R(G,T)$ be the roots of T in G. If $\alpha\in R(G,T)$ then $\alpha^\vee\in Q_*(G,T)$ denotes the corresponding coroot.

Choose a Cartan subgroup ${}^dT$ of ${}^\vee G$ and a Borel subgroup ${}^dB$ containing ${}^dT$. Let $\rho\in P_*({}^\vee G,{}^dT)$ be one-half the sum of the coroots of ${}^dT$ in ${}^dB$. Let ${}^\vee z_\rho$ be its image in $Z({}^\vee G)$ under the isomorphism $Z({}^\vee G)\approx P_*({}^\vee G,{}^dT)/X_*({}^dT)$. This is independent of the choices.

1.3 Definition:

An L-group for G is an extended group $({}^\vee G^\Gamma,{}^\vee\mathcal{D})$ containing ${}^\vee G$ where the action of ${}^\vee\delta\in{}^\vee\mathcal{D}$ on ${}^\vee G$ by conjugation is inner to ${}^\vee\gamma$, and ${}^\vee\delta^2={}^\vee z_\rho$. Explicitly:

(1) ${}^\vee G^\Gamma$ is an algebraic group containing ${}^\vee G$ as a subgroup of index 2.

(2) ${}^\vee\mathcal{D}$ is the conjugacy class of pairs $({}^\vee\delta,{}^\vee B)$, where:

(a) $^\vee\delta$ is an element of $^\vee G^\Gamma$ not contained in $^\vee G$,

(b) $^\vee B$ is a Borel subgroup of $^\vee G$.

(3) Let $^\vee\theta(g)={}^\vee\delta g{}^\vee\delta^{-1}$ $(g\in{}^\vee G)$. Then

(a) $^\vee\theta$ is a principal involution inner to $^\vee\gamma$,

(b) $^\vee\theta$ normalizes $^\vee B$.

(4) $^\vee B$ is large with respect to $^\vee\theta$.

(5) $^\vee\delta^2={}^\vee z_\rho\in Z({}^\vee G)$.

The isomorphism class of $({}^\vee G^\Gamma,{}^\vee\mathcal{D})$ is independent of all choices, and $^\vee G^\Gamma$ is isomorphic to ^LG ([AV2], discussion following Definition 9.6). We will often drop the extra data from the notation and refer to $^\vee G^\Gamma$ as an L-group for G. Note that it depends only on the inner class of G (as a group defined over $\mathbb{R}$); we say $^\vee G^\Gamma$ is associated to an inner class.

The analogous situation with $^\vee G$ in place of G allows us to define an extended group G^Γ containing G as a subgroup of index two. That is, G^Γ is an extended group containing G associated to the inner class defined by $\mathcal{C}(\gamma)$. Alternatively, note that $^\vee\theta$ is the Cartan involution of a real form of $^\vee G$. Then G^Γ is an L-group for (the real group) $^\vee G$.

Thus our initial data will always be the following: a reductive algebraic group G and an inner class $\mathcal{C}(\gamma)$ of real forms of G. Associated to this we obtain G^Γ and $^\vee G^\Gamma$, unique up to isomorphism. Let θ be an involution of G defined by 1.1(3a). Note that this is the Cartan involution for a quasi-split real form of G.

The irreducible representations of real forms of G in a given inner class $\mathcal{C}(\gamma)$ are parametrized by homomorphisms of $W_\mathbb{R}$, the Weil group of $\mathbb{R}$, into $^\vee G^\Gamma$. Representations of certain covering groups of G play an important role. These are parametrized by homomorphisms of $W_{\mathbb{R}}$ into other extended groups containing $^\vee G$. Furthermore these extended groups for certain subgroups (e.g. Levi subgroups) of G arise naturally as subgroups of $^\vee G^\Gamma$.

1.4 Definition:

An E-group for G is an extended group $(^\vee G^\Gamma, {}^\vee\mathfrak{D})$ containing $^\vee G$ where the action of $^\vee\delta$ on $^\vee G$ by conjugation is inner to γ.

Thus we have merely dropped the assumption $^\vee\delta^2={}^\vee z_\rho$ from Definition 1.3. The following Lemma is immediate.

1.5 Lemma:

Suppose G and an inner class $\mathcal{C}(\gamma)$ of real forms have been given.

(1) An E-group for G is determined uniquely (up to an isomorphism which is canonical modulo conjugation by $^\vee G$) by an element contained in $Z(^\vee G)^{^\vee\theta}$. (Here $^\vee\theta\in Aut(^\vee G)$ is any involution of $^\vee G$ mapping to $^\vee\gamma\in Out(^\vee G)$.)

(2) Given $^\vee z_0\in Z(^\vee G)^{^\vee\theta}$, let $(^\vee G^\Gamma, {}^\vee\mathfrak{D}_0)$ be a corresponding E-group for G. Then the possible E-group structures $(^\vee G^\Gamma, {}^\vee\mathfrak{D})$ on $^\vee G^\Gamma$ are parametrized by

$$\{\, ^\vee z\in Z(^\vee G) \mid {}^\vee z\, {}^\vee\theta(^\vee z)=1\}/\{\, ^\vee w\, {}^\vee\theta(^\vee w^{-1}) \mid {}^\vee w\in Z(^\vee G)\,\}.$$

Explicitly, the structure $(^\vee G^\Gamma, {}^\vee\mathfrak{D})$ parametrized by the class of z as above has $^\vee\mathfrak{D}=\{(^\vee\delta z, {}^\vee B) \mid (^\vee\delta, B)\in {}^\vee\mathfrak{D}_0\}$.

For example an L-group for G is an extended group determined by $^\vee z_\rho$.

In terms of the Langlands classification, changing $^\vee\mathfrak{D}$ (by an element of the set in (2)) corresponds to a permutation of the admissible representations of a real form of G which takes π to $\pi\otimes\tau$, for some fixed one-dimensional representation τ.

We next discuss real forms of a given group. Recall that a real form of G is by definition the group of fixed points of an anti-holomorphic involution γ_0 of G; or simply the involution itself. We write $G(\mathbb{R})=G^{\gamma_0}$. Two real forms γ_0 and γ'_0 are said to be equivalent if $\gamma_0'=\mathrm{int}(g)\circ\gamma_0\circ\mathrm{int}(g)^{-1}$ for some $g\in G$.

Following [AV2] we prefer to parameterize equivalence classes of real forms by conjugacy classes of holomorphic involutions. Given a real form we obtain a holomorphic involution θ: let θ be a corresponding Cartan involution as in [AV2] (6.7).

1.6 Lemma ([AV2], Proposition 6.9):

Every holomorphic involution of G is the Cartan involution of some real form of G. Two holomorphic involutions are conjugate if and only if the corresponding real forms are equivalent. Thus there is a bijection between the set of equivalence classes of real forms, and conjugacy classes of holomorphic involutions.

Let $K=G^{\theta}$. Given θ we recover $G(\mathbb{R})$ as a real form of G which satisfies: $G(\mathbb{R})\cap K$ is a maximal compact subgroup of $G(\mathbb{R})$. Then the subgroup $G(\mathbb{R})$ of G is determined up to conjugation by K.

Let G^{Γ} be an extended group containing G. Let x be an element of $G^{\Gamma}-G$ such that x^2 is contained in the center $Z(G)$ of G, and let $\bar{x}$ denote the coset $xZ(G)$. Define an involution θ_x of G by $\theta_x(g)=xgx^{-1}$ $(g\in G)$. This defines a map from pairs $(G^{\Gamma}, \bar{x})$ to involutions of G. Say $(G^{\Gamma}, \bar{x})$ is equivalent to $(H^{\Gamma},\bar{x}')$ if there is an isomorphism $G^{\Gamma}/Z(G)\approx H^{\Gamma}/Z(H)$ taking the conjugacy class of $\bar{x}$ to the conjugacy class of $\bar{x}'$. The next Proposition follows immediately.

1.7 Proposition:

There is a bijection between equivalence classes of pairs $(G^\Gamma,\bar{x})$ and equivalence classes of real forms.

We need to eliminate the ambiguity due to the center of G in the preceding Proposition. We keep track not just of the Cartan involution θ, but of the element $x\in G^\Gamma$ (not just the coset $xZ(G)$) representing it. Thus:

1.8 Definition:

A strong real form of G is a pair (G^Γ,x), for G^Γ an extended group containing G, and $x\in G^\Gamma-G$, $x^2\in Z(G)$. Two strong real forms (G^Γ,x) and (G^Γ,y) are equivalent if y is conjugate to x by an element of G..

If G^Γ is given we will refer to x as a strong real form.

We will not need a notion of equivalence of strong real forms attached to different extension groups (we will generally have in mind a fixed group G^Γ).

For example SL(2) has two (equivalence classes) of real forms, but 3 strong real forms corresponding to SU(2,0), SU(1,1) and SU(0,2) respectively (cf. section 11).

Given (G^Γ,x) let θ denote the Cartan involution $\theta(g)=xgx^{-1}$ of G, and let $K_x=G^\theta$. We write θ_x when we want to emphasize the dependence on x. Write $G(\mathbb{R})$ for a real form of G with Cartan involution θ (i.e. $G(\mathbb{R})\cap K_x$ is a maximal compact subgroup of $G(\mathbb{R})$); this is defined up to conjugation by K_x. This defines a map, not necessarily injective, from equivalence classes of strong real forms, to equivalence classes of real forms. We write $G(\mathbb{R})_x$ when we want to emphasize the dependence on x. Note that the inner class of G is determined by G^Γ, and the equivalence class of its real form by the coset $xZ(G)$.

We will mostly work in the category $\mathcal{M}(\mathfrak{g},K_x)$ of $(\mathfrak{g},K_x)$-modules instead of $G(\mathbb{R})$-modules, so it is usually not necessary to choose $G(\mathbb{R})$ (the main exception is in section 7). The main results will all hold with "admissible representation of $G(\mathbb{R})$" in place of "admissible $(\mathfrak{g},K_x)$-module".

For later use we note that if π is a $(\mathfrak{g},K_x)$-module and $g\in G$, then we obtain a $(\mathfrak{g},K_{x'})$module in the natural way, with $x'=\mathrm{int}(g)x=gxg^{-1}$. We denote this module $g\pi$. A crucial point is that $g\pi$ depends only on x', and not on the particular element g taking x to x'.

We recall some facts about Cartan subgroups of G^Γ ([AV2],§9). By definition a Cartan subgroup of an extended group G^Γ is a subgroup T^Γ meeting both components of G^Γ, such that $T^\Gamma\cap G$ is a Cartan subgroup of G. Suppose (G^Γ,x) is a strong real form of G. A Cartan subgroup T^Γ is said to be θ_x-stable if $x\in T^\Gamma$; equivalently if T^Γ is generated by a θ_x-stable Cartan subgroup of G and by x. As the notation indicates given T^Γ, T will denote $T^\Gamma\cap G$; and given T θ_x-stable, we let $T^\Gamma=T\cup Tx$.

Let T^Γ be a θ_x-stable Cartan subgroup of G^Γ. Fix ${}^\vee\delta\in\mathcal{D}$, and let ${}^dT^\Gamma$ be a ${}^\vee\theta$-stable Cartan subgroup of ${}^\vee G^\Gamma$. Let ${}^\vee T^\Gamma$ be the L-group of the underlying Cartan subgroup $T=T^\Gamma\cap G$ of G, equipped with the real form associated to $\theta_x|_T$. Now ${}^\vee T$ comes equipped with a Cartan involution, call it ${}^\vee\theta_T$. Choosing Borel subgroups B and ${}^dB$ of G and ${}^\vee G$, containing T and ${}^dT$, we obtain a map ${}^\vee T\rightarrow {}^dT$ ([AV2],9.7). By changing the choice of Borel subgroups this may be modified arbitrarily by the Weyl group of T or ${}^dT$.

We say T is <u>dual</u> to ^dT if this map may be chosen to take $^\vee\theta_T$ to $(^\vee\theta)|_{^dT}$; we say <u>dual via $(B,{}^dB)$</u> if it is necessary to specify the role of the Borel subgroups. Given T^Γ, there exists $^dT^\Gamma$ such that T^Γ is dual to $^dT^\Gamma$ ([AV2], Lemma 9.13). Given x we say $^dT^\Gamma$ is <u>relevant</u> if there exists a θ_x-stable Cartan subgroup T^Γ such that T^Γ and $^dT^\Gamma$ are dual.

Given T^Γ and $^dT^\Gamma$ dual via $(B,{}^dB)$, we obtain an isomorphism $^\vee T\approx{}^dT$ as above. This induces isomorphisms $X^*(T)\approx X_*({}^dT)$, $W(G,T)\approx W(^\vee G,{}^dT)$, etc. We denote these isomorphism Υ, or $\Upsilon_{B,{}^dB}$ if it is necessary to specify the Borel subgroups.

Note that from the construction of $^\vee G$ we obtain a unique $G\times{}^\vee G$-conjugacy class $\mathcal{Z}_0$ of isomorphisms $^\vee T\approx{}^dT$ for T a Cartan subgroup of G and ^dT a Cartan subgroup of $^\vee G$. We refer to the elements $\Upsilon\in\mathcal{Z}_0$ as distinguished.

In general we will use the superscript ^{d} to indicate an object having to do with $^\vee G$, not necessarily originating from G in any specified way; for example ^dT denotes a general Cartan subgroup of $^\vee G$. We reserve the notation $^\vee$ for an object on the dual side related to G by some isomorphism as above. For example if T is a Cartan subgroup of G, then the dual torus $^\vee T$ is isomorphic to any Cartan subgroup ^dT of $^\vee G$, but this isomorphism is not canonical. Once $\Upsilon:{}^\vee T\rightarrow{}^dT$ is fixed for $\alpha\in R(G,T)$ we let $^\vee\alpha=\Upsilon(\alpha^\vee)\in Q^*(^\vee G,{}^dT)$ be the corresponding root of $^\vee T$ in $^\vee G$. (This is not to be confused with $\alpha^\vee\in Q_*(G,T)$, which is the corresponding coroot of T in G.) Similarly given ^dT, let $^dW=W(^\vee G,{}^dT)$, and write ^dw for a typical element of ^dW. Given T and $(B,{}^dB)$ as above, we obtain $\Upsilon:W\approx{}^dW$. Then for $w\in W$ we let $^\vee w=\Upsilon(w)$.

§2

The Translation Principle

We are going to make extensive use of the results of [V2]. Those results deal not so much with individual representations as with "translation families" of representations. To establish notation it is convenient to recall the necessary ideas. We work in a rather unusual setting, the usefulness of which will appear only later. For now we simply mention that most of the difficulties of this chapter do not arise when the representations in question have integral infinitesimal character.

As before we fix a complex reductive algebraic group G and a dual group ${}^\vee G$. Recall from section 1 that there is a distinguished $G\times{}^\vee G$-conjugacy class $\mathcal{Z}_0$ of isomorphisms

$$\mathfrak{J}: {}^\vee T \approx {}^d T$$

each of which identifies the dual group of a maximal torus T in G with some maximal torus ${}^d T$ in ${}^\vee G$. The following observation is just a reformulation of Harish-Chandra's theorem on the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$.

2.1 Proposition:

Let T and ${}^d T$ be maximal tori in G and ${}^\vee G$ respectively. The following four sets are in natural one-to-one correspondence:

a) infinitesimal characters for G; that is, homomorphisms from $\mathfrak{Z}(\mathfrak{g})$ to $\mathbb{C}$;

b) $W(G,T)$ orbits in $\mathfrak{t}^*$;

c) $W({}^\vee G,{}^d T)$ orbits in ${}^d\mathfrak{t}$; and

d) semisimple $Ad({}^\vee G)$-orbits on ${}^\vee\mathfrak{g}$.

If λ is an element of $\mathfrak{t}^*$ or ${}^d\mathfrak{t}$, or a semisimple element of ${}^\vee\mathfrak{g}$, we write χ_λ for the corresponding infinitesimal character for G.

We will be considering representations of subgroups G_c and ${}^\vee G_{{}^\vee c}$ of G and ${}^\vee G$ (or rather their Lie algebras). Thus we fix semisimple elements

2.2 (a) $c \in G$

$\quad {}^\vee c \in {}^\vee G.$

Define

2.2 (b) G_c = identity component of centralizer of c in G,

${}^\vee G_{{}^\vee c}$ = identity component of centralizer of ${}^\vee c$ in ${}^\vee G$.

The representations we consider will be $\mathfrak{g}_c$-modules for the most part.

We need a way to identify duals of tori in G_c with tori in ${}^\vee G_{{}^\vee c}$. Of course some of the isomorphisms in $\mathcal{Z}_0$ do exactly that; but there are too many of these isomorphisms. We therefore fix a $G_c \times {}^\vee G_{{}^\vee c}$-conjugacy class $\mathcal{Z}$ of isomorphisms

2.3 $\quad \zeta: {}^\vee T \to {}^d T;$

here for each ζ, T is a maximal torus in G_c (with dual torus ${}^\vee T$), and ${}^d T$ is a maximal torus in ${}^\vee G_{{}^\vee c}$. We assume that $\mathcal{Z}$ is contained in $\mathcal{Z}_0$ (of course if $G_c = G$ and ${}^\vee G_c = {}^\vee G$ this says $\mathcal{Z} = \mathcal{Z}_0$). If T and ${}^d T$ are fixed maximal tori in G_c and ${}^\vee G_{{}^\vee c}$ then the set of isomorphisms in $\mathcal{Z}$ from ${}^\vee T$ to ${}^d T$ is a single orbit of $W(G_c, T) \times W({}^\vee G_{{}^\vee c}, {}^d T)$.

Fix $\zeta \in \mathcal{Z}$ mapping ${}^\vee T$ to ${}^dT$. Then ζ identifies coroots of T in G with roots of ${}^dT$ in ${}^\vee G$. We can therefore define

2.4 $R_{c^\vee c}(\zeta) = \{ \alpha \in R(G_c,T) \mid \zeta(\alpha^\vee)({}^\vee c) = 1 \}$.

The corresponding set of roots of ${}^dT$ in ${}^\vee G_{^\vee c}$ is written ${}^dR_{c^\vee c}(\zeta)$. The definition is symmetric in G and ${}^\vee G$, as is clear if we rewrite (2.4) as

2.4' $R_{c^\vee c}(\zeta) = \{ \alpha \in R(G,T) \mid \alpha(c)=1 \text{ and } \zeta(\alpha^\vee)({}^\vee c) = 1 \}$.

Here is a fairly standard formulation of the Jantzen-Zuckerman translation principle. Most of the proof is in [Z] and [SV].

2.5 <u>Theorem</u>:

With notation as above, suppose T is a maximal torus of G_c. Fix a weight $\lambda \in \mathfrak{t}^*$, and a weight $\nu \in X^*(T)$ of a finite-dimensional representation of G. Assume that

i) for every coroot $\alpha^\vee$ of T in G_c such that $\lambda(\alpha^\vee)$ is a positive integer (respectively zero), $(\lambda+\nu)(\alpha^\vee)$ is a non-negative integer (respectively zero).

Then there is a translation functor $\Psi = \Psi_\lambda^{\lambda+\nu}$ from the category $\mathcal{M}(\mathfrak{g}_c, \chi_\lambda)$ (of $\mathfrak{g}_c$-modules with infinitesimal character χ_λ) to $\mathcal{M}(\mathfrak{g}_c, \chi_{\lambda+\nu})$. The functor Ψ is exact, and takes any irreducible Harish-Chandra module to an irreducible module or zero. Any irreducible Harish-Chandra module in $\mathcal{M}(\mathfrak{g}_c, \chi_{\lambda+\nu})$ has a unique irreducible preimage under Ψ. Suppose in addition that

ii) for every coroot $\alpha^\vee$ of T in G_c such that $\lambda(\alpha^\vee)$ is a positive integer, $(\lambda+\nu)(\alpha^\vee)$ is a positive integer. Then Ψ is an equivalence of categories, with inverse $\Psi^{\lambda}_{\lambda+\nu}$.

We are being deliberately vague about what kind of Harish-Chandra modules are permitted, partly to allow for possible future applications in very general settings. The main requirement is that the group K_c (with respect to which Harish-Chandra modules are defined) should come equipped with an appropriate map to the complex group G_c. In this paper K_c will always be a covering of a subgroup of G_c, so there is no difficulty.

Translation families of representations will be collections of representations differing by translation functors. The next definition provides a first kind of parameter set for such a family.

2.6 <u>Definition</u>:

Suppose λ_0 and T are as in Theorem 2.5. Define

$$R_c(\lambda_0) = \{ \alpha \in R(G_c,T) \mid \alpha^\vee(\lambda_0) \in Z \},$$

the set of integral roots for λ_0. Fix a set P of positive roots for $R_c(\lambda_0)$ making λ_0 dominant. Define

$$\mathcal{P} = \mathcal{P}(\lambda_0,P) = \{ \lambda \in \mathfrak{t}^* \mid \lambda-\lambda_0 \in X^*(T), \text{ and } \lambda \text{ is P-dominant} \}.$$

A <u>translation family based on $\mathcal{P}$</u> is a map π from $\mathcal{P}$ to $\mathfrak{g}_c$-modules, with the following properties.

a) For all $\lambda \in \mathcal{P}$, $\pi(\lambda)$ has infinitesimal character λ.

b) Suppose λ and $\lambda+\nu$ belong to $\mathcal{P}$, and the translation functor $\Psi=\Psi_{\lambda}^{\lambda+\nu}$ is defined. Then $\Psi\pi(\lambda)$ is isomorphic to $\pi(\lambda+\nu)$.

We say that the family π is <u>irreducible</u> if $\pi(\lambda)$ is irreducible for regular λ. Two irreducible modules are called <u>translation equivalent</u> if they occur in the same irreducible translation family.

The set we will actually use to parametrize a translation family of representations will be a variant of this one (cf. Definition 2.9); we include this because it corresponds more closely to the usual notion.

It is not quite a formal consequence of Theorem 2.5 that many translation families exist, but this follows from [Z] and [SV]. Here is a precise statement.

2.7 <u>Proposition</u>:

In the setting of Definition 2.6, suppose π_0 is an irreducible Harish-Chandra module for $\mathfrak{g}_c$ of infinitesimal character corresponding to some element of $\mathcal{P}$. Fix $\lambda\in\mathcal{P}$ so that π has infinitesimal character χ_λ. Then there is a unique translation family π of irreducible Harish-Chandra modules based on $\mathcal{P}$, such that $\pi(\lambda_0) = \pi_0$.

A technical complication here is that λ may fail to be uniquely determined by π_0; two different choices will give rise to two different translation families. This can only happen if λ is not integral. The simplest example is for $G=SL(2)$, and $\lambda_0=\frac{1}{2}\alpha$, for α a root. The problem is that there are non-isomorphic principal series with this infinitesimal character, but they differ by a translation functor.

2.8 Definition:

In the setting of (2.2), we say that an infinitesimal character χ for G_c is associated to ${}^\vee c$ under the following condition: there are an isomorphism $\zeta \in \mathcal{Z}$ and an element λ in $\mathfrak{t}^*$, such that

a) $\chi = \chi_\lambda$, and

b) $\exp(2\pi i \zeta(\lambda)) = {}^\vee c$.

In (b), we are regarding ζ as an isomorphism from $\mathfrak{t}^*$ to ${}^d\mathfrak{t}$.

Now ζ maps $2\pi i X^*(T)$ to the kernel of the exponential map on ${}^dT$; so condition (b) is unaffected if λ is replaced by $\lambda+\nu$, with $\nu \in X^*(T)$. The infinitesimal characters of the representations in a translation family π are therefore all associated to ${}^\vee c$, or none of them are. In the former case we say π is associated to ${}^\vee c$.

Definition 2.8 becomes particularly simple when c is central in G (as it will be in most of our applications) and ${}^\vee c$ is central in ${}^\vee G$ (which corresponds to integral infinitesimal character for G). Then

2.8' the infinitesimal character χ_λ is associated to ${}^\vee c$ if and only if $\exp(2\pi i\lambda) = {}^\vee c$.

Here we identify a weight λ in the dual of a maximal torus in $\mathfrak{g}$ with an element of ${}^\vee\mathfrak{g}$ using any of the distinguished isomorphisms ζ.

We also parametrize a translation family by data for ${}^\vee G$.

2.9 Definition:

In the setting of (2.4), note that

$${}^dR_{{}^\vee c}(\Upsilon) = \{ \alpha \in R({}^\vee G_{{}^\vee c}, {}^dT) \mid \Upsilon^{-1}(\alpha^\vee)(c) = 1 \}.$$

Choose a set ${}^dP$ of positive roots for ${}^dR_{{}^\vee c}(\Upsilon)$. Define

$$\begin{aligned} \mathcal{P} &= \mathcal{P}({}^\vee c, {}^dP) \\ &= \{ \lambda \in {}^d\mathfrak{t} \mid \exp(2\pi i\lambda) = {}^\vee c, \text{ and } \lambda \text{ is } {}^dP\text{-dominant} \}. \end{aligned}$$

A <u>translation family based on</u> $(\mathcal{P},\Upsilon)$ is a map π from $\mathcal{P}$ to $\mathfrak{g}_c$-modules, with the following properties. (To keep the notation manageable, we will use Υ implicitly to identify ${}^d\mathfrak{t}$ with $\mathfrak{t}^*$.)

a) For all $\lambda \in \mathcal{P}$, $\pi(\lambda)$ has infinitesimal character χ_λ (cf. Proposition 2.3).

b) Suppose λ and $\lambda+\nu$ belong to $\mathcal{P}$, and the translation functor $\Psi=\Psi_\lambda^{\lambda+\nu}$ is defined. Then $\Psi\pi(\lambda)$ is isomorphic to $\pi(\lambda+\nu)$.

A <u>translation family</u> is a triple $(\mathcal{P},\Upsilon,\pi)$ for $\mathcal{P},\Upsilon$ and π as above. We say two translation families $(\mathcal{P},\Upsilon,\pi)$ and $(\mathcal{P}',\Upsilon',\pi')$ are <u>equivalent</u> if there exists $g \in G_c$ and ${}^\vee g \in {}^\vee G_{{}^\vee c}$ such that:

1) $\mathrm{int}(g)T = T'$ and $\mathrm{int}({}^\vee g){}^dT={}^dT'$,

2) $\mathrm{int}(g)\circ\Upsilon = \Upsilon'\circ\mathrm{int}({}^\vee g)$,

3) $\mathrm{int}(g)P = P'$ (equivalently, $\mathrm{int}({}^\vee g){}^dP={}^dP'$),

4) $g(\pi(\lambda)) = \pi'(\mathrm{int}({}^\vee g)^\vee\lambda)$.

If (P,Υ) is given we will write π for a translation family.

In the setting of Definition 2.9 fix an element $\lambda_0 \in {}^dT$ such that $\exp(2\pi i\lambda_0) = {}^\vee c$. Let ${}^dP$ be a set of positive roots making λ_0 dominant. Then

2.10 $\mathcal{P} = \{ \lambda \in {}^d\mathfrak{t} \mid \lambda-\lambda_0 \in X_*(T) \text{ and } \lambda \text{ is } {}^dP\text{-dominant} \}.$

By inspection of (2.4) and Definition 2.6, we have

$$R_{c^\vee c}(\mathfrak{I}) = R_c(\mathfrak{I}^{-1}(\lambda)_0),$$

and the corresponding set of coroots is identified by $\mathfrak{I}$ with ${}^dR_{c^\vee c}(\mathfrak{I})$. Of course $\mathfrak{I}$ identifies $\mathcal{P}(\lambda_0,P)$ with $\mathcal{P}(\exp(2\pi i\mathfrak{I}(\lambda_0), {}^dP)$, so we have immediately the analogue of Proposition 2.7.

In the applications either c or ${}^\vee c$ will be central, in which case we have the following result.

2.11 Lemma:

Suppose $c\in Z(G)$ or ${}^\vee c\in Z({}^\vee G)$. Let $(\mathcal{P},\mathfrak{I})$ and $(\mathcal{P}',\mathfrak{I}')$ be as in 2.9. Then there is a unique isomorphism $\xi:{}^dT\to{}^dT'$ satisfying the following properties.

(1) ξ is inner for ${}^\vee G_{^\vee c}$

(2) $\xi({}^dP)={}^dP'$

(3) there exists $g\in G_c$ such that $\xi\circ\mathfrak{I}=\mathfrak{I}'\circ int(g)$.

Thus there is a canonical bijection $\xi:\mathcal{P}\to\mathcal{P}'$. Consequently there is associated to c and ${}^\vee c$ a canonical parameter set $\mathcal{P}$.

We omit the elementary proof.

§3

Character Duality Revisited

Our goal in this section is to recast the results of [V2] in a form convenient for our present purposes. Let ${}^{\vee}G^{\Gamma}$ and G^{Γ} be L-groups for G and ${}^{\vee}G$ as in §1. We begin by describing (quasi-admissible - see [AV2], Definition 9.8) maps of the Weil group into ${}^{\vee}G^{\Gamma}$ in terms of "roots and weights" (Proposition 3.4). Recall first that $W_{\mathbb{R}}$ is generated by $\mathbb{C}^*$ and a distinguished element j. These are subject to the relations

3.1 $j^2 = -1 \in \mathbb{C}^*$

$jzj^{-1} = \bar{z}$.

We consider triples $(y, {}^dT^{\Gamma}, \lambda)$ satisfying

3.2 (1) $y \in {}^{\vee}G^{\Gamma} - {}^{\vee}G$, and y^2 is semisimple;

(2) ${}^dT^{\Gamma}$ is a Cartan subgroup of ${}^{\vee}G^{\Gamma}$ containing y;

(3) $\lambda \in {}^d\mathfrak{t} \approx X_*({}^dT) \otimes \mathbb{C}$ (where ${}^d\mathfrak{t} = \mathrm{Lie}({}^dT)$; and

(4) $\exp(2\pi i\lambda) = y^2$.

Define a quasi-admissible homomorphism $\varphi = \varphi(y, {}^dT^{\Gamma}, \lambda)$ from $W_{\mathbb{R}}$ to ${}^{\vee}G^{\Gamma}$ by

3.3 (1) $\varphi(z) = z^{\lambda}\bar{z}^{\mathrm{Ad}(y)\lambda}$ (where $z^{\lambda} = \exp(\lambda \log(z))$, for $z \in \mathbb{C}^* \subset W_{\mathbb{R}}$)

(2) $\varphi(j) = \exp(-\pi i\lambda)y$.

We must check that 3.3(1) is well defined, and that the relations 3.1 are preserved. First note that 3.2(4) implies

$y(\exp(2\pi i\lambda))y^{-1} = \exp(2\pi i\lambda)$,

which in turn implies that $\lambda - \mathrm{Ad}(y)\lambda \in X_*({}^dT)$. Thus 3.3(1) provides a well-defined homomorphism from $\mathbb{C}^*$ to ${}^\vee G^\Gamma$. From 3.3(2) and 3.2(4) we have $\varphi(j)^2 = \exp(\pi i(\lambda - Ad(y)\lambda))$. By 3.3(1), the right side is equal to $\varphi(-1)$, so $\varphi(j)^2 = \varphi(-1)$. Similarly one calculates that $\varphi(j)\varphi(z)\varphi(j)^{-1} = \varphi(\bar{z})$. This shows that φ preserves the relations defining $W_{\mathbb{R}}$.

3.4 Proposition:

For any quasi-admissible homomorphism φ from $W_{\mathbb{R}}$ to ${}^\vee G^\Gamma$, there exists a triple $(y, {}^dT^\Gamma, \lambda)$ satisfying 3.2 such that $\varphi = \varphi(y, {}^dT^\Gamma, \lambda)$. The elements y and λ are uniquely determined; if λ is regular, then the triple is unique. Furthermore there exists ${}^\vee\delta \in {}^\vee\mathcal{D}$ such that ${}^dT^\Gamma$ is ${}^\vee\theta$-stable (where ${}^\vee\theta = \mathrm{int}({}^\vee\delta)$, conjugation by ${}^\vee\delta$).

Proof:

By [AV2], Lemma 9.15, the image of φ is contained in some Cartan subgroup ${}^dT^\Gamma$ of ${}^\vee G^\Gamma$. Then we can write $\varphi(z) = z^\lambda \bar{z}^\nu$, for some λ and ν in ${}^dT$. Set $y = \exp(\pi i\lambda)\varphi(j)$; then 3.3(2) holds. The relation $\varphi(j)\varphi(z)\varphi(j)^{-1} = \varphi(\bar{z})$ implies that $\nu = Ad(y)\lambda$, so 3.3(1) holds. Similarly $\varphi(j)^2 = \varphi(-1)$ leads to 3.2(4). The other conditions in 3.2 are immediate.

We have identified λ as the image of the holomorphic part of the differential of φ, and given a formula for y in terms of λ; so λ and y are unique. The Cartan subgroup ${}^dT^\Gamma$ can be the group generated by y and any maximal torus ${}^dT$ containing λ. If λ is regular, then ${}^dT$ is unique. The final statement follows from [AV2], Lemma 9.16, proving the proposition.

If $(y, {}^dT^\Gamma, \lambda_0)$ satisfies (3.2), then $(y, {}^dT^\Gamma, \lambda)$ does as well if and only if

3.5 $\quad \lambda \in \lambda_0 + X_*({}^dT)$.

In the context of Definition 2.9 (cf. 2.10), this suggests that the family of maps φ obtained by varying λ in this way should parametrize something like a translation family of representations. This is nearly true, although a variety of complications can arise for singular λ. Here is a precise statement, essentially proved in [SV].

3.6 Theorem:

Suppose $(y,{}^dT^\Gamma,\lambda_0)$ satisfies (3.2). Then any representation associated to $\varphi(y,{}^dT^\Gamma,\lambda_0)$ by the Langlands classification has infinitesimal character λ_0 (Proposition 2.1); in particular, its infinitesimal character is associated to y^2 (Definition 2.8).

Suppose in addition that λ_0 is regular. Write ${}^dP$ for the set of positive roots of ${}^dT$ in ${}^\vee G_{y^2}$ making λ_0 dominant, and define $\mathcal{P}=\mathcal{P}(y^2,{}^dP)$ as in Definition 2.9. Fix a strong real form x of G (Definition 1.8).

(a) Suppose $I(\lambda_0)$ is a standard $(\mathfrak{g},K_x)$-module attached to $\varphi(y,{}^dT^\Gamma,\lambda_0)$ by the Langlands classification; write I for the corresponding translation family parametrized by $\mathcal{P}$. Then for any $\lambda \in \mathcal{P}$, $I(\lambda)$ is a direct summand of some standard $(\mathfrak{g},K_x)$-module attached to $\varphi(y,{}^dT^\Gamma,\lambda)$. If λ is regular, $I(\lambda)$ is itself standard.

(b) Suppose $J(\lambda_0)$ is an irreducible $(\mathfrak{g},K_x)$-module attached to $\varphi(y,{}^dT^\Gamma,\lambda_0)$ by the Langlands classification; write J for the corresponding translation family parametrized by $\mathcal{P}$. Then for any $\lambda\in\mathcal{P}$, $J(\lambda)$ is either zero, or it is an irreducible $(\mathfrak{g},K_x)$-module attached to $\varphi(y,{}^dT^\Gamma,\lambda)$. If λ is regular, the latter is always the case.

We now wish to extend the data $(y, {}^dT^\Gamma, \lambda)$ in such a way as to specify a unique representation within an L-packet. To motivate Definition 3.8, we need to recall a little about Langlands' construction of representations as described in [AV2]. Suppose $(y, {}^dT^\Gamma, \lambda)$ are as in (3.2). Fix a strong real form x of G (Definition 1.8) and assume that ${}^dT^\Gamma$ is relevant with respect to x ([AV2], Definition 9.11). (If this hypothesis is not satisfied, then the Langlands classification attaches no $(\mathfrak{g}, K_x)$-modules to $\varphi(y, {}^dT^\Gamma, \lambda)$.) Write $\theta = \mathrm{int}(x)$ for the corresponding Cartan involution. Then we can find a θ-stable Cartan subgroup T^Γ of G^Γ dual to ${}^dT^\Gamma$ ([AV2], Definition 9.11). By the definition of dual, we can choose a distinguished isomorphism $\mathcal{I}$ from ${}^\vee T$ to ${}^dT$ taking ${}^\vee\theta$ to $(\mathrm{int}\, y)|_{{}^dT}$. This $\mathcal{I}$ is not unique; varying it will produce the various representations in the L-packet. Fix a set Ψ of positive real roots of T in G; Ψ corresponds by $\mathcal{I}$ to a set ${}^d\Psi$ of positive imaginary coroots of ${}^dT$ in ${}^\vee G$. As in Definition 9.22 of [AV2], we can find an element ${}^\vee\delta$ in ${}^dT^\Gamma \cap {}^\vee\mathcal{D}$ making ${}^d\Psi$ into a special set of positive imaginary roots for ${}^\vee\delta$ ([AV2], Definition 6.29). The element ${}^\vee\delta$ is unique up to conjugation by the centralizer of y (or, equivalently, of ${}^\vee\delta$) in ${}^dT$. Write ρ for half the sum of some set of positive roots of T in G, and ${}^\vee z_\rho$ for the corresponding element of ${}^\vee T$. Sending ${}^\vee\delta$ to a distinguished element now gives an isomorphism

$$\iota = \iota(\mathcal{I}, \Psi) : {}^dT^\Gamma \to {}^\vee T^\Gamma,$$

where ${}^\vee T^\Gamma$ is an E-group for T determined by ${}^\vee z_\rho$ ([AV2], Definition 5.9). Composing with ι, we get a map $\iota \circ \varphi$ from $W_{\mathbb{R}}$ to ${}^\vee T^\Gamma$. By the Langlands classification for tori ([AV2], §5), this map defines a one-dimensional genuine $(\mathfrak{t}, (T \cap K_x)_\rho)$-module

$$\Lambda = \Lambda(\mathcal{I}, \Psi).$$

Here $(T \cap K_x)_\rho$ is the two-fold cover determined by ρ (cf. [AV1], §2).

(Writing $T(\mathbb{R})$ for the real form of T with Cartan involution θ, we may also regard Λ as a genuine character of $T(\mathbb{R})_\rho$.) Note that $d\Lambda=\lambda$.

Finally, we want to use cohomological induction from Λ to define a standard limit $(\mathfrak{g},K_x)$-module. When λ is regular, there is no difficulty in doing this; but for singular λ we must also specify a set of positive imaginary roots for T in G ([AV2],Definition 8.18 and Proposition 8.20). Through the isomorphism $\mathfrak{I}$,this amounts to picking a set of positive real roots for ${}^dT$ in ${}^\vee G$. The next lemma shows that this is subsumed in the choice of positive roots needed in Theorem 3.6 to discuss translation families.

3.7 Lemma:

Suppose y and ${}^dT^\Gamma$ are as in (3.2). Then every real root α of ${}^dT$ in ${}^\vee G$ satisfies $\alpha(y^2) = 1$, and so is a root in ${}^\vee G_{y^2}$.

Proof:

Necessarily y is of the form $t\,{}^\vee\delta$, for some t in ${}^dT$. Since the square of ${}^\vee\delta$ is the central element z_ρ , y^2 must be $t({}^\vee\theta t)z_\rho$, and $\alpha(y^2) = \alpha(t)({}^\vee\theta\alpha)(t)$. Since α is real, ${}^\vee\theta\alpha = -\alpha$; so $\alpha(y^2) = 1$, as we wished to show.

We turn now to the main definition of this section; these are the sets that will parametrize translation families of representations.

3.8 Definition:

A set of strong L-data for G^Γ and ${}^\vee G^\Gamma$ is a 7-tuple $S=(x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\mathfrak{I})$ subject to the following conditions.

a) T^Γ is a Cartan subgroup of G^Γ, and x is an element of $T^\Gamma - T$.

b) ${}^dT^\Gamma$ is a Cartan subgroup of ${}^\vee G^\Gamma$, and y is an element of ${}^dT^\Gamma - {}^dT$.

c) Υ is a distinguished isomorphism from ${}^\vee T$ to ${}^dT$ making T^Γ dual to ${}^dT^\Gamma$ (that is, carrying the inverse transpose of int(x) to int(y)).

d) P is a set of positive roots for the root system $R_{x^2y^2}(\Upsilon)$ defined in (2.4).

e) ${}^dP$ is the image under Υ of the coroots corresponding to P; this is a set of positive roots for the system ${}^dR_{x^2y^2}(\Upsilon)$ defined after (2.4).

f) $x^2 \in Z(G)$.

We say that S is <u>integral</u> if $y^2 \in Z({}^\vee G)$. The set S is said to be <u>equivalent</u> to S' if S is conjugate to S' under the obvious action of $G \times G'$.

Because of condition (e), P contains all the imaginary roots of T in G. It is not hard to deduce that conditions (c)-(d) therefore determine Υ up to conjugation by an element of G. Up to equivalence, we could therefore omit Υ from the definition. It will be needed in the generalization considered in §4, however, so we retain it now for symmetry.

Note that ${}^dP$ corresponds to a choice of Borel subgroup ${}^dB \subset {}^\vee G_{y^2}$, so we write ${}^dB$ in place of ${}^dP$ on occasion. Furthermore is S is integral, then P corresponds to a Borel subgroup $B \subset G$, and a similar convention holds.

Fix S as in Definition 3.8, and write

3.9 (a) $\mathcal{P}(S) = \mathcal{P}(y^2, {}^dP)$

(Definition 2.9). Suppose λ belongs to $\mathcal{P}(S)$. Choose a set Ψ of positive real roots for T in G, and define $\Lambda = \Lambda(\Upsilon, \Psi)$ as in the discussion before Lemma 3.7. Write P_{im} for the set of imaginary roots of T in P. Define

3.9 (b) $I(S, \lambda) = I(\Psi, P_{im}, \Lambda(\Upsilon, \Psi))$,

a standard limit representation ([AV2], Definition 8.27). If λ is regular, we define

3.9 (c) $J(S,\lambda)$ = Langlands subquotient of $I(S,\lambda)$.

We extend the definition of J to all λ in $\mathcal{P}(S)$ by requiring $J(S)$ to be a translation family. This is possible by Theorem 3.6.

The omission of Ψ from the notation on the left of 3.9 (b) and (c) is justified by

3.10 Lemma:

In the setting of 3.9, any two choices of Ψ lead to isomorphic representations $I(S,\lambda)$.

proof:

This assertion was verified in the course of the proof of the Langlands classification in [AV2]. The argument is given (in a slightly disguised form) in Lemmas 8.24 and 9.28 of [AV2]. We leave the remaining details to the reader.

Here is a sharper version of Theorem 3.6. It combines the Langlands classification with Theorem 8.2.1 of [V1].

3.11 Theorem:

Suppose S is a set of strong L-data. Then $I(S,*)$ and $J(S,*)$ are translation families based on $(\mathcal{P}(S),\mathfrak{T})$ (Definition 2.9). The map from S to (x,J) defines a bijection between equivalence classes of strong L-data, and equivalence classes of pairs (x,π), with x a strong real form of G and π a translation family of irreducible $(\mathfrak{g},K_x)$-modules.

The theorem says that two non-zero irreducible modules $J(S,\lambda)$ and $J(S',\lambda')$ are isomorphic if and only if (S,λ) is conjugate to (S',λ'). A small but important subtlety is that two non-zero standard limit representations $I(S,\lambda)$ and $I(S',\lambda')$ may be isomorphic even if the corresponding Cartan subgroups T^Γ and $(T')^\Gamma$ are not conjugate. This can happen only if λ is singular.

We can now conveniently define "super" L-packets. Given a quasi-admissible homomorphism $\varphi: W_{\mathbb{R}} \to {}^\vee G^\Gamma$, and a strong real form x of G, we a obtain a (possibly empty) L-packet Π_φ of $(\mathfrak{g},K_x)$-modules. We write $\Pi_{\varphi,x}$ to indicate the dependence on x.

3.12 Definition:

The <u>super</u> <u>L</u>-<u>packet</u> Π_φ associated to φ is the union of the sets $\Pi_{\varphi,x}$ as x varies over a set of representatives of equivalence classes of strong real forms of G.

Suppose now that $\varphi = \varphi(y, {}^dT^\Gamma, \lambda)$, and that λ is regular. Write ${}^dP$ for the corresponding set of positive roots of ${}^dT$ in ${}^\vee G_{y^2}$. Choose a Cartan subgroup T^Γ of G^Γ that is dual to ${}^dT^\Gamma$ via an isomorphism Υ from ${}^\vee T$ to ${}^dT$. Write P for the positive integral roots of T in G corresponding to ${}^dP$ by Υ. Let $S'=(x',(T')^\Gamma,P',y,{}^dT^\Gamma,{}^dP,\Upsilon')$ be a set of strong L-data such that $J(S',\lambda)$ belongs to Π_φ. After conjugating S by G, we may assume that $(T')^\Gamma = T^\Gamma$, $\Upsilon' = \Upsilon$, and $P' = P$. The entire set S is therefore determined by x; and x in turn is determined by S up to conjugation by T. The following lemma is an immediate consequence.

3.13 Lemma:

In the setting just described,

(1) Π_φ is in bijection with $\{ x \in T^\Gamma \mid x^2 \in Z(G) \}$ / (conjugation by T).

(2) Fix δ in $T^\Gamma - T$ with $\delta^2 \in Z(G)$; for example, take δ in $\mathfrak{D} \cap T^\Gamma$. Write $\theta = \mathrm{int}(\delta)$. Then Π_φ is in bijection with

$$\{ t \in T \mid t\theta t \in Z(G) \} / \{ s\theta(s^{-1}) \mid s \in T \}$$

Notice that the second parameter set is a group.

We want next to understand blocks of $(\mathfrak{g},K_x)$-modules. Write $\mathcal{U}(\mathfrak{g},K_x)$ for the Grothendieck group of finite length $(\mathfrak{g},K_x)$-modules, or virtual modules. This is a free Z-module having as basis the set of irreducible $(\mathfrak{g},K_x)$-modules. A less obvious but equally important basis is provided by the set of indecomposable standard modules. These are the standard limit representations admitting a unique Langlands subrepresentation; passage to this subrepresentation defines a bijection from the indecomposable standard modules to the irreducible modules. For regular infinitesimal character, the indecomposable standard modules are just the standard modules.

3.14 Definition:

Suppose x is a strong real form of G^Γ. Block equivalence is the smallest equivalence relation on irreducible $(\mathfrak{g},K_x)$-modules with the following property: if X and Y are irreducible $(\mathfrak{g},K_x)$-modules, F is a finite dimensional representation of G, and Y is a subquotient of $X\otimes F$, then X is block equivalent to Y.

3.15 Definition:

(1) A block $\mathcal{B}$ of $(\mathfrak{g},K_x)$-modules is the Z-module spanned by a block equivalence class of irreducible $(\mathfrak{g},K_x)$-modules. We say a representation π is contained in $\mathcal{B}$ if this holds in the Grothendieck group (i.e. all the irreducible constituents of π are contained in $\mathcal{B}$).

Fix a block $\mathcal{B}$.

(2) We let $\mathcal{B}^{irr}$ be the set of irreducible representations contained in $\mathcal{B}$. If I is a standard limit representation, all of the irreducible factors of I are contained in the same block. Thus we let $\mathcal{B}^{std}$ denote the set of standard limit representations contained in $\mathcal{B}$.

Note that $\mathcal{B}^{irr}$ is a basis of $\mathcal{B}$.

(3) Define translation families of virtual modules in the obvious way. Block equivalence is weaker than translation equivalence. Thus it makes sense to say that a translation family π is contained in $\mathcal{B}$ (i.e. $\pi(\lambda)\in\mathcal{B}$ for all (equivalently any) $\lambda\in\mathcal{P}$). We let $\mathcal{B}_{TF}$ denote the translation families contained in $\mathcal{B}$. We define $\mathcal{B}_{TF}^{irr}$ and $\mathcal{B}_{TF}^{std}$ similarly.

Note that $\mathcal{B}_{TF}$ is a Z-module of finite length, with bases $\mathcal{B}_{TF}^{irr}$ and $\mathcal{B}_{TF}^{std}$.

(4) Given $\lambda \in \mathcal{P}$, we let

$$\mathcal{B}(\lambda) = \{ \pi(\lambda) \mid \pi \in \mathcal{B}_{TF} \}$$
$$= \{ \sigma \in \mathcal{B} \mid \sigma \text{ has infinitesimal character } \chi_\lambda \}.$$

It is possible that $\mathcal{B}(\lambda)=\mathcal{B}(\lambda')$ for $\lambda,\lambda' \in \mathcal{P}$ for $\lambda \neq \lambda'$. This can happen only if λ is conjugate to λ' via $W({}^\vee G, {}^\vee T)$. Since λ and λ' are P-dominant, this cannot happen if they are integral. In fact it can only arise if ${}^\vee G_{{}^\vee c}$ is not connected, and is rare in general.

We will see that any L-packet is contained in a single block.

3.16 Theorem:

Suppose $S=(x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\Upsilon)$ and $S'=(x,(T^\Gamma)',P',y',({}^dT^\Gamma)',{}^dP',\Upsilon')$ are two sets of strong L-data for the strong real form x of G^Γ. Then J(S) is block equivalent to J(S') if and only if y is conjugate to y'.

This is easily deduced from the results of [V2], and we will omit most of the proof. Some of the ideas will be needed in §6 to compare our results with Shelstad's; we will therefore outline those ideas.

3.17 Definition:

Fix a maximal torus T^a of G and a regular weight λ^a in $(\mathfrak{t}^a)^*$. Set

$$W(\lambda^a) = \{ w \in W(G,T^a) \mid w\lambda^a - \lambda^a \in Q^*(G,T^a) \},$$

the integral Weyl group for λ^a. Suppose now that T is any other maximal torus in G, and that $\lambda \in \mathfrak{t}^*$ is a weight conjugate to λ^a. Because λ^a is regular, there is a unique isomorphism

$$\xi = \xi(\lambda^a,\lambda): T^a \to T$$

that takes λ^a to λ and is given by the restriction of Ad(g) for some g in G. We use this isomorphism to identify $W(\lambda^a)$ with $W(\lambda)$.

Suppose now that x is a strong real form of G^Γ, T^Γ is a Cartan subgroup of G^Γ containing x, Ψ is a set of positive real roots for T in G, and Λ is a genuine character of $T(\mathbb{R})_\rho$. Write λ for the differential of Λ, and assume that λ is conjugate to λ^a. Fix w in $W(\lambda)$, and write Ω for the holomorphic character of T with differential $w\lambda - \lambda$; this is a sum of roots. We can regard Ω as a character of $T(\mathbb{R})$, so

$$w \times \Lambda = \Lambda \otimes \Omega$$

is a genuine character of $T(\mathbb{R})_\rho$ (cf. [V1], Definition 8.3.1).

Because λ is regular, the standard module $I = I(\Psi,\Lambda)$ is defined. The <u>cross action</u> of $W(\lambda^a)$ on standard representations with infinitesimal character λ^a is defined by

$$w^a \times I = I(\Psi, w^{-1} \times \Lambda)$$

([V2], Definition 4.1). Here w^a is an element of $W(\lambda^a)$, and $w = \xi(\lambda^a,\lambda)(w^a)$ is the corresponding element of $W(\lambda)$. The definition in [V1] is somewhat more complicated than this one, because a less natural parametrization of representations is used there.

Here are two results on the relevance of the cross action.

3.18 <u>Lemma:</u>

In the setting of Definition 3.17, write W_{im} for the Weyl group of the imaginary roots of T in G. Then W_{im} is contained in $W(\lambda)$. If λ is regular, the L-packet containing $J(\Psi,\Lambda)$ is the set of representations $\{ I(\Psi, w \times \Lambda) \mid w \in W_{im} \}$.

This follows from Proposition 9.23 of [AV2].

3.19 Lemma ([V1], Theorem 9.2.11):

In the setting of Definition 3.17, the Langlands subquotients of I and $w^{\alpha} \times I$ are block equivalent.

Because of these lemmas, it is important to understand the cross action on the level of L-data.

3.20 Lemma:

Suppose $S = (x, T^{\Gamma}, P, y, {}^dT^{\Gamma}, {}^dP, \Upsilon)$ is a set of strong L-data as in Definition 3.8, and λ^{α} is as in Definition 3.17. Fix λ in $\mathcal{P}(S)$ regular, and assume that λ and λ^{α} define the same infinitesimal character for G. Use Υ to identify λ with an element of $\mathfrak{t}^*$. The group $W(\lambda^{\alpha})$ is identified by ξ with $W(\lambda)$, then by Υ with a subgroup of $W({}^\vee G, {}^dT)$. For z^{α} in $W(\lambda)$, write z and ${}^\vee z$ for the corresponding elements of these latter groups. For $z^{\alpha} \in W(\lambda^{\alpha})$, define $z^{\alpha} \times S = (x, T^{\Gamma}, z^{-1}P, y, {}^dT^{\Gamma}, ({}^\vee z^{-1})^dP, \Upsilon)$.
Then

$$z^{\alpha} \times I(S, \lambda) = I(z^{\alpha} \times S, ({}^\vee z^{-1})\lambda).$$

The proof, which is an exercise in the definitions, is left to the reader.

This lemma suggests studying other ways in which L-data may be modified using the Weyl groups of G and ${}^\vee G$. Thus we compute the action of W^{θ} and ${}^\vee W^{{}^\vee\theta}$ on S via the action on P and ${}^dP$. The resulting lemma is useful in computing examples.

In the setting of Lemma 3.20, write W^θ for the subgroup of $W(G,T)$ consisting of elements commuting with θ. This clearly contains the subgroup $W(K_x,T)$ of elements having a representative in K_x. (This latter group is well-defined even though K_x need not contain T.) If $w \in W^\theta$, then w acts naturally on $T(\mathbb{R})$ and $T(\mathbb{R})_\rho$, and so on the genuine characters of $T(\mathbb{R})_\rho$. If Λ is such a character, we write $w\Lambda$ for the action. Since w commutes with θ, $w\Psi$ is another set of positive real roots for T in G. Similar coments apply to $W({}^\vee G, {}^dT)$.

3.21 Lemma:

Suppose we are in the setting of Lemma 3.20.

(1) For $w \in W^\theta$, let $wS = (x, T^\Gamma, wP, y, {}^dT^\Gamma, {}^dP, w \circ \mathfrak{J})$.

Then $I(w\Psi, w\Lambda) = I(wS, w\lambda)$.

Furthermore if $w \in W(K_x,T)$ then $I(w\Psi, w\Lambda) \approx I(\Psi, \Lambda)$.

(2) For ${}^\vee w \in W({}^\vee G, {}^dT)$, let ${}^\vee wS = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^\vee w {}^dP, \mathfrak{J} \circ {}^\vee w)$.

Let $w = \mathfrak{J}^{-1}({}^\vee w)$.

Then $I(\Psi, w^{-1}(w \times \Lambda)) = I({}^\vee wS, {}^\vee w\lambda)$.

Again we leave this as an exercise for the reader.

Combining these results with the one in ([AV2], Lemma 8.24) on changing Ψ leads to a complete description of the effects of modifying the various positive root systems involved in L-data. In particular, one sees that changing P and ${}^dP$ does not change the block of $J(S)$; this is part of Theorem 3.16. As indicated earlier, we are omitting the rest of the proof of that Theorem.

Theorem 3.16 can be reformulated to sharpen the analogy with Theorem 3.11.

3.22 Definition:

Suppose $x \in G^\Gamma - G$, and $y \in {}^\vee G^\Gamma - {}^\vee G$. We say that the pair (x,y) is (strongly) admissible if it can be extended to a set of (strong) L-data. It is equivalent to require that x^2 be central in G, and that there exist dual Cartan subgroups T^Γ and ${}^dT^\Gamma$ of G^Γ and ${}^\vee G^\Gamma$ containing x and y respectively. Two pairs are said to be equivalent if they are conjugate by $G \times {}^\vee G$. The pair (x,y) is called integral if $x^2 \in Z(G)$ and $y^2 \in Z({}^\vee G)$.

3.23 Corollary:

There is a natural bijection from equivalence classes of strongly admissible pairs (x,y) onto equivalence classes of pairs $(x, \mathcal{B})$, with x a strong real form of G and $\mathcal{B}$ a block of $(\mathfrak{g}, K_x)$-modules.

We write

13.24 $\mathcal{B}(x,y)$ = block corresponding to the pair (x,y).

For the remainder of this section we restrict attention to the case of representations of integral infinitesimal character; that is, to integral L-data (Definition 3.8). Notice that the definition of integral L-data is entirely symmetric in G and ${}^\vee G$.

3.25 Definition:

Suppose $S = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \mathfrak{I})$ is a set of integral L-data. The set of dual L-data is

$$^{\vee}S = (y, {}^{\vee}T^{\Gamma}, {}^{d}P, x, T^{\Gamma}, P, \mathfrak{I}^{-1}).$$

If $J = J(S)$ (respectively $I = I(S)$) is the corresponding translation family of irreducible (respectively standard) $(\mathfrak{g}, K_x)$-modules, we define the <u>dual translation family</u> to be the translation family ${}^{\vee}J = J({}^{\vee}S)$ (respectively ${}^{\vee}I = I({}^{\vee}S)$) of $({}^{\vee}\mathfrak{g}, K_y)$-modules. If we fix regular elements λ in $\mathcal{P}(S)$ and ${}^{\vee}\lambda$ in $\mathcal{P}({}^{\vee}S)$, then we may speak of $J({}^{\vee}S, {}^{\vee}\lambda)$ as the <u>dual module</u> to $J(S, \lambda)$. Suppose (x,y) is an integral admissible pair (Definition 3.22), corresponding to a block $\mathcal{B}$ for G (Corollary 3.23). The <u>dual block</u> ${}^{\vee}\mathcal{B}$ for ${}^{\vee}G$ is the one corresponding to the (integral admissible) pair (y,x). Similary we obtain ${}^{\vee}\mathcal{B}_{TF}$, the block of translation families dual to $\mathcal{B}_{TF}$.

In accordance with the discussion at the end of §1, we will write ${}^{d}\pi$ (respectively ${}^{d}I$) for a typical irreducible (respectively standard) module in ${}^{\vee}\mathcal{B}$.

Fix a block $\mathcal{B}$, and fix a constant c_0 (depending on $\mathcal{B}$). Suppose $S=(x, T^{\Gamma}, P, y, {}^{d}T^{\Gamma}, {}^{d}P, \mathfrak{I})$ is a set of L-data for a translation family I of standard modules in $\mathcal{B}$. Let A be the split part of T. The <u>length</u> of I (or S) is defined to be

3.26 $\ell(I) = \frac{1}{2}|\{\, \alpha \in P \mid \theta\alpha \notin P \,\}| + \frac{1}{2}\dim A - c_0.$

We may choose c_0 in such way that $\ell(I)$ is an integer for all $I \in \mathcal{B}$. For the time being we allow any such normalization. If J is the translation family of irreducible modules attached to S, we define $\ell(J) = \ell(I)$.

For every pair (π, I) of an irreducible and an indecomposable standard module in $\mathcal{B}$, we define integers $M(I,\pi)$ and $m(\pi, I)$ so that

3.27 $I = \Sigma_{\pi\in\mathcal{B}}\, m(\pi,I)\,\pi$

$\pi = \Sigma_{I\in\mathcal{B}}\, M(I,\pi)\, I.$

The matrix m is the multiplicity matrix, and M is its inverse; since standard modules have relatively simple characters, the entries of M can be interpreted as coefficients in character formulas for irreducibles. We define $m({}^d\pi,{}^dI)$ and $M({}^dI,{}^d\pi)$ on the dual block ${}^\vee\mathcal{B}$ similarly.

We are now able to define $m(\pi,I)$ and $M(I,\pi)$ for translation families contained in a block $\mathcal{B}$.

3.28 Definition:

Fix an integral admissible pair (x,y) with the corresponding pair of dual blocks $\mathcal{B}$ and ${}^\vee\mathcal{B}$. Let $(\mathcal{P},\mathfrak{I},\pi)$ and $(\mathcal{P}',\mathfrak{I}',I)$ denote translation families of irreducible and standard modules respectively in $\mathcal{B}$. Choose a regular element λ of $\mathcal{P}$ and let λ also denote the corresponding element of $\mathcal{P}'$ via the canonical bijection $\mathcal{P}\to\mathcal{P}'$ of Lemma 2.11. We define:

(1) $m(\pi,I) = m(\pi(\lambda),I(\lambda))$

(2) $M(I,\pi) = M(I(\lambda),\pi(\lambda))$.

This definition is independent of the choice of (regular) λ.

The next theorem, Kazhdan-Lusztig duality for Harish-Chandra modules, relates these matrices for $\mathcal{B}$ and ${}^\vee\mathcal{B}$. It is the main result (Theorem 13.13) of [V2].

3.29 Theorem:

Fix an integral admisible pair (x,y) with the corresponding pair of dual blocks $\mathcal{B}$ and ${}^\vee\mathcal{B}$. Let $\mathcal{B}_{TF}$ and ${}^\vee\mathcal{B}_{TF}$ be the corresponding dual blocks of translation families (Definition 3.15). Suppose $(\mathcal{P},\mathfrak{I},\pi)$ (respectively $(\mathcal{P}',\mathfrak{I}',I)$) is a translation families of irreducible (resp. standard) standard modules contained in $\mathcal{B}_{TF}$. Let $({}^{\vee}\mathcal{P},\mathfrak{I}^{-1},{}^{\vee}\pi)$ and $({}^{\vee}\mathcal{P}',\mathfrak{I}'^{-1},{}^{\vee}I)$ be the dual translation families (Definition 3.25).

Then:

(1) $M(I,\pi) = (-1)^{\ell(I)-\ell(\pi)} m({}^{\vee}\pi,{}^{\vee}I)$

(2) $m(\pi,I) = (-1)^{\ell(I)-\ell(\pi)} M({}^{\vee}I,{}^{\vee}\pi)$.

The statement is evidently independent of the constant c_0 used to normalize the length function.

3.30 Definition:

In the setting of Theorem 3.29, define a perfect bilinear pairing $\mathcal{B}_{TF}\times{}^{\vee}\mathcal{B}_{TF}\to Z$ by defining it on the basis of irreducible translation families as follows. Suppose $\pi\in\mathcal{B}_{TF}^{irr}$ is an irreducible translation family with dual ${}^{\vee}\pi\in{}^{\vee}\mathcal{B}_{TF}^{irr}$. Let ${}^{d}\sigma$ be any irreducible translation family in ${}^{\vee}\mathcal{B}_{TF}^{irr}$, and set

$$\langle\pi,{}^{d}\sigma\rangle \begin{cases} = (-1)^{\ell(\pi)} & ({}^{d}\sigma \text{ is equivalent to } {}^{\vee}\pi) \\ = 0 & ({}^{d}\sigma \text{ is not equivalent to } {}^{d}\pi). \end{cases}$$

Now the following Corollary to Theorem 3.29 is immediate (and is in fact is easily seen to be equivalent to the theorem).

3.31 <u>Corollary</u>:

In the setting of Definition 3.28, suppose I is a standard translation family in $\mathcal{B}_{TF}^{std}$, with dual translation family $^{\vee}I$ in $^{\vee}\mathcal{B}_{TF}^{std}$. Suppose $^{d}I'$ is any other standard translation family in $^{\vee}\mathcal{B}_{TF}^{std}$. Then

$$\langle I, {}^{d}I' \rangle \begin{cases} = (-1)^{\ell(I)} & ({}^{d}I' \text{ is equivalent to } {}^{\vee}I) \\ = 0 & ({}^{d}I' \text{ is not equivalent to } {}^{\vee}I). \end{cases}$$

We restate 3.29, 3.30 and 3.31 in terms of virtual modules instead of translation families. By specializing translation families to particular infinitesimal characters, we obtain the following Corollary.

3.32 <u>Corollary</u>:

In the setting of Theorem 3.29, fix regular elements $\lambda \in \mathcal{P}$ and ${}^{d}\lambda \in {}^{d}\mathcal{P}$. Let π (resp. I) be an irreducible (resp. standard) module in $\mathcal{B}(\lambda)$. Similarly choose $^{\vee}\pi$, $^{\vee}I$ and ^{d}I in $^{\vee}\mathcal{B}({}^{d}\lambda)$.

(1) Theorem 3.29 holds for π, I , $^{\vee}\pi$, $^{\vee}I$.

(2) Define a perfect pairing $\langle\ ,\ \rangle$: $\mathcal{B}(\lambda) \times {}^{\vee}\mathcal{B}({}^{d}\lambda) \rightarrow \mathbb{Z}$ as in Definition 3.30. Then Corollary 3.31 holds for π, I, $^{\vee}\pi$, $^{\vee}I$, and ^{d}I.

§4

Covering Groups and Non-Integral Infinitesimal Character

Theorem 3.29 is based on Definition 3.25, which appears to make sense only in the case of integral infinitesimal character: if we wish to interchange the roles of x and y, then y ought to define an involution of $^\vee G$, and this amounts to requiring y^2 to be central in $^\vee G$. To extend the result, we therefore have to drop the assumption that x^2 is central in G. Then x will define an involution only on the centralizer G_{x^2} of x^2 in G. This immediately suggests the setting of §2, with x^2 and y^2 playing the roles of the elements c and $^\vee c$ in that section. We will consider representations not of G and $^\vee G$ but of the centralizers of x^2 and y^2. In the applications x^2 will always be central; or rather (because of the duality we want to exploit) either x^2 or y^2 will be central; but at first there is no particular advantage in retaining that assumption.

While we make these generalizations, it is convenient at the same time to replace our pair of L-groups by E-groups (cf. §1). We therefore fix an inner class of real forms of G, and dual E-groups G^Γ and $^\vee G^\Gamma$ associated to given elements

4.1 $\quad z \in Z(G)^\theta, \quad {}^\vee z \in Z({}^\vee G)^{\vee\theta}$

(cf. Lemma 1.5). Here θ represents the action on Z(G) of any element of $G^\Gamma - G$; this is well-defined. As explained in [AV2], maps of the Weil group into an E-group for G are related to representations of a certain covering of G.

Throughout this section, we will be considering a triple

4.2 (a) $(c, {}^\vee c, \mathcal{Z})$

as in (2.2). In particular, we consider the identity components G_c and ${}^\vee G_{{}^\vee c}$ of the centralizers of c and ${}^\vee c$. After Definition 4.9 we will also assume:

4.2 (b) either $c \in Z(G)$ or ${}^\vee c \in Z({}^\vee G)$.

Here is the main definition.

4.3 Definition:

A set of L-data for G^Γ and ${}^\vee G^\Gamma$ is a 7-tuple $S = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \mathfrak{I})$ satisfying conditions (a)-(e) of Definition 3.8. Explicitly, it is subject to the following conditions.

a) T^Γ is a Cartan subgroup of G^Γ, and x is an element of $T^\Gamma - T$.

b) ${}^dT^\Gamma$ is a Cartan subgroup of ${}^\vee G^\Gamma$, and y is an element of ${}^dT^\Gamma - {}^dT$.

c) $\mathfrak{I}$ is a distinguished isomorphism from ${}^\vee T$ to ${}^dT$ making T^Γ dual to ${}^dT^\Gamma$.

d) P is a set of positive roots for the root system $R_{x^2 y^2}(\mathfrak{I})$.

e) ${}^dP$ is the image under $\mathfrak{I}$ of the coroots corresponding to P.

We say that S is of type $(c, {}^\vee c, \mathcal{Z})$ if $x^2 = c$, $y^2 = {}^\vee c$, and $\mathfrak{I} \in \mathcal{Z}$. The set S is said to be equivalent to S' if S is conjugate to S' under the obvious action of $G_{x^2} \times {}^\vee G_{y^2}$. (In particular, this forces S' to be of the same type as S.) As in Definition 3.22, the pair (x,y) is called admissible if it can be extended to a set of L-data.

In order to discuss representations attached to L-data, we must recall some covering groups. Let $\pi_1(G_c)$ be the fundamental group of G_c. When no confusion can arise, we will call it simply π_1. Fix x as above, and write $\theta = int(x)$ for the corresponding Cartan involution of G_c. In analogy with the notation after (1.8), write

4.4 $K_x = (G_c)^\theta$, the centralizer of x in G_c.

(Since the centralizer of c may fail to be connected, K_x may be slightly smaller than the centralizer of x in all of G.) Pick a θ-stable Cartan subgroup T of G_c; π_1 is identified with $X_*(T)/Q_*(G_c,T)$. The involution θ acts on π_1 by its action on this quotient. Following ([AV2], 7.11(a)) let

4.5 $\pi_1(G_c)(\mathbb{R}) = \pi_1/(1+\theta)\pi_1$.

When there is no ambiguity we write this simply as $\pi_1(\mathbb{R})$. It depends only on c and $(G_c)^\Gamma$ (and not on x or T).

Let G_c^{can} be the covering of G_c with group $\pi_1(\mathbb{R})$ ([AV2],7.11(b)). Write K_x^{can} for the inverse image of K_x in G_c^{can}. There are exact sequences

4.6 $1 \to \pi_1(\mathbb{R}) \to G_c^{can} \to G_c \to 1$

$$1 \to \pi_1(\mathbb{R}) \to K_x^{can} \to K_x \to 1.$$

Recall that the group of characters of $\pi_1(\mathbb{R})$ may be identified with $Z(^\vee(G_c))^{\vee\theta}$. Let $\tau(^\vee z,c)$ denote the character of $\pi_1(\mathbb{R})$ corresponding to the element $^\vee z \in Z(^\vee(G_c))^{\vee\theta}$. Identifying $Z(^\vee(G_c))^{\vee\theta}$ in terms of root data, we see that it contains a subgroup naturally isomorphic to $Z(^\vee G)^{\vee\theta}$. The element $^\vee z$ of (4.1) may therefore be regarded as an element of $Z(^\vee(G_c))^{\vee\theta}$. Write $^\vee z_{\rho(c)}$ for the element corresponding to half the sum of the positive roots in G_c. Finally, let

4.7 $\quad \tau = \tau(^\vee z\,^\vee z_{\rho(c)},c)$

be the character of $\pi_1(\mathbb{R})$ associated to $^\vee z\,^\vee z_{\rho(c)}$. (Similarly we can define $^\vee\tau$ for $^\vee G$.) A $(\mathfrak{g}_c,K_x^{can})$-module is said to be of type τ if its restriction to $\pi_1(\mathbb{R}) \subset K_x^{can}$ is a multiple of τ.

In terms of real groups the situation is as follows. Fix a real form $G_c(\mathbb{R})$ of G_c corresponding to x as in Lemma 1.6, and let $G_c(\mathbb{R})^{can}$ be the preimage of $G_c(\mathbb{R})$ in G_c^{can}. Then $(\mathfrak{g}_c,K_x^{can})$-modules correspond to representations of $G_c(\mathbb{R})^{can}$. There is an exact sequence

$$1 \to \pi_1(\mathbb{R}) \to G_c(\mathbb{R})^{can} \to G_c(\mathbb{R}) \to 1.$$

We say a representation of $G_c(\mathbb{R})^{can}$ is of type τ if its restriction to $\pi_1(\mathbb{R})$ is a multiple of τ. The property "of type τ" is preserved by the correspondence between representations of $G_c(\mathbb{R})^{can}$ and $(\mathfrak{g}_c,K_x^{can})$-modules.

Here is the basic construction of representations.

4.8 <u>Theorem</u> (cf. [V2], Proposition 15.7):

Suppose S is a set of L-data of type $(c, {}^\vee c, \mathcal{Z})$ (Definition 4.3). Write $\mathcal{P}(S)$ for the set $\mathcal{P}({}^\vee c, {}^d P)$ of Definition 2.9. Then there are translation families $I(S,*)$ and $J(S,*)$ of $(\mathfrak{g}_c, K_x^{can})$-modules of type τ (cf. (4.7)) based on $\mathcal{P}(S)$. Here each $I(S,\lambda)$ is a standard limit representation; if λ is regular, $J(S,\lambda)$ is its Langlands quotient.

This correspondence is a bijection from the set of equivalence classes of L-data of type $(c, {}^\vee c, \mathcal{Z})$ containing x, to the set of equivalence classes of translation families of pairs (x,π) where π is an irreducible $(\mathfrak{g}_c, K_x^{can})$-module of type τ, with infinitesimal character associated to ${}^\vee c$.

If (x,y) is admissible (Definition 4.3; that is, if there are any sets of L-data of this type) then the set of classes of L-data containing x and y corresponds to a single block of $(\mathfrak{g}_c, K_x^{can})$-modules (Definition 3.14).

<u>Sketch of proof</u>:

Fix a set S of L-data of type $(c, {}^\vee c, \mathcal{Z})$. We will imitate the construction of [AV2], as recalled after Theorem 3.6. Fix a set Ψ of positive real roots for T in G_c. By Lemma 3.7 (with the roles of G and ${}^\vee G$ reversed) Ψ is a full set of positive real roots of T in G. Via $\mathfrak{I}$, Ψ corresponds to a set ${}^d\Psi$ of positive imaginary roots of ${}^dT$ in ${}^\vee G$. Choose ${}^\vee\delta$ in ${}^dT^\Gamma \cap {}^\vee\mathcal{D}$ making ${}^d\Psi$ a special set of positive imaginary roots. (Notice that this uses only the E-group structure on ${}^\vee G^\Gamma$, not one (which we do not have) related to ${}^{\vee}G_{{}^{\vee}c}$.) Sending ${}^{\vee}\delta$ to a distinguished element now defines an isomorphism

$$\iota = \iota(\mathfrak{I},\Psi): {}^dT^\Gamma \to {}^{\vee}T^\Gamma,$$

where ${}^{\vee}T^\Gamma$ is the E-group for T determined by the central element ${}^{\vee}z$ of (4.1). For each λ in $\mathcal{P}(S)$, Proposition 3.4 shows that $(y,{}^dT^\Gamma,\lambda)$ defines a quasi-admissible homomorphism $\varphi = \varphi(y,{}^dT^\Gamma,\lambda)$ from $W_{\mathbb{R}}$ into ${}^dT^\Gamma$. We can now proceed exactly as in the construction given after Theorem 3.6 to get a character Λ for an appropriate cover of $T(\mathbb{R})$, and then (by induction) a $(\mathfrak{g}_c,K_x^{can})$-module $I(\Psi,P_{im},\Lambda)$; this is $I(S,\lambda)$. We omit the verification that it is independent of the choices of Ψ and ${}^{\vee}\delta$. The claims in the first paragraph of the proposition are immediate.

Next, suppose I is a translation family of standard limit $(\mathfrak{g}_c,K_x^{can})$-modules of type τ, based on $\mathcal{P}({}^{\vee}c,{}^dP)$. Specifying this parameter set implicitly specifies a maximal torus ${}^dT$ in ${}^{\vee}G$. The Langlands classification for G_c (Theorem 3.11, or rather its analogue for covering groups) guarantees the existence of an int(x)-stable maximal torus T in G_c to which I is attached. (We let T^Γ be the group generated by T and x.) This means the following. Fix a set Ψ of positive real roots for T in G. Then there is an isomorphism $\mathfrak{I}$ in $\mathcal{Z}$ from ${}^{\vee}T$ to ${}^dT$ so that to each λ in $\mathcal{P}$ we can associate a character Λ of an appropriate covering of $T(\mathbb{R})$, having the properties that $d\Lambda$ corresponds to λ by $\mathfrak{I}$; and

$$I(\lambda) = I(\Psi,\Lambda).$$

Now $\mathfrak{I}$ and Ψ together determine an involution ${}^d\theta$ on ${}^dT$, and a set ${}^d\Psi$ of positive imaginary roots for ${}^dT$ in ${}^{\vee}G$. Choose ${}^{\vee}\delta$ in $\mathfrak{D}$

normalizing ${}^dT$ and acting there by ${}^d\theta$, making ${}^d\Psi$ special; this is possible as in the proof of the Langlands classification. We let ${}^dT^\Gamma$ be the group generated by ${}^dT$ and ${}^\vee\delta$. The ${}^dT$-conjugacy class of δ defines an E-group structure on ${}^dT^\Gamma$. Since I was assumed to be of type $\tau({}^\vee z,c)$, and $({}^\vee\delta)^2 = {}^\vee z$, the character Λ corresponds to a map φ of $W_{\mathbb{R}}$ into ${}^dT^\Gamma$. By Proposition 3.4, φ determines an element y of ${}^dT^\Gamma$. We have now constructed from I all the elements of a set S of L-data; we omit the verification that $I(S,\lambda)=I(\lambda)$, and that the equivalence class of S is well-defined.

It remains to discuss the claim about blocks. Here we already omitted proofs even in the setting of section 3, and we will not supply them now. The main point is that moving among the various translation families in a block (by the cross action, Cayley transforms, and so on) involves only roots on which the parameter λ is integral. For G_c and infinitesimal character associated to ${}^\vee c$, the integral roots are those in the set $R_{c^\vee c}$ of Definition 2.9. All of these roots are roots of ${}^dT$ in $G_{\vee c}$; so the necessary operations can be carried out on L-data in the sense of Definition 4.3. This completes the sketch of the proof.

With Theorem 4.8 in hand, it is an easy matter to extend the results at the end of section 3.

4.9 <u>Definition</u>:

Suppose $S = (x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\Upsilon)$ is a set of L-data. The set of <u>dual L-data</u> is

$^{\vee}S = (y, {}^dT^\Gamma, {}^dP, x, T^\Gamma, P, \mathfrak{I}^{-1})$.

Using $^{\vee}S$, define dual translation families, dual modules, and dual blocks (for G_c and $^{\vee}G_{^{\vee}c}$) as in Definition 3.25.

Using Lemma 2.11 we may now, as in Definition 3.28, define $m(\pi, I)$ and $M(I, \pi)$, where π (respectively I) denotes a translation family of irreducible (respectively standard) modules. Furthermore Theorem 3.29 and Corollaries 3.31 and 3.32 hold as stated (with the restriction of integrality omitted).

We conclude this section with some elementary remarks about another description of some of the projective representations appearing. For simplicity we speak about G itself, although these ideas will generally be applied to groups such as G_c. So fix $\xi \in P^*(G,T)$, and assume that the corresponding element $^{\vee}b$ in $Z(^{\vee}G)$ satisfies

4.10 $\quad {}^{\vee}\theta({}^{\vee}b) = {}^{\vee}b, \; ({}^{\vee}b)^2 = 1.$

The second condition amounts to $2\xi \in X^*(G,T)$, so there is a two-fold cover $\tilde{G} = G_\xi$ ([AV2], 7.9(b)). If x is a strong real form of G, write $(K_x)^\sim$ for the preimage of K_x in $\tilde{G}$. By ([AV2], Proposition 7.12), there is a natural bijection between genuine $(\mathfrak{g}, (K_x)^\sim)$-modules and $(\mathfrak{g}, K_x^{can})$-modules of type $\tau({}^{\vee}b, 1)$. Using this bijection, one can reformulate Theorem 4.8 (in the case when $({}^{\vee}z)^2 = 1$) in terms of genuine representations of a certain two-fold cover of G_x. The role of the element $^{\vee}b$ above is played by $^{\vee}z\,^{\vee}z_{\rho(c)}$.

§5

Parametrization of L-packets

We discuss the coefficents which arise in the lifting of a stable distribution from an endoscopic group. Given an admissible map φ of the Weil group into ${}^\vee G^\Gamma$ we will obtain a perfect pairing $\Pi\times\tilde{\mathbb{S}}_\varphi \to \mathbb{C}^*$, where Π is the super L-packet defined by φ, and $\tilde{\mathbb{S}}_\varphi$ is a certain component group on the dual side. A Weyl group acts on Π, and the terms "$\varkappa$" of [S1] are obtained from this pairing. This will permit us, in §9, to compare our results with those of [S1].

Suppose we are given G and an inner class of real forms of G, and L-groups ${}^\vee G^\Gamma$ and G^Γ as usual. Let $\varphi=\varphi(y,{}^dT^\Gamma,\lambda):W_{\mathbb{R}} \to {}^\vee G^\Gamma$ be a quasi-admissible homomorphism, with λ regular. Let S_φ denote the centralizer of φ in ${}^\vee G$. Therefore $S_\varphi=({}^dT)^{{}^\vee\theta}$, where ${}^\vee\theta=\text{int}(y)$. Recall the covering group ${}^\vee G^{can}$ of ${}^\vee G$ (cf. §4). Let $\tilde{S}_\varphi$ be the inverse image of S_φ in ${}^\vee G^{can}$, let $\tilde{\mathbb{S}}_\varphi=\tilde{S}_\varphi/(\tilde{S}_\varphi)^0$, and let $(\tilde{\mathbb{S}}_\varphi)\hat{}$ denote the group of characters of $\tilde{\mathbb{S}}_\varphi$. Let Π be the super L-packet (Definition 3.12) defined by φ, which we consider here as a set of standard modules.

5.1 <u>Theorem</u>:

There is a canonical bijection $\Pi \overset{1-1}{\leftrightarrows} (\tilde{\mathbb{S}}_\varphi)\hat{}$.

The extension of this Theorem to include L-packets with singular infinitesimal character is carried out in section 10 (cf. Theorem 10.19).

<u>proof</u>:

We use Lemma 3.13, so choose ${}^dP$, T^Γ, and P and Υ as in that lemma. Thus the standard modules in Π are parametrized by

$\mathcal{E} = \{x\epsilon T^{\Gamma}-T \mid x^2\epsilon Z(G)\}/\text{conjugation by } T.$

Choose $\delta\epsilon T^{\Gamma}\cap\mathfrak{D}$ so that the imaginary roots of P are distinguished with respect to δ. Then the map $t_0\delta\to t_0$ induces a bijection between $\mathcal{E}$ and

$\mathcal{T}=\{t\epsilon T \mid t\theta(t)\epsilon Z(G)\}/\{s\theta(s^{-1}) \mid s\epsilon T\}$

(where $\theta=\text{int}(\delta)$). This bijection is independent of the choice of δ (subject to the condition on it). Note that $\mathcal{T}$ is a group.

We compute $\mathcal{T}^{\wedge}$, the group of (holomorphic) characters of $\mathcal{T}$. Any character of $\{t|t\theta(t)\epsilon Z(G)\}$ is the restriction of a holomorphic character of T. Thus $\mathcal{T}^{\wedge}$ is the set of holomorphic characters of T, trivial on $\{s\theta(s^{-1})\}$, modulo characters trivial on $\{t \mid t\theta(t)\epsilon Z(G)\}$. It follows that

$$\mathcal{T}^{\wedge}\approx\{\gamma\epsilon X^*(T) \mid \gamma-\theta(\gamma)=0\}/\{\mu+\theta\mu \mid \mu\epsilon Q^*(T)\}.$$

Via the isomorphism Υ, this becomes

$$\mathcal{T}^{\wedge}\approx\{\gamma\epsilon X_*({}^dT) \mid \gamma+{}^{\vee}\theta(\gamma)=0\}/\{\mu-{}^{\vee}\theta\mu \mid \mu\epsilon Q_*({}^dT)\} = L_1/L_2.$$

Given γ in L_1, let $\tilde{s}=\exp(\frac{1}{2}\gamma)\epsilon\,{}^dT^{can}$. We obtain a map $L_1 \to \tilde{S}_\varphi$; composing with projection gives a map $L_1 \to \tilde{\mathbb{S}}_\varphi$. The kernel is L_2 (by a straightforward calculation), so we have an isomorphism $\mathcal{T}^{\wedge}\approx\tilde{\mathbb{S}}_\varphi$. By Pontryagin duality we have $\mathcal{T}\approx(\tilde{\mathbb{S}}_\varphi)^{\wedge}$; define $\Pi\to(\tilde{\mathbb{S}}_\varphi)^{\wedge}$ by the sequence of maps: $\Pi\overset{1-1}{\leftrightarrow}\mathcal{T}\approx(\tilde{\mathbb{S}}_\varphi)^{\wedge}$.

These two bijections depend on the choice of (T^{Γ},Υ,P), but it is easy to see that the bijection $\Pi\overset{1-1}{\leftrightarrow}(\tilde{\mathbb{S}}_\varphi)^{\wedge}$ does not. Explicitly, write I for the $(\mathfrak{g},K_{t\delta})$-module in Π corresponding to $t\epsilon\mathcal{T}$ using the isomorphism Υ and the choice of δ made earlier. Tracing through the definitions, we find that the corresponding character χ of $\tilde{\mathbb{S}}_\varphi$ is given by

$$\chi(\exp(\tfrac{1}{2}\gamma)) = \Upsilon^{-1}(\gamma)(t).$$

Now suppose (T^{Γ},Υ,P) is replaced by a G-conjugate $(gT^{\Gamma},g\Upsilon,gP)$. Then I defines the same element of Π as the $(\mathfrak{g},K_{g(t\delta)})$-module gI. The character χ' of $\tilde{\mathbb{S}}_\varphi$ defined using the new choices is therefore

$$\chi'(\exp(\tfrac{1}{2}\gamma)) = (g\mathfrak{I})^{-1}(\gamma)(gt)$$
$$= \mathfrak{I}^{-1}(\gamma)(t) = \chi(\exp(\tfrac{1}{2}\gamma)).$$

Thus the bijection $\Pi \to (\tilde{\mathbb{S}}_\varphi)^\wedge$ is canonical, proving the Theorem.

Write $<\ ,\ >$ for the pairing $\Pi\times\tilde{\mathbb{S}}_\varphi\to \mathbb{C}^*$ given by the Theorem. Note that each super L-packet Π contains a distinguished standard module I, corresponding to the trivial character of $\tilde{\mathbb{S}}_\varphi$. For example if Π is a super L-packet of discrete series representations, then I is the distinguished large discrete series of the quasisplit form. For this element I, $<I,\tilde{s}>=1$ for all $\tilde{s}\in\tilde{\mathbb{S}}_\varphi$.

Choose λ^a, T^a, etc. as in Definition 3.17 so that λ^a is conjugate to λ. Write W^a_{im} for the subgroup of $W(\lambda^a)$ corresponding to W_{im} (Lemma 3.17). Recall W^a_{im} acts on Π via the cross action, denoted $\times$.

5.2 <u>Definition</u>:

For $I\in\Pi$, $\tilde{s}\in\tilde{\mathbb{S}}_\varphi$, let $\delta_{I,\tilde{s}}(w)=<w\times I,\tilde{s}>/<I,\tilde{s}>$ $\quad(w\in W^a_{im})$.

For $s\in S_\varphi$, let s' be an inverse image of s in $\tilde{S}_\varphi$, and let $\tilde{s}$ denote the image of this element in $\tilde{\mathbb{S}}_\varphi$. Then we define

5.3 $\quad\delta_{I,s}(w)= \delta_{I,\tilde{s}}(w)$.

This is independent of the choice of s' (by Lemma 5.6 below).

If I and s are fixed we let $\delta(\ )=\delta_{I,s}(\ )$, which is a map from W^a_{im} to $\mathbb{C}^*$. It is immediate that δ satisfies the cocycle condition:

5.4 $\quad\delta_{I,s}(xy)=\delta_{I,s}(y)\delta_{y\times I,s}(x)$.

We will see below that $\delta_{I,s}(\ )$ takes values in $\{\pm1\}$.

Fix $S=(x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\Upsilon)$ as above so I is the standard module corresponding to S. Fix a regular weight ${}^d\lambda\epsilon({}^d\mathfrak{t}^*)$, dominant for ${}^dP$, so that $x^2 = \exp(\Upsilon^{-1}({}^d\lambda))$. Let ${}^d\Lambda$ be the character of the two-fold cover of ${}^dT(\mathbb{R})$ defined by ${}^\vee\rho$ constructed in the proof of Theorem 4.8 (or Theorem 3.6 if the infinitesimal character of I is integral) applied to ${}^\vee G$. We consider this as a character of the inverse image ${}^dT(\mathbb{R})^{can}$ of ${}^dT(\mathbb{R})$ in ${}^\vee G^{can}$. From the definition of ${}^d\Lambda$ ([AV2], Theorem 5.11) it is straightforward to conclude:

5.5 <u>Lemma</u>:

$$<I,\tilde{s}> = \frac{{}^d\Lambda}{e^{{}^d\lambda}}(\tilde{s}) \quad .$$

Notice that $x^2\epsilon Z(G)^\theta$. This implies ${}^d\lambda\epsilon P^*({}^dT,{}^\vee G)$ and ${}^\vee\theta{}^d\lambda-{}^d\lambda\epsilon X^*({}^dT)$, so $e^{{}^d\lambda}$ is a well-defined character of ${}^dT^{can}$. Also note that ${}^d\Lambda/e^{{}^d\lambda}$ has zero differential.

For later use we note the following lemma, which follows immediately from 5.5.

5.6 <u>Lemma</u>:

For x a strong real form of G, let τ denote the character of $\pi_1({}^\vee G)(\mathbb{R})\subset Z({}^\vee G^{can})$ corresponding to the element $z_{{}^\vee\rho}x^{-2}\epsilon Z(G)$ (cf. §4). If I is a standard module for the strong real form x, then $<I,\tilde{z}\tilde{s}>=\tau_x(\tilde{z})<I,\tilde{s}>$ for all $\tilde{z}\epsilon\pi_1({}^\vee G)(\mathbb{R})$, $\tilde{s}\epsilon\tilde{\mathbb{S}}_\varphi$.

For $w \in W^a_{im}$ define ${}^\vee w$ as in Lemma 3.19. It follows immediately from Lemma 5.5, and Lemma 3.19 that

$$5.7 \qquad \delta_{I,s}(w) = \frac{{}^\vee w({}^\vee w^{-1} \times {}^d\Lambda)}{{}^d\Lambda}(s) .$$

Note that the numerator and denominator of the right hand side are genuine characters of ${}^dT(\mathbb{R})^\sim_\rho$ with the same differential. It follows that the quotient is a character of ${}^dT(\mathbb{R})$, trivial on the identity component, so it takes values in $\{\pm 1\}$.

5.8 Lemma:

(1) For $u, v \in W^a_{im}$, $\delta_{I,s}(uv) = \delta_{I,s}(v)\delta_{v \times I,s}(u)$.

(2) For α root of T^a corresponding to a simple imaginary root of T, let ${}^\vee\alpha = \mathfrak{I}(\alpha^\vee) \in Q^*({}^\vee G, {}^dT)$. Then

$$\delta_{I,s}(s_\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is compact} \\ \operatorname{sgn}({}^\vee\alpha(s)) & \text{if } \alpha \text{ is non-compact} \end{cases} .$$

We prove this in a moment. One may use this Lemma to calculate $\delta(w)$ explicitly. These conditions are the same as ([S3], Propositions 2.1 and 3.1) and we use them in Lemma 5.14 to show $\delta(w) = \kappa(w^{-1})$, for κ as in [S3].

We need a few preliminaries. Given a standard module $I(\Psi, \Lambda)$ and a real root α, let $m_\alpha \in T$ be as in ([V1], Definition 8.3.8).

5.9 Definition:

The real root α satisfies the parity condition if

$$(\Lambda\otimes e^{\rho})(m_{\alpha}) = -(-1)^{{}^{\vee}\alpha(d\Lambda+\rho-\rho(\Psi))}.$$

To see that this agrees with Definition 8.3.11 of [V1], notice first that it is independent of the positive system used to define ρ. We may therefore choose this system to be maximally θ-stable and compatible with Ψ. In that case $I(\Psi,P_{im},\Lambda)$ can be constructed by induction by stages as in [V1], and $\rho-\rho(\Psi)$ is orthogonal to all real roots. We can therefore reduce to the case when all roots are real, in which case this definition is nearly identical to the one in [V1], and proving their equivalence is easy.

As is shown in [V2], duality has the property that:

5.10 α satisfies the parity condition $\Leftrightarrow$ ${}^{\vee}\alpha$ is non-compact.

proof of Lemma 5.8:

We have already mentioned (1). For (2), by 5.10 and the remark after Lemma 5.6 we need to show:

$$s_{{}^{\vee}\alpha}(s_{{}^{\vee}\alpha}\times{}^{d}\Lambda)/{}^{d}\Lambda\,(s) = \begin{cases} 1 & {}^{\vee}\alpha \text{ does not satisfy the parity condition} \\ \operatorname{sgn}({}^{\vee}\alpha(s)) & {}^{\vee}\alpha \text{ does satisfy the parity condition}. \end{cases}$$

The proof is now a transcription of ([V1], Lemma 8.3.17), but since it is of interest we sketch the argument. To conserve notation we drop the dual notation and prove the corresponding statement for G.

Recall ([AV1], §2) T_{ρ} is the set of pairs (s,z) such that $e^{2\rho}(s)=z^2$ ($s\in T$, $z\in\mathbb{C}^*$); and the genuine character e^{ρ} of T_{ρ} is given by $e^{\rho}(s,z)=z$. For α a root, define $\alpha: T_{\rho}\to\mathbb{C}$ by $\alpha(s,z)=\alpha(s)$, and define $\alpha^{\vee}:\mathbb{C}^*\to T_{\rho}$ by $\alpha^{\vee}(z)=(\alpha^{\vee}(z),z\rho(\alpha^{\vee}))$. Then if $t=(s,z)\in T(\mathbb{R})_{\rho}$, $s_{\alpha}(t)=t(\alpha^{\vee}(\alpha(s^{-1})))$. From the definition of the cross action we have $(s_\alpha \times \Lambda)(t)=\Lambda(t)\alpha(s^{-1})^n$, where $n=\alpha^\vee(d\Lambda)$.

Let $\nu=1$ (resp. 0) if α satisfies (resp. does not satisfy) the parity condition. We need to show $s_\alpha(s_\alpha \times \Lambda)(t)=\mathrm{sgn}(\alpha(t))^\nu \Lambda(t)$. Write the left-hand side as

$$\begin{aligned}(s_\alpha \times \Lambda)(s_\alpha(t)) &= (s_\alpha \times \Lambda)(t)(s_\alpha \times \Lambda)(\alpha^\vee(\alpha(s^{-1}))\\ &= \Lambda(t)\Lambda(\alpha^\vee(\alpha(s^{-1}))\alpha(s)^n.\end{aligned}$$

Thus we need to show:

5.11 $$\Lambda(\alpha^\vee(\alpha(s^{-1}))\alpha(s)^n=\mathrm{sgn}(\alpha(s))^\nu.$$

This is immediate if s is in the identity component of $T(\mathbb{R})$, so it is enough to show this for t an element of order 2. If $\alpha(s)=1$ the result is immediate. Suppose $\alpha(s)=-1$; we need to show:

$$\Lambda(\alpha^\vee(-1)) = (-1)^\nu(-1)^n.$$

It is immediate that

$$\begin{aligned}\Lambda(\alpha^\vee(-1)) &= (-1)^{\alpha^\vee(\rho)}(\Lambda\otimes e^\rho)(m_\alpha)\\ &= -(-1)^{n+\alpha^\vee(\rho(\Psi))+\nu},\end{aligned}$$

where the second equality is by the definition of the parity condition. But α is simple among real roots, so $\alpha^\vee(\rho(\Psi))=1$. Thus $\Lambda(\alpha^\vee(-1))=(-1)^{n+\nu}$, proving the Lemma.

We recall the construction of κ [S1]. Let strong real forms x and y be fixed, with $\theta=\mathrm{int}(x)$ and ${}^\vee\theta=\mathrm{int}(y)$. Suppose we are given:

5.12 (1) Cartan subgroups T^Γ and ${}^dT^\Gamma$

(2) $s\in{}^dT$, ${}^\vee\theta s=s$

(3) an isomorphism $\Upsilon: X_*(T)\approx X^*({}^dT)$ taking θ to $-{}^\vee\theta$

(4) $w\in W_{im}$.

(For (3), note that the choice of such an isomorphism is exactly equivalent to the choice of Υ in a set of L-data.)

Let $\iota(w)$ be the image of w in

$\{\, \gamma \in Q_*(G,T) \mid \theta(\gamma)-\gamma=0\}/Q_*(G,T)\cap\{\mu+\theta\mu \mid \mu\in X_*(G,T)\}$

under Tate-Nakayama duality ([S3], §2). We define $\kappa_{\Upsilon,s}(w)=\Upsilon(\iota(w))(s)$. Note that

$$\begin{aligned} \Upsilon(\iota(w))(s)^2 &= \Upsilon(\iota(w))(s\,{}^\vee\theta(s)) \\ &= \Upsilon(\iota(w)+{}^\vee\theta(\iota(w))(s) \\ &= 1, \end{aligned}$$

so $\kappa_{\Upsilon,s}(w)\in\{\pm 1\}$.

By [S3], Propositions 2.1 and 3.1) we have:

5.13 Lemma:

(1) $\kappa_{\Upsilon,s}(uv) = \kappa_{\Upsilon,s}(v)\kappa_{\Upsilon\circ v^{-1},s}(u) \quad (v \in W_{im})$

(2) $\kappa_{\Upsilon,s}(s_\alpha) = \begin{cases} 1 & \alpha \text{ is compact} \\ \Upsilon(\alpha^\vee)(s) & \alpha \text{ is non-compact}. \end{cases}$

5.14 Corollary:

Given the data of 5.12, fix P and ${}^dP$ so that the L-data $S=(x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\Upsilon)$ is a set of L-data. With I any representation given by the L-data S, we have

$\delta_{I,s}(w) = \kappa_{\Upsilon,s}(w)$.

proof:

Observe that a representation associated to the L-data $(x,T^\Gamma,wP,y,{}^dT^\Gamma,{}^dP,\Upsilon\circ w^{-1})$ is $w^{-1}\times I$. The Corollary follows from Lemmas 5.7 and 5.13.

It is interesting to compare these results with those of [S1] and [S3]. For fixed x, [S1] duality provides an embedding $\Pi_\varphi \hookrightarrow (S_\varphi/S_\varphi^0 Z^\Gamma)^\wedge$. This embedding is non-canonical, depending on a choice of $\pi\in\Pi$ (going to the trivial character). Furthermore the images of Π_φ as the real form varies are not disjoint; in particular the trivial character is in the image of Π_φ for all real forms. On the other hand, in our situation we have a canonical bijection $\Pi=\cup\Pi_\varphi \overset{1-1}{\leftrightarrow} (\tilde{\mathbb{S}}_\varphi)^\wedge$ (disjoint union).

In terms of Weyl groups this may be described as follows. Note that W_{im}^a acts on $\mathcal{E}$, and the stabilizer of a point x is W_{K_x}. This is canonical; and choosing a point in each orbit we have identified $\mathcal{E}$ with the union of W_{im}^a/W_{K_x}, where the union runs over equivalence classes of strong real forms. Under the bijection $\Pi \overset{1-1}{\leftrightarrow} \mathcal{E}$ this corresponds to the decomposition of Π under the cross action of W_{im}^a.

Fix $x\in\mathcal{E}$, and let $\theta=\theta_x$. Then $\mathcal{E} \overset{1-1}{\leftrightarrow} \mathcal{T}$ as in the proof of Theorem 5.1 . The resulting action of W_{im} on $\mathcal{T}$ is twisted: if $g\in G$ represents w, then g takes $t\in\mathcal{T}$ to $gt\theta(g^{-1})$. We obtain an embedding $W_{im}/W_{K_x} \hookrightarrow \mathcal{T}$ via $g\to g\bullet 1=g\theta(g^{-1})$. This gives a map $W_{im}\to H^1(T)$ and the cocyle condition follows as in ([S3], §2 and Proposition 3.1).

§6

Endoscopic Groups

We now define endoscopic groups in this context, and discuss their structural aspects.

As usual we are given G and an inner class of real forms, with L-groups for G and $^\vee G$. Thus we have specified $({}^\vee G^\Gamma, {}^\vee\mathcal{D})$ and $(G^\Gamma, \mathcal{D})$ (cf. §1). If $(\delta, B) \in \mathcal{D}$ we write $\delta \in \mathcal{D}$ as usual.

6.1 Definition:

An element s in an algebraic group is said to be elliptic if it is semisimple and for any algebraic representation π the eigenvalues of $\pi(s)$ have absolute value 1.

The next lemma is well known; the only point requiring care is that we do not assume that K is connected.

6.2 Lemma:

Let K be a reductive algebraic group endowed with a compact real form (so $K(\mathbb{R})$ is a maximal compact subgroup of K). Let s be an elliptic element of K.

(1) There exists $k \in K$ such that $ksk^{-1} \in K(\mathbb{R})$

(2) Suppose $s,t \in K(\mathbb{R})$, and $t = ksk^{-1}$ for some $k \in K$. Then there exists $\ell \in K(\mathbb{R})$ such that $t = \ell s \ell^{-1}$.

Recall (cf. §4) the canonical cover ${}^\vee G^{can}$ of ${}^\vee G$ defined in ([AV2], 7.11(b)). Let π denote the covering map ${}^\vee G^{can} \to {}^\vee G$.

6.3 Definition:

A set of endoscopic data for G is a triple $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee \mathcal{D}_H)$, where:

(1) $\tilde{s} \in {}^\vee G^{can}$, and $s = \pi(\tilde{s})$ is elliptic. Let ${}^\vee H$ denote the identity component of the centralizer of s in ${}^\vee G$.

(2) ${}^\vee H^\Gamma$ is a subgroup of ${}^\vee G^\Gamma$ satisfying:

(i) ${}^\vee H^\Gamma$ has two connected components,

(ii) ${}^\vee H$ is the identity component of ${}^\vee H^\Gamma$,

(iii) ${}^\vee H^\Gamma$ meets both components of ${}^\vee G^\Gamma$,

(3) $s \in Z({}^\vee H^\Gamma)$.

(4) $({}^\vee H^\Gamma, {}^\vee \mathcal{D}_H)$ is an extended group determined by z_ρ (Definition 1.1).

A set of weak endoscopic data for G is a pair $(\tilde{s}, {}^\vee H^\Gamma)$ satisfying (1)-(2).

The appropriate notion of equivalence of endoscopic data is a little subtle, and it is best stated in two parts. The equivalence relation we want is the smallest one containing the two kinds of equivalence in the next definition.

6.4 Definition:

(1) We say $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee \mathcal{D}_H)$ and $(\tilde{s}', {}^\vee H'^\Gamma, {}^\vee \mathcal{D}'_H)$ are equivalent if there exists $g \in {}^\vee G^{can}$ such that $\mathrm{int}(g)\tilde{s} = \tilde{s}'$, $\mathrm{int}(\pi(g)){}^\vee H^\Gamma = {}^\vee H'^\Gamma$, and $\mathrm{int}(\pi(g))({}^\vee \mathcal{D}_H) = {}^\vee \mathcal{D}'_H$.

(2) We say $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$ is <u>equivalent</u> to $(\tilde{z}\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}'_H)$ for $\tilde{z}$ contained in the identity component of the inverse image of $Z({}^\vee H^\Gamma)\cap {}^\vee H$.

Similarly we define equivalence of weak endoscopic data.

We write S for an equivalence class of endoscopic data, and let $\mathcal{S}$ denote the set of equivalence classes. We write $\mathcal{S}_{weak}$ for the equivalence classes of weak data. These sets depend only on the inner class of G, not on the particular real form. In Theorem 6.14 we will show there is a surjective map from $\mathcal{S}_{weak}$ to equivalence classes of endoscopic data as defined in ([S1], §2).

Note that given conditions 6.3 (1) and (2), (3) says: $\mathrm{int}(y)s=s$ for all (equivalently any) $y\in {}^\vee H^\Gamma - {}^\vee H$.

6.5 <u>Lemma</u>:

Suppose $(\tilde{s}, {}^\vee H^\Gamma)$ is a set of weak endoscopic data for G. Then there is an extended group structure ${}^\vee\mathcal{D}_H$ determined by z_ρ on ${}^\vee H^\Gamma$; in fact we may arrange that ${}^\vee\delta_H\in {}^\vee\mathcal{D}$ whenever ${}^\vee\delta_H\in {}^\vee\mathcal{D}_H$.

<u>proof</u>:

Choose $y\in {}^\vee H^\Gamma$ so that $\mathrm{int}(y)$ is a principal involution of ${}^\vee H$. (This is possible since every element of $\mathrm{Out}({}^\vee H)$ of order 2 is represented by a principal involution.) Choose a Cartan subgroup ${}^\vee T$ of ${}^\vee H$ maximally split with respect to $\mathrm{int}(y)$. By ([AV2], Lemma 9.14) we may extend $\mathrm{int}(y)|_{{}^\vee T}$ to a principal involution γ of ${}^\vee G$. Therefore $\gamma=\mathrm{int}({}^\vee\delta)$ for some ${}^\vee\delta\in\mathcal{D}$. Consequently ${}^\vee\delta=ty$ for some $t\in {}^\vee T\subset {}^\vee H$, so ${}^\vee\delta\in {}^\vee H^\Gamma$. Since ${}^\vee T$ was maximally split for the principal involution int(y) of $^\vee H$, and $\mathrm{int}(^\vee\delta)|_{^\vee T} = \mathrm{int}(y)|_{^\vee T}$, $\mathrm{int}(^\vee\delta)$ must be principal for $^\vee H$ ([AV2], Proposition 6.24). Let ^dB_H be any large Borel subgroup of H with respect to $\mathrm{int}(^\vee\delta)$, and define

$$^\vee\mathcal{D}_H = {}^\vee H\text{-conjugacy class of } (^\vee\delta, {}^dB_H).$$

Clearly this satisfies the conditions of the theorem.

6.6 Corollary (cf. Lemma 1.5):

Suppose $(\tilde{s}, {}^\vee H^\Gamma)$ is a set of weak endoscopic data for G. Then the set of possible extended group structures $^\vee\mathcal{D}_H$ making $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$ endoscopic data for G is a principal homogeneous space for

$$\{z \in Z(^\vee H) \mid z\,^\vee\theta_H(z) = 1\} / \{ w\,^\vee\theta_H(w^{-1}) \mid w \in Z(^\vee H) \}.$$

(Here the automorphism $^\vee\theta_H$ of $Z(^\vee H)$ is defined by the action of any element of $^\vee H^\Gamma - {}^\vee H$).

This is an immediate consequence of Lemmas 1.5 and 6.5. Notice that the set in question is finite. Recall that changing the choice of $^\vee\mathcal{D}_H$ has the effect of altering the Langlands parametrization of representations (by maps of $W_{\mathbb{R}}$ into $^\vee H^\Gamma$) by tensoring with a one-dimensional character.

Because of Lemma 6.5, it is often enough to study weak endoscopic data. Here are two alternative descriptions of such data.

6.7 Lemma:

The set of weak endoscopic data is in bijection with the set of equivalence classes of pairs $(\tilde{s}, {}^\vee\delta)$ where:

(1) $\tilde{s} \in {}^\vee G^{can}$, $s = \pi(\tilde{s})$ is elliptic,

(2) ${}^\vee\delta\in{}^\vee\mathfrak{D}$ commutes with s.

Let ${}^\vee H$ denote the identity component of the centralizer of s in ${}^\vee G$, and let ${}^\vee H^\Gamma=\langle {}^\vee H, {}^\vee\delta\rangle$.

(1) $(\tilde{s}, {}^\vee\delta)$ and $(\tilde{s}', {}^\vee\delta')$ are equivalent if there exists $g\in{}^\vee G^{can}$ such that $int(g)\tilde{s}=\tilde{s}'$ and $int(\pi(g))^\vee H^\Gamma={}^\vee H'^\Gamma$, and

(2) $(\tilde{s}, {}^\vee\delta)$ is equivalent to $(\tilde{z}\tilde{s}, {}^\vee\delta)$ for $\tilde{z}$ as in 6.4(2).

<u>proof</u>:

Given $(\tilde{s}, {}^\vee\delta)$, let ${}^\vee H^\Gamma=\langle {}^\vee H, {}^\vee\delta\rangle$. It is immediate that $(\tilde{s}, {}^\vee H^\Gamma)$ is a set of weak endoscopic data. The converse follows from Lemma 6.5; notice that we may require $int({}^\vee\delta)|_{{}^\vee H}$ to define a principal involution of ${}^\vee H$.

6.8 <u>Corollary</u>:

Fix ${}^\vee\delta_0\in{}^\vee\mathfrak{D}$, let ${}^\vee\theta=int({}^\vee\delta_0)$, and let ${}^\vee K=({}^\vee G)^{{}^\vee\theta}$. Then a set of representatives for the pairs in Lemma 6.7 is given by $\{(\tilde{s}, {}^\vee\delta_0)\mid \tilde{s}\in{}^\vee G^{can}\}$ satisfying:

(1) $\pi(\tilde{s})$ is an elliptic element of ${}^\vee K$

(2) ${}^\vee\theta|_{{}^\vee H}$ is a principal involution.

Two such elements $\tilde{s}$ and $\tilde{s}'$ correspond to equivalent weak endoscopic data if there exists $k\in{}^\vee K^{can}$ such that $\pi(k)\in{}^\vee K$, and $int(k)\tilde{z}\tilde{s}=\tilde{s}'$ for some $\tilde{z}$ as in 6.4(2).

We do not know whether the condition given here is necessary for the equivalence of $(\tilde{s}, {}^\vee\delta_0)$ and $\tilde{s}', {}^\vee\delta_0)$. Nevertheless Corollary 6.8 is often the most convenient way to construct endoscopic groups.

6.9 Definition:

Let S be an equivalence class of endoscopic data, with a representative $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$. An endoscopic group determined by S is a quasisplit group H which has an E-group isomorphic to $({}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$.

Of course the isomorphism class of H depends only on S. More precisely, if H' is another endoscopic group determined by S, then there is an isomorphism from H to H', unique up to composition with an inner automorphism of H. Note that H could be defined as a quasisplit group with dual group ${}^\vee H$, and inner class given by conjugation by ${}^\vee\delta$ (via $\mathrm{Aut}({}^\vee H)\rightarrow \mathrm{Out}({}^\vee H) \approx \mathrm{Out}(H)\hookrightarrow \mathrm{Aut}(H)$).

Recall (cf. §4) that maps of $W_{\mathbb{R}}$ into ${}^\vee H^\Gamma$ parametrize certain representations of covering groups of real forms of H. In the present case we can use the $\rho-\rho_H$ cover of H. We proceed to make this precise.

Let θ_H be a principal involution of a Cartan subgroup T_H of H, and choose a corresponding Cartan subgroup T of G. (This means that one chooses a dual Cartan subgroup ${}^\vee T_H$ of ${}^\vee H\subset {}^\vee G$, and then a Cartan subgroup $T\subset G$ dual to ${}^\vee T_H$.) Fix positive root systems for T in G and T_H in H, and let ρ (resp. ρ_H) be one-half the sum of the positive roots of T (resp. T_H). Choose one of the distinguished isomorphisms $T\approx T_H$ which intertwines the action of θ and θ_H; exponentiating the resulting homomorphism $P_*(G,T)\rightarrow P_*(H,T_H)$ we obtain a homomorphism $Z(G)\hookrightarrow Z(H)$ independent of all choices. This also permits us to identify $\gamma=\rho-\rho_H$ with an element of $P^*(H,T_H)$ (defined up to $X^*(H,T_H)$).

Let $K_H = H^{\theta_H}$. Let $\tilde{K}_H$ be the inverse image of K_H in the two-fold cover of H determined by γ. The isomorphism class of $\tilde{K}_H$ is independent of the choices. The next lemma follows immediately from the definitions and Theorem 4.8, and the remarks at (4.10).

6.10 Lemma:

Let S be an equivalence class of endoscopic data, with $({}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$ an associated extended group (Definition 6.7), and H an associated endoscopic group. Then maps of $W_{\mathbb{R}}$ into ${}^\vee H^\Gamma$ parametrize L-packets of genuine $(\mathfrak{h}, \tilde{K}_H)$-modules.

We also construct an extended group $(H^\Gamma, \mathcal{D}^H)$ containing H. By Lemma 1.5 to define the pair $(H^\Gamma, \mathcal{D}^H)$ up to isomorphism it is enough to choose $z \in Z(H)^{\theta_H}$.

6.11 Definition:

Given an equivalence class S of endoscopic data, and H a corresponding endoscopic group. Let x be a strong real form of G. An extended group defined by S and x is an extended group H^Γ containing H, determined by the element $x^2 \in Z(G) \hookrightarrow Z(H)$.

For later use we note that Proposition 2.1(a) and (b) provides a map from infinitesimal characters for $\mathfrak{h}$ to those for $\mathfrak{g}$. We say χ corresponds to χ_H if they may both be written χ_λ, for some $\lambda \in {}^d\mathfrak{t}$, ${}^d\mathfrak{t}$ a Cartan subalgebra of both ${}^\vee\mathfrak{h}$ and ${}^\vee\mathfrak{g}$. This is a several to one mapping from infinitesimal characters for $\mathfrak{h}$ to those for $\mathfrak{g}$.

We describe the connection between endoscopic groups defined above and as defined in [S1] and [LS]. Let $S_{[S1]}$ and $S_{[LS]}$ denote the set of equivalence classes of endoscopic data as defined by ([S1], §2.1) and ([LS], §2.1) respectively.

Let ${}^LG$ be an L-group for G in the conventional sense [Bo]. Recall ([AV2],§9) ${}^\vee G^\Gamma \approx {}^LG$ via $g \to g$ ($g \in {}^LG^0$) and ${}^\vee\delta \to m_\rho \sigma$. (Here $(\mathfrak{s}, {}^dB) \in {}^\vee\mathcal{D}$, we fix an isomorphism ${}^LG \approx {}^LG^0 \rtimes \Gamma$, $m_\rho \in {}^LG^0$ is defined in ([AV2],9.4) from ${}^dB$, and σ= the non-trivial element of Γ.) Note that int(${}^\vee\delta$) is inner to int(σ) on ${}^\vee G \approx {}^LG^0$.

In ([S1], §2.1) endoscopic data is defined as follows. A set of endoscopic data is defined to be a sextuple $(s, {}^LH^0, {}^LB_H{}^0, {}^LT_H{}^0, \{Y_{\alpha^\vee}\}, \rho_s)$. Here s is the Z^Γ-coset of a semisimple element of ${}^LG^0$ (Z^Γ is the Γ-invariants of the center of ${}^LG^0$) and ${}^LH^0$ is the identity component of the centralizer of s. Then ${}^LB_H{}^0$, ${}^LT_H{}^0$ and $\{Y_{\alpha^\vee}\}$ are a Borel subgroup, Cartan subgroup, and set of simple root vectors for ${}^LH^0$. Most significantly ρ_s is a map of Γ into Aut(${}^LH^0$) satisfying certain conditions. We refer the reader to the above reference for the details of the definition, and for the definition of equivalence of such data.

In the preceding discussion we have used the Galois form of the L-group, and have modified the definitions of [S1] accordingly. Aside from considerations of embeddings of L-groups of endoscopic groups this is not important. With this in mind we have the following lemma.

6.12 Lemma:

The set $\mathcal{S}_{[S1]}$ is in canonical bijection with the set of equivalence classes of pairs (s,ω) defined as follows.

6.13 (1) s is a semisimple element of ${}^LG^0$; let ${}^LH^0$ be the identity component of the centralizer of s in ${}^LG^0$.

(2) ω is a component of the centralizer of s in ${}^LG-{}^LG^0$. For $g\in\omega$ consider int(g) as an automorphism of ${}^LH^0$; we require that the corresponding element of Out(${}^LH^0$) have order two.

(3) Data (s,ω) and (s',ω') are equivalent if there exists $g\in{}^LG^0$ such that $g{}^LH^0g^{-1}={}^LH'^0$, and $g\omega g^{-1}=\omega'$.

sketch of proof:

This is an exercise using ([S1] Lemma 2.1.6 and Lemma 2.1.7). Note that the elements of ω are of the form $g\times\sigma$, $g\in{}^LG^0$. Hence there exists such a component ω if and only if the conjugacy class of s is fixed by int($1\times\sigma$), or equivalently s is conjugate to $\sigma(s)$ via ${}^LG^0$.

Suppose (s,ω) is given. We obtain endoscopic data by taking sZ^Γ, ${}^LH^0$, any choice of Borel and Cartan subgroups, and set of simple root vectors for ${}^LH^0$. We define the map ρ_s as follows. Choose $g\in\omega$. Then int(g) is an element of Aut(${}^LH^0$). Let γ denote the image of int(g) in Out(${}^LH^0$), and using the preceding data define a splitting φ:Out(${}^LH^0$) $\rightarrow$ Aut(${}^LH^0$). This gives us an element τ of Aut(${}^LH^0$) of order two; let ρ_s map σ to τ. This defines a map from pairs (s,ω) to sets of endoscopic data, which yields the desired bijection of equivalence classes.

Using Lemma 6.12 we refer to (s,ω) as a set of endoscopic data.

Note that ${}^LH^0$ together with γ (as in the proof of the Lemma) determine an L-group ${}^LH$. However the group generated by ${}^LH^0$ and

$g\times\sigma$ is not necessarily isomorphic to ${}^LH$. In fact the former is an E-group, and not necessarily an L-group.

Given weak endoscopic data $(\tilde{s},{}^\vee H^\Gamma)$, let $s=\pi(\tilde{s})$ and let $\omega={}^\vee H^\Gamma-{}^\vee H$. Since ${}^LG\approx{}^\vee G^\Gamma$, we obtain endoscopic data (s,ω) in the sense of [S1]. It is immediate that this defines a map Ψ which is well defined and surjective on equivalence classes. Thus we have:

6.14 Theorem:

The map Ψ is a surjection $\Psi\colon\ \mathcal{S}_{\mathbf{weak}}\to\mathcal{S}_{[S1]}$.

The map Ψ fails to be injective for two reasons. The first is our use of the covering group ${}^\vee G^{can}$. Secondly, the notion of equivalence in [S1] is weaker than ours. Our "extra" endoscopic data will be used to eliminate the need for choices in lifting, to be defined in section 9. We discuss a few examples at the end of this section. For further examples see section 11.

We briefly discuss the connection between this data and that of [LS]. The definition of $\mathcal{S}_{[LS]}$ incorporates an embedding of the group $\mathcal{H}$ into ${}^LG$ (in most instances $\mathcal{H}$ is the L-group of the endoscopic group). As noted above this requires the use of the Weil form of the L-group. If we incorporate these changes into our definition of $\mathcal{S}$, we obtain a surjective map $\Psi\colon\mathcal{S}_{\mathbf{weak}}\to\mathcal{S}_{[LS]}$. Again Ψ fails to be injective because of our use of ${}^\vee G^{can}$. Our notion of equivalence is close to that of [LS] (cf. [LS], §2.1,(iii)), although we do not allow variation of $\tilde{s}$ by arbitrary elements in $Z({}^\vee G)$.

It is worth mentioning that the group ${}^\vee H^\Gamma$ is an E-group for H, and <u>not</u> necessarily an L-group, even in cases when it is possible to embed ${}^LH$ in ${}^LG$. Thus we do not need an extra choice of an embedding of the L-group of H in ${}^LG$ (we do not even assume that such an embedding exists), and the embedding of the E-group ${}^\vee H^\Gamma$ is canonical; the tradeoff is that we introduce a covering group of H.

Consider the notion of weak endoscopic data given by Corollary 6.8. Thus we are considering simply elements $\tilde{s}\in{}^\vee K^{can}$. It is interesting to consider the map $\Psi:S\to S_{[S1]}$ in this context. The crucial point is that Shelstad's automorphism of ${}^LH^0$ given by the component ω is replaced by our $int({}^\vee\delta_0)$, where now we have the advantage that ${}^\vee\delta_0$ is fixed.

Suppose for example that G is split (or inner to a split group). Given a semsimple element s of ${}^LG^0$, the choice of component ω is then canonically equivalent to a choice of a component of the centralizer of s in ${}^LG^0$. Furthermore given (s,ω) the corresponding endoscopic group H is split if and only if ω is taken to be the identity component. Hence note that every endoscopic group is split if all such centralizers are connected.

The corresponding facts for weak endoscopic data in the context of Corollary 6.8 are the following. Suppose we are given $\tilde{s}$, $\pi(\tilde{s})=s\in{}^\vee K$. Then the corresponding endoscopic group is split if and only if s is contained in the identity component of ${}^\vee K$. Thus every endoscopic group is split if and only if ${}^\vee K$ is connected.

For example, suppose G is SL(2). Then ${}^\vee G=PGL(2)$, and ${}^\vee G^{can}=SL(2)$. We write elements of $PGL(2,\mathbb{C})$ via representatives in GL(2). We may take ${}^\vee\delta$ acting by $int(diag(1,-1))$, so

$${}^{\vee}K = \langle \{ \mathrm{diag}(z, z^{-1}) \mid z \in \mathbb{C} \}, \varepsilon \rangle / \{\pm I\} \approx O(2)$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\mathcal{S}_{\mathbf{weak}}$ has representatives {I, -I, diag(i,-i), ε }. The corresponding endoscopic groups are $SL(2,\mathbb{R})$, $SL(2,\mathbb{R})$, $\mathbb{R}^*$, and S^1 respectively. Note that ε is the only element not contained in the identity component of ${}^{\vee}K$, so S^1 is the only non-split endoscopic group. The map Ψ takes the same value on I and -I, but not on any other pair of distinct elements of $\mathcal{S}_{\mathbf{weak}}$.

Suppose G is PGL(2), so ${}^{\vee}G = {}^{\vee}G^{can} = SL(2)$. Suppose ${}^{\vee}\delta$ acts by int(diag(1,-1)). Then ${}^{\vee}K \approx SO(2)$, and is connected. Thus all endoscopic groups are split. We may take as representatives for $\mathcal{S}_{\mathbf{weak}}$ {I, -I, ε}, with corresponding endoscopic groups $PGL(2,\mathbb{R})$, $PGL(2,\mathbb{R})$, and $\mathbb{R}^*$ respectively. Again the map Ψ takes the same value only on I and -I. (In this case there are two equivalence classes of endoscopic data corresponding to I and -I, because of the two possible extended group structures on $SL(2) \times \Gamma$.)

In both of these examples the endoscopic data I and -I give the same ordinary lifting, but we will see that they give different super-lifting.

See §11 for more examples.

§7

Method of Descent

We restate some results of [AV1] in the setting of this paper. Let G, an inner class of real forms, and L-groups G^Γ and ${}^\vee G^\Gamma$ be given as usual. Let x be a strong real form of G. Let $\theta = int(x)$ and $K_x = G^\theta$ be as usual, and suppose s is an elliptic element of K_x.

Choose an antiholomorphic involution γ of G fixing s, corresponding to the Cartan involution θ. Thus $G^\gamma \cap K_x$ is a maximal compact subgroup of G^γ (cf. §1). Such a γ exists by Lemma 6.2. Write $G(\mathbb{R}) = G^\gamma$.

Let H be the identity component of the centralizer of s in G. Then γ and θ stabilize H, and $\gamma|_H$ is a real form of H corresponding to the Cartan involution $\theta|_H$. Let $H(\mathbb{R}) = H^\gamma$; then $s \in H(\mathbb{R})$. Write $\tilde{G}$ for the canonical covering of G (cf. (4.6)). (We will need eventually to consider also the canonical covering of a certain subgroup of G, and we prefer to reserve the superscript "can" for that.)

Now suppose we are given a semisimple element $\tilde{s} \in \tilde{G}$ such that $s = \pi(\tilde{s})$. Furthermore suppose we are given a Borel subgroup B of G containing s. Let χ be an infinitesimal character for $\mathfrak{g}$.

Suppose Θ is a virtual $(\mathfrak{g}, K_x)$-module. We assume the infinitesimal character of Θ is the same as that of a finite-dimensional representation of $\tilde{G}$. (It is not difficult to show that this is equivalent to the apparently weaker assumption that the infinitesimal character be regular and integral.) This guarantees that we may use the covering group $\tilde{G}$ instead of G^{sc} in [AV1].

7.1 Definition:

The descent of Θ, written $Des_{(\tilde{s},B,\chi)}(\Theta)$, is defined by ([AV1], Definition 2.4). It is an $H(\mathbb{R})$-invariant eigendistribution defined in a neighborhood of the identity Ω in $H(\mathbb{R})$.

The equivalence class of the pair $(H(\mathbb{R}), Des_{(\tilde{s},B,\chi)}(\Theta))$ is independent of the choice of Υ, and depends only on the H-conjugacy class of B. Furthermore if $\pi(\tilde{s})=\pi(\tilde{s}')$ then $Des_{(\tilde{s},B,\chi)}(\ )=CDes_{(\tilde{s}',B,\chi)}(\ )$ for some constant C.

We obtain an infinitesimal character χ_H for H as in [AV1]. That is define $B_H=B\cap H$ and choose a Cartan subgroup T_H of H contained in B_H. Write $\chi=\chi_\lambda$ for $\lambda\in\mathfrak{t}^*$, dominant with respect to the positive root system defined by B. Then we define χ_H to be the infinitesimal character for $\mathfrak{h}$ defined by λ. Thus in the the setting of Definition 7.1, if Θ has infinitesimal character χ then $Des_{(\tilde{s},B,\chi)}(\Theta)$ has infinitesimal character χ_H.

Let $\rho_H\in P^*(H,T_H)$, $\rho\in P^*(G,T_H)\hookrightarrow P^*(H,T_H)$ be as usual, given by the positive roots defined by B_H and B respectively. Let $\tilde{H}(\mathbb{R})$ be the canonical cover of $H(\mathbb{R})$.

Recall that a virtual character is a Z-linear combination of irreducible characters. A $\mathbb{C}$-linear combination of irreducible characters will be called a complex virtual character.

7.2 Theorem ([AV1], Corollary 2.13):

(1) There exists a complex virtual character Θ_H of $\tilde{H}(\mathbb{R})$ such that Θ_H restricted to Ω is equal to $Des_{(\tilde{s},B,\chi)}(\Theta)$. We may choose Θ_H to be of type $\tau({}^\vee z_\rho {}^\vee z_{\rho_H},1)$ (cf. (4.7)).

(2) If Θ is a standard module then $Des_{(\xi,B,\chi)}(\Theta)=c\Theta_H$, where Θ_H is a standard module and c is a root of unity. Both c and Θ_H may be computed explicitly.

We need to express this result in terms of $(\mathfrak{h},K_H)$-modules. Thus let $K_H=K_x\cap H$, and let $\tilde{K}_H$ be the preimage of K_H in $\tilde{H}$; this is a complexified maximal compact subgroup of $\tilde{H}(\mathbb{R})$.

Note that the complex virtual character Θ_H of Theorem 7.2(1) is not unique: it may be replaced by Θ_H+Z for any complex virtual character Z which vanishes on Ω. Thus (by abuse of notation) we redefine Descent as follows:

7.3 Definition:

$Des_{(\xi,B,\chi)}(\Theta)$ is the complex virtual $(\mathfrak{h},\tilde{K}_H)$-module Θ_H defined by 7.2(1), and is defined modulo the space of complex virtual characters vanishing on Ω.

Note that by 7.2(2) $Des_{(\xi,B,\chi)}(\Theta)$ may be explicitly computed for any character Θ for which there is a character formula expressing Θ as a linear combination of standard modules. In principal this holds for any irreducible character by the Kazhdan-Lusztig conjecture.

We compute the descent of a standard module in terms of the parameters of the previous sections. First we define extended groups containing H and $^{\vee}H$.

Let $H^{\Gamma}=H\cup Hx\subset G^{\Gamma}$ (note that H is int(x)-stable). Let δ_H^0 be any element in Hx such that $int(\delta_H^0)$ is a principal involution of H. Let B_H^0 be any Borel subgroup of H which is large with respect to δ_H^0, and let $\mathfrak{D}_H$ be the H-conjugacy class of (δ_H^0, B_H^0).

Let ${}^\vee H^\Gamma$ be an E-group containing ${}^\vee H$ determined by the element ${}^\vee z_\rho$ (Lemma 1.5). By Theorem 4.8 and the remarks at the end of section 4, strong L-data for the pair H^Γ, ${}^\vee H^\Gamma$ (containing x) parametrize $(\mathfrak{h}, \tilde{K}_H)$ modules of type $\tau({}^\vee z_\rho {}^\vee z_{\rho_H})$. Let $S=(x, T^\Gamma, B, y, {}^dT^\Gamma, {}^dB, \Upsilon)$ be a set of (strong) integral L-data for G. Choose an element ${}^\vee\delta \in {}^dT^\Gamma \cap {}^\vee\mathfrak{D}$ making the positive imaginary roots in ${}^dB$ into a special set of positive imaginary roots for ${}^\vee\delta$ (see the discussion after Theorem 3.6 above). Then ${}^\vee\delta$ gives an isomorphism

$$\eta_G: {}^dT^\Gamma \to {}^\vee T^\Gamma.$$

Assume now that $s \in T$, i.e. that T is contained in H. By ([AV2], Lemma 9.16) choose a Cartan subgroup ${}^dT_H^\Gamma$ of ${}^\vee H^\Gamma$ dual to T^Γ. Fix an isomorphism $\Upsilon_H: {}^\vee T \to {}^dT_H$ distinguished for H and respecting Cartan involutions. Define ${}^dB_H$ to be the Borel subgroup of ${}^\vee H$ containing ${}^dT_H$ and corresponding to $B \cap H$ under Υ_H. Choose ${}^\vee\delta_H \in {}^dT_H^\Gamma \cap {}^\vee\mathfrak{D}_H$ making the positive imaginary roots in ${}^dB_H$ special. We obtain an isomorphism

$$\eta_H: {}^dT_H^\Gamma \to {}^\vee T^\Gamma.$$

(Here we use the fact that $({}^\vee\delta_H)^2 = {}^\vee z_\rho$ by the construction of ${}^\vee H^\Gamma$.) Finally, we get an isomorphism

$$\eta_G^{-1} \circ \eta_H = \eta_{H,G}: {}^dT_H^\Gamma \to {}^dT^\Gamma.$$

Define $y_H = (\eta_{H,G})^{-1}(y)$.

7.4 <u>Theorem</u>:

Let $S=(x, T^\Gamma, B, y, {}^dT^\Gamma, {}^dB, \Upsilon)$ be a set of (strong) integral L-data. Fix a regular parameter $\lambda \in \mathcal{P}(S)$ and let $\chi = \chi_\lambda$. Let Θ be the standard

$(\mathfrak{g},K_x)$-module $I(S,\lambda)$ defined by S. Let $\tilde{s}$ and B' be given as in Definition 7.1, so $Des_{(\tilde{s},B',\chi)}$ is defined. Let $s=\pi(\tilde{s})$.

(1) Suppose s is not conjugate via K_x to an element of T, or B is not conjugate via HK_x to B'. Then $Des_{(\tilde{s},B',\chi)}(\Theta)=0$.

(2) Suppose s is conjugate via K_x to an element of T, and $int(k)B=int(h)B'$ for some $k\in K_x$ and $h\in H$. By replacing B by $int(k)B$, we may assume $B=int(h)B'$. Choose ${}^dT^\Gamma_H$ and define ${}^dB_H$, η_H, y_H, and Υ_H as in the paragraph preceding the theorem.

Set

$$S_H=(x,T^\Gamma, B\cap H, y_H, {}^dT^\Gamma_H, {}^dB_H, \Upsilon_H),$$

and let Θ_H be the corresponding standard $(\mathfrak{h},\tilde{K}_H)$ module with infinitesimal character χ_H.

Then $Des_{(\tilde{s},B,\chi)}(\Theta)=\sigma\Theta_H$ for some $\sigma\in\mathbb{C}$.

proof:

We refer to ([AV1], Theorem 2.11) as (*). Case (1) follows immediately from (*) and Lemma 6.2.

Suppose we are in case (2). Let $Des(\Theta)=Des_{(\tilde{s},B,\chi)}(\Theta)$. By conjugating by K_x we may assume $s\in T$, and furthermore that $B=int(h)B'$ for some $h\in H$. Recall that Θ is constructed from a genuine character Λ of $T(\mathbb{R})_\rho$, which in turn depends on y,λ, and the isomorphism η_G. One can check that, in the notation of [AV1], $\Theta=\Theta_G(T(\mathbb{R}),\Lambda)$. Now (*) says that $Des(\Theta)=(\text{constant})\Theta_H(T(\mathbb{R}),\Lambda)$. On the other hand, Λ is obviously equal to the character of $T(\mathbb{R})_\rho$ constructed using y_H, λ, and the isomorphism η_H. Consequently $\Theta_H(T(\mathbb{R}),\Lambda)$ is the standard module associated to the L-data S_H. This completes the proof of the Theorem.

The first thing to notice about this result is that the dependence of Des(Θ) on the element $\tilde{s}$ is quite weak: except for the constant σ in front, Des(Θ) depends only on the groups H and B.

For the applications to lifting, we will need to regard Des(Θ) as belonging to the block of a finite-dimensional representation. The character Θ_H of Theorem 7.4 belongs to such a block if and only if y_H defines a principal involution of ${}^\vee H^\Gamma$, which is not true in general. What we use is the fact that Des(Θ) is defined only near the identity, so the corresponding L-data is not well-defined. The next Lemma considers this situation.

7.5 <u>Lemma</u>:

Suppose $G(\mathbb{R})$ is a real reductive group with abelian Cartan subgroups, and $T(\mathbb{R}) \subset G(\mathbb{R})$ is a Cartan subgroup.

(1) Let Λ_1 and Λ_2 be genuine characters of $T(\mathbb{R})_\rho$ having the same G-regular differential $\lambda \in \mathfrak{t}^*$. Then $\Theta(T(\mathbb{R}),\Lambda_1)$ and $\Theta(T(\mathbb{R}),\Lambda_2)$ agree on a neighborhood of the identity.

(2) The space of virtual characters of regular infinitesimal character vanishing in a neighborhood of the identity is spanned by the set of all $\Theta(T(\mathbb{R}),\Lambda_1)-\Theta(T(\mathbb{R}),\Lambda_2)$, as $T(\mathbb{R})$ and Λ_i vary as in (1).

<u>proof</u>:

Part (1) follows immediately from [D1]; see also ([AV1], 3.8). Part (2) follows from the relationship between Langlands parameters and leading terms of character formulas, established in [HS].

7.6 Corollary:

Suppose $S = (x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\zeta)$ and $S'=(x,T^\Gamma,P,y',({}^dT^\Gamma)',{}^dP',\zeta')$ are two sets of L-data for G, attached to E-groups ${}^\vee G^\Gamma$ and $({}^\vee G^\Gamma)'$ respectively. Assume that ${}^dT={}^dT'$, ${}^dP={}^dP'$, $\zeta=\zeta'$, and $y^2=(y')^2$. Fix $\lambda\in\mathcal{P}(S)=\mathcal{P}(S')$, and let Θ (respectively Θ') denote the character of the standard $(\mathfrak{g},\tilde{K}_x)$-module attached to S and λ (respectively S' and λ). Then $\Theta-\Theta'$ vanishes near the identity.

proof:

The character Θ is $\Theta(\tilde{T}(\mathbb{R}),\Lambda)$, where Λ has differential $\zeta^{-1}(\lambda)\in{}^\vee\mathfrak{t}\approx\mathfrak{t}^*$; and Θ' is $\Theta(\tilde{T}(\mathbb{R}),\Lambda')$. Since $\zeta=\zeta'$, the result is an immediate consequence of Lemma 7.5(1).

7.7 Corollary:

Let $S = (x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\zeta)$ be a set of integral L-data, and assume that $s\in T$. Fix a regular parameter $\lambda\in\mathcal{P}(S)$, and let $\Theta=\Theta(S,\lambda)$. Let $({}^\vee H^\Gamma)'$ be an extenced group containing ${}^\vee H$ defined by $y^2\in Z({}^\vee G)\hookrightarrow Z({}^\vee H)$. Choose a Cartan subgroup $({}^dT_H^\Gamma)'$ of $({}^\vee H^\Gamma)'$ dual to T^Γ, and a distinguished isomorphism ζ'_H: ${}^\vee T\rightarrow{}^dT'_H$ implementing the duality. Define ${}^dB'_H$ to be the Borel subgroup of ${}^\vee H$ containing ${}^dT'_H$ corresponding to $B\cap H$ under ζ'_H. Choose ${}^\vee\delta'_H\in({}^dT_H^\Gamma)'\cap{}^\vee\mathcal{D}'_H$ making the positive imaginary roots in ${}^dB_H$ special. Set

$$S'_H= (x,T^\Gamma, B\cap H, {}^\vee\delta'_H, ({}^dT_H^\Gamma)', {}^dB'_H, \zeta'_H).$$

Then $\zeta'_H\circ\zeta^{-1}$: ${}^d\mathfrak{t}\rightarrow{}^d\mathfrak{t}'_H$ induces an inclusion $\mathcal{P}(S)\hookrightarrow\mathcal{P}(S'_H)$; write λ_H for the image of λ. Then

$$\mathrm{Des}_{(\tilde{s},B,\chi)}(\Theta) = \sigma\Theta_H(S'_H,\lambda_H).$$

The constant σ is the same as in Theorem 7.4. The standard $(\mathfrak{h},\tilde{K}_H)$-module on the right has type $\tau(y^{2^\vee}z_{\rho_H})$.

proof:

This is an immediate consequence of Theorem 7.4 and Corollary 7.6 (applied to H). The main point is that ${}^\vee\delta_H=y^2$ by the construction of $({}^\vee H^\Gamma)'$.

For our applications we need to drop the assumption that $x^2 \in Z(G)$. We continue to assume integral infinitesimal character for G, i.e. $y^2 \in Z({}^\vee G)$, and that x^2 belongs to H. Let $c=x^2$, and let G_c be the identity component of the centralizer of c in G, θ=int(x), $K_x=(G_c)^\theta$, etc. as in §4. Let s be an elliptic element of K_x. Let H (respectively H_c) be the identity component of the centralizer of s in G (respectively G_c). Let $K_{H,x}=H \cap K_x$, and let $H(\mathbb{R})_c=(H_c)^\gamma$; here γ is an antiholomorphic involution of G_c fixing s, corresponding to the Cartan involution θ.

Recall that $\tilde{G}$ is the canonical covering of G, with group $\pi_1(G)(\mathbb{R})=\pi_1(G)/(1+\Theta)\pi_1(G)$. Write G_c^{can} for the canonical covering of G^c, which has group $\pi_1(G_c)(\mathbb{R})=\pi_1(G_c)/(1+\Theta)\pi_1(G_c)$. Since G_c is a subgroup of G sharing a maximal torus, there is a natural surjection $\pi_1(G_c)\rightarrow\pi_1(G)$. This induces $\pi_1(G_c)(\mathbb{R})\twoheadrightarrow\pi_1(G)(\mathbb{R})$ and (writing $\tilde{G}_c$ for the preimage of G_c in $\tilde{G}$) $G_c^{can} \twoheadrightarrow \tilde{G}_c$.

We want to descend from G_c to H_c. To do this using the theory outlined above requires the choice of a preimage $\tilde{s}^{can}$ of $\tilde{s}$ in G_c^{can}. We will see eventually (Corollary 7.11) that descent is independent of this choice; but it is not yet convenient to prove that, so we simply carry along the extra choice temporarily. Fix a Borel subgroup B_c of G_c and a G_c-regular infinitesimal character χ_c for G_c. For Θ a virtual $(\mathfrak{g}_c, K_x^{can})$-module we may then define $Des(\Theta) = Des_{(\tilde{s}^{can}, B_c, \chi_c)}(\Theta)$ as before; it is a virtual $(\mathfrak{h}_c, K_{Hx}^{can})$-module defined near the identity in $H_c(\mathbb{R})^{can}$. (This last group and K_{Hx}^{can} are related to the canonical cover of H_c, not of G_c.)

We could now formulate generalizations of Theorem 7.4 and Corollary 7.7. We state only the latter, since that is the one used in endoscopy.

7.8 <u>Corollary</u>:

Let $S = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \mathcal{T})$ be a set of L-data for G^Γ and ${}^\vee G^\Gamma$ (Definition 4.3). Assume that $y^2 \in Z({}^\vee G)$, and write $c = x^2 \in G$. Fix a G_c-regular weight $\lambda \in \mathcal{P}(S)$; then $\Theta = \Theta(S, \lambda)$ is the character of a standard $(\mathfrak{g}_c, K_x^{can})$-module of of type $\tau({}^\vee z_\rho {}^\vee z_{\rho(c)})$ (Theorem 4.8).

Now assume that $s \in T \cap K_x$, and that $\tilde{s}^{can}$ is a preimage of s in G_c^{can}. Write H_c for the identity component of the centralizer of s in G_c, B_c for the Borel subgroup of G_c corresponding to P, and χ_c for the G_c-infinitesimal character attached to λ. Write P_H for the roots of T in H belonging to P.

Construct an E-group $({}^\vee H^\Gamma)'$ for H defined by $y^2 \in Z({}^\vee G) \subset Z({}^\vee H)$. As in Corollary 7.7, choose $({}^dT_H^\Gamma)' \subset ({}^\vee H^\Gamma)'$ dual to T^Γ, and $\mathcal{T}_H: {}^\vee T \to {}^dT'_H$. Define ${}^dP'_H$ to be the roots of ${}^dT_H$ in ${}^\vee H$ corresponding to P_H, and choose ${}^\vee\delta'_H$ as in Corollary 7.7. Set

$$S'_H = (x, T^\Gamma, P_H, {}^\vee\delta'_H, ({}^dT_H^\Gamma)', {}^dP'_H, \mathcal{T}'_H)$$

$$\lambda_H = \mathcal{T}_H(\mathcal{T}^{-1}(\lambda)) \in \mathcal{P}(S'_H).$$

Then

$$\mathrm{Des}_{(\mathfrak{s}^{can},B_c,\chi_c)}(\Theta) = \sigma\Theta_H(S'_H,\lambda_H)$$

for some complex number σ. The standard $(\mathfrak{h}_c, K^{can}_{Hx})$-module on the right has type $\tau(y^{2^\vee}z_{\rho_H}(c))$.

We next need to specify the constant σ appearing in Theorem 7.4 and Corollary 7.8.

7.9 Definition:

(1) Given G and a Cartan involution θ, let $K=G^\theta$, and let A_f be the split part of a fundamental Cartan subgroup of G. We define:

$q(G,\theta) = \frac{1}{2}[\dim(G/K) - \dim(A_f)]$.

(2) If $x \in G^\Gamma - G$, we define

$q(G,x) = q(G_{x^2},\theta)$

$= \frac{1}{2}[\dim(G_{x^2}/K_x) - \dim(A_f)]$

where G_{x^2}, K_x, $\theta=\mathrm{int}(x)$ are as usual (cf. §3) and A_f is the split part of a fundamental Cartan subgroup of G_{x^2}. If we do not specify x, then q(G) is defined to be $q(G,\delta)$ with δ defining the involution for the quasisplit form of G.

(3) Suppose π is a standard or irreducible $(\mathfrak{g}_{x^2},K_x)$-module with integral infinitesimal character. We define $\ell_0(\pi)$ by Definition 3.24, with $c_0=\frac{1}{2}\dim(A_f)-q(G,x)$ and A_f as in (2).

Note that $2q(G,x)$ is the dimension of $\mathfrak{p}_x/\mathfrak{a}_f$, where $\mathfrak{p}_x$ is the -1 eigenspace of θ on $\mathfrak{g}_{x^2}$, and $\mathfrak{a}_f$ is the Lie algebra of A_f. This decomposes into root spaces for the action of the compact part of the fundamental Cartan subgroup. Since these these roots come in pairs this number is even, so $q(G,x) \in \mathbb{Z}$.

We have included the next lemma for comparison with ([AV1], Lemma 2.12).

7.10 Lemma:

Given $\pi \in \mathfrak{M}(\mathfrak{g}_{x^2}, K_x)$ having regular integral infinitesimal character, let $I = I(\Psi, P_{im}, \Lambda)$ be the corresponding standard module. Let Δ^+ be the positive roots of T in G_{x^2} determined by $d\Lambda$. Then

$$\ell_0(\pi) = 2q(G,x) - |\{\alpha \in \Delta^+ \mid \alpha \text{ is imaginary non-compact}\}| - \tfrac{1}{2}|\{\alpha \in \Delta^+ \mid \alpha \text{ is complex}, \theta\alpha \in \Delta^+\}| .$$

proof:

In the notation of 3.26, let a,b,c,d and e denote the number of: imaginary non-compact roots $\alpha \in \Delta^+$; imaginary compact roots $\alpha \in \Delta^+$; pairs of complex roots $\alpha, \theta\alpha \in \Delta^+$; pairs of complex roots $\alpha, -\theta\alpha \in \Delta^+$, and real roots $\alpha \in \Delta^+$ respectively. Then $q(G,x) = \frac{1}{2}\dim(G_x/K_x) - \frac{1}{2}\dim(A_f) = \frac{1}{2}\dim(A) + a + c + d + \frac{1}{2}e - \frac{1}{2}\dim(A_f)$. Thus, with A the split part of T,

$$\begin{aligned} &\ell_0(\pi) - 2q(G,x) \\ &\quad = [d + \tfrac{1}{2}e + \tfrac{1}{2}\dim(A)] - \tfrac{1}{2}\dim(A_f) \\ &\qquad\quad - [\tfrac{1}{2}\dim(A) + a + c + d + \tfrac{1}{2}e - \tfrac{1}{2}\dim(A_f)] \\ &\quad = -a - c , \end{aligned}$$

proving the Lemma.

With this normalization of lengths (on G and H) we have:

7.11 Corollary:

(1) Suppose we are in the setting of Theorem 7.4. Let Λ be as in the proof, considered as a character of the inverse image $\tilde{T}(\mathbb{R})$ of $T(\mathbb{R})$ in $\tilde{G}$. Then the constant σ of Theorem 7.4 is given by:

$$\sigma = (-1)^{\ell_0(\Theta)+\ell_0(\Theta_H)} \frac{\Lambda}{e^{d\Lambda}}(\tilde{s})$$

(2) Suppose we are in the setting of Corollary 7.8. Now Λ is a character of the inverse image of $T(\mathbb{R})$ in $(G_c)^{can}$, and $\Lambda/e^{d\Lambda}$ factors to $\tilde{T}(\mathbb{R})$ (cf. the discussion preceding Corollary 7.7). Then (1) holds as stated. (In particular $Des(\Theta)$ depends only on $\tilde{s}$, not on $\tilde{s}^{can}$).

In all cases, the dependence of $Des(\Theta)$ on $\pi_1(G)(\mathbb{R})$ (as $\tilde{s}$ varies over the preimage of s in $\tilde{G}$) is of type $\tau({}^{\vee}z_\rho y^{-2})$). In particular if $y^2={}^{\vee}z_\rho$, then $Des(\Theta)$ depends only on s itself.

proof:

By Lemma 7.9,

$\ell_0(\pi) \equiv |\{\alpha\in\Delta^+ \mid \alpha \text{ is imaginary non-compact}\}| -$

$\frac{1}{2}|\{\alpha\in\Delta^+ \mid \alpha \text{ is complex, } \theta\alpha\in\Delta^+\}| \bmod(2\mathbb{Z})$.

Part (1) follows immediately from ([AV1], Theorem 2.11). For (2) and the last claim, notice the $e^{d\Lambda}$ defines a character of $\tilde{T}$ of type y^2 (by the definition of the bijection between characters of $\pi_1(G)(\mathbb{R})$ and $Z({}^{\vee}G)^{\theta}$). By the construction of Λ in the Langlands classification, it is a character of $\tilde{T}(\mathbb{R})$ of type ${}^{\vee}z_\rho$ (the type of the E-group ${}^{\vee}G^{\Gamma}$). This proves the Corollary.

7.12 Corollary:

Suppose $(\tilde{s},B,\chi)$ and $(\tilde{s}',B',\chi')$ are two choices of the data of Definition 7.1, and $\Theta\in\mathcal{U}(g,K_x)$.

(1) Suppose there exists $\tilde{k}\in\tilde{G}$ such that $\pi(\tilde{k})\in K_x$, and $\mathrm{int}(\tilde{k})(\tilde{s}, B, \chi)= (\tilde{s}', B',\chi')$.

Then $\mathrm{Des}_{(\tilde{s},B,\chi)}(\Theta)\approx\mathrm{Des}_{(\tilde{s}',B',\chi')}(\Theta)$

(2) Assume $\mathrm{Cent}(s)^0=\mathrm{Cent}(s')^0$, and call this group H. Suppse B=B', $\chi=\chi'$, and $\tilde{s}=\tilde{z}\tilde{s}'$, for some $\tilde{z}$ contained in the identity component of the inverse image of $Z(H)^\theta$ in $\tilde{G}$.

Then $\mathrm{Des}_{(\tilde{s},B,\chi)}(\Theta)\approx\mathrm{Des}_{(\tilde{s}',B',\chi')}(\Theta)$.

proof:

The first part is immediate. For (2) we only need to check the constant of Corollary 7.11. We need to show $\Lambda/e^{d\Lambda}(\tilde{z})=1$. This follows immediately since Λ and $e^{d\Lambda}$ have the same differential, and $\tilde{z}$ is contained in the identity component.

§8

Stable Characters

Once again fix G and an inner class of real forms, and L-groups $^\vee G^\Gamma$ and G^Γ usual.

Let x be a strong real form of G. By a real form γ corresponding to x we mean a real form of G corresponding to θ=int(x) by Lemma 1.6. Thus γ is an antiholmorphic involution of G, and letting $G(\mathbb{R})=G^\gamma$ we have a bijection between virtual $G(\mathbb{R})$-modules and $(\mathfrak{g},K_x)$-modules (cf. §1). We say a virtual $(\mathfrak{g},K_x)$-module vanishes near the identity if this holds for the corresponding $G(\mathbb{R})$-module; of course this is independent of the choice of γ.

Recall (cf. §2) $\mathcal{U}(\mathfrak{g},K_x)$ is the direct sum of blocks, and projection on a block is defined with respect to this decomposition.

8.1 Definition:

(1) Fix an admissible pair (x,y), with $x^2\in Z(G)$, and let $\mathcal{B}$ and $^\vee\mathcal{B}$ be the corresponding blocks. Fix $\mathcal{P}$ and $^d\mathcal{P}$ as in Lemma 2.11. Fix regular elements $\lambda\in\mathcal{P}$ and $^d\lambda\in{}^d\mathcal{P}$. Suppose Θ is a virtual $(\mathfrak{g},K_x)$ module contained in $\mathcal{B}(\lambda)$.

We say Θ is stable if $<\Theta, {}^dZ>=0$ for all $^dZ\in{}^\vee\mathcal{B}({}^d\lambda)$ such that ^dZ vanishes near the identity.

(2) In general (still for regular infinitesimal character) we say Θ is stable if the projection of Θ on a block $\mathcal{B}$ and the infinitesimal character χ_λ is stable for all $\mathcal{B}$ and λ.

It is immediate that the notion of stability is independent of the normalization of length.

Recall an L-packet Π is a finite set of irreducible $(\mathfrak{g},K_x)$-modules, or alternatively the Z-module of virtual modules spanned by this set. Given a standard module I, we write $I\in\Pi$ if the Langlands submodule π of I is contained in Π. This does not imply all the constituents of I are contained in Π.

8.2 Theorem:

(1) Let Π be an L-packet of $(\mathfrak{g},K_x)$-modules with regular infinitesimal character. Then $\Theta_\Pi=\Sigma_{I\in\Pi}$ I is stable.

(2) Any stable virtual $(\mathfrak{g},K_x)$-module is a finite sum of terms of the form (1).

proof:

We may assume $\Theta_\pi\in\mathcal{B}(\lambda)$ for some $\mathcal{B}$ and λ. Choose x and y so that $\mathcal{B}$ is the block defined by (x,y) via Corollary 3.22. Let ${}^\vee\mathcal{B}$ denote the dual block. The sum in (1) is the sum of the standard modules associated to inequivalent data of the form $(x,T^\Gamma,P,y,{}^dT^\Gamma,{}^dP,\Upsilon)$, where ${}^dT^\Gamma$ and ${}^dP$ are fixed. The Theorem follows immediately from Lemma 7.5 (cf. Corollary 7.6).

Recall a semisimple element $g\in G$ is strongly regular if the centralizer of g is a Cartan subgroup. The strongly regular elements are dense in G, and for such elements "stable" conjugacy is the same as conjugacy by G [St]. Given a strong real form x, let $G^{\#}$ be the set of strongly regular semimple elements g of G satisfying the following condition: there exists a real form γ

corresponding to a conjuate of x such that $g\epsilon G^{\gamma}$. Note that $G^{\#}$ is invariant under conjugation by G.

If $\Theta \epsilon \mathcal{U}(\mathfrak{g},K_x)$, we may try to define a function Φ_{Θ} on $G^{\#}$ as follows. For $g\epsilon G^{\#}$, choose γ corresponding to some conjugate x' of x such that, with $G(\mathbb{R})=G^{\gamma}$, we have $g\epsilon G(\mathbb{R})$. Consider Θ as an element of $\mathcal{U}(\mathfrak{g},K_{x'})$ and let F_{Θ} denote the function on the strongly regular semisimple elements of $G(\mathbb{R})$ which represents Θ [HC]. Let $\Phi_{\Theta}(g)=F_{\Theta}(g)$. This is not necessarily well defined: it may depend on the choice of γ.

8.3 <u>Theorem</u>:

The following conditions on a virtual $(\mathfrak{g},K_x)$-module Θ are equivalent.

(1) Fix γ associated to x, and let $G(\mathbb{R})=G^{\gamma}$. Then $F_{\Theta}(h')=F_{\Theta}(h)$ for all strongly regular semisimple elements $h,h'\epsilon G(\mathbb{R})$ such that there exists $g\epsilon G$, $ghg^{-1}=h'$.

(2) Φ_{Θ} is a well defined function on $G^{\#}$.

(3) $\Theta = \Sigma_{\Pi}\, \alpha_{\Pi}\, \Theta_{\Pi}$ $\quad(\alpha_{\Pi}\epsilon Z)$, where the sum is over a finite set of L-packets.

(4) Θ is stable.

<u>proof</u>:

Suppose $\gamma'=int(g)\gamma$, for some $g\epsilon G$. Then if F_{Θ} (resp. F'_{Θ}) represents Θ on G^{γ} (resp. $G^{\gamma'}$), then by construction $F_{\Theta}(h)=F'_{\Theta}(ghg^{-1})$. Now if γ and γ' are real forms associated to x, then G^{γ} is conjugate to $G^{\gamma'}$ via G. Composing with this conjugation we see condition (1) is the only obstruction to defining Φ_{Θ}; hence (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (1) is similar. Theorem 8.2 says (3) $\Leftrightarrow$ (4); it remains to show (1) $\Leftrightarrow$ (3). Lemmas 5.1 and 5.3 of [S4] show (1) $\Leftrightarrow$ (3) for tempered representations. The arguments carry over in general since Θ_{Π} is a sum of the standard modules contained in Π (cf. [A1]). We omit the details.

Note that condition (1) is (one of the) usual definitions of stability.

Theorem 8.3 suggests a stronger version of the notion of stability, which takes into account different real forms of G. By a virtual $(\mathfrak{g},K)$-module Θ, we mean a formal sum $\Theta = \Sigma_x \Theta_x$, where Θ_x is a virtual $(\mathfrak{g},K_x)$-module, and the sum runs over representatives of the equivalence classes of strong real forms of G. We allow formal infinite sums (which may only occur if the center of G is infinite). Note that K is merely a formal symbol, and does not denote the fixed points of an involution of G. Recall each of these real forms is in the fixed inner class of real forms. We refer to Θ_x as the projection of Θ on $\mathcal{U}(\mathfrak{g},K_x)$. Let $\mathcal{U}(\mathfrak{g},K)$ denote the vector space of virtual $(\mathfrak{g},K)$-modules: $\mathcal{U}(\mathfrak{g},K)=\Pi_x \mathcal{U}(\mathfrak{g},K_x)$. Projection of $\mathcal{U}(\mathfrak{g},K)$ onto $\mathcal{U}(\mathfrak{g},K_x)$ is defined with respect to this decomposition.

8.4 Definition:

Super-Block Equivalence on $(\mathfrak{g},K)$-modules is the smallest equivalence relation generated by block equivalence and (super)-L-indistinguishability. That is, a super-block of irreducible modules is the union of the super L-packets of all the irreducible modules in a single block. By Corollary 3.23, a super-block is associated to each semisimple element $y \in {}^\vee G^\Gamma - {}^\vee G$:

$$\mathcal{B}^{irr}(y) = \cup_x \mathcal{B}^{irr}(x,y)$$

Here and below, x runs over a set of representatives for the equivalence classes of strong real forms of G.

$$\mathcal{B}(y) = \Pi_x \mathcal{B}(x,y)$$

$\mathcal{B}_{TF}(y) = \prod_x \mathcal{B}_{TF}(x,y)$.

(We define $\mathcal{B}(x,y)$ to be zero if (x,y) is not admissible.)

The following lemma is an immediate consequence of Definitions 3.30 and 8.4.

8.5 Lemma:

In the setting of Definition 8.4, there is a perfect pairing

$$\mathcal{B}_{TF}(y) \times (\oplus_x {}^{\vee}\mathcal{B}_{TF}(x,y)) \to \mathbb{Z}.$$

The first factor is the dual $\mathbb{Z}$-module of the second.

We want to introduce super-stability using virtual characters in $\oplus_x {}^{\vee}\mathcal{B}(x,y)({}^d\lambda)$ that vanish near the identity, in analogy with Definition 8.1. Unfortunately this expression does not even make sense as written, for the following reason. Recall from sections 2 and 4 that a translation family of $({}^{\vee}\mathfrak{g}_{{}^{\vee}c}, {}^{\vee}K_y^{can})$-modules corresponding to an admissible pair (x,y) is parametrized by a set

$${}^d\mathcal{P}(x^2, P) = \{ {}^d\lambda \in \mathfrak{t} \mid \exp(2\pi i {}^d\lambda) = x^2, \text{ and } {}^d\lambda \text{ is } P\text{-dominant} \}.$$

It can easily happen that there are two admissible pairs (x,y) and (x',y) with $x^2 \neq x'^2$. In that case ${}^d\mathcal{P}(x^2,P)$ and ${}^d\mathcal{P}(x'^2,P)$ will have no elements in common, so there is no possible choice of ${}^d\lambda$ to make sense of $\oplus_x {}^{\vee}\mathcal{B}(x,y)({}^d\lambda)$.

The solution is to extend our translation families. On the dual side this amounts to replacing ${}^{\vee}G$ by its canonical cover ${}^{\vee}G^{can}$. As in section 7 to avoid confusion with the canonical cover ${}^{\vee}G_{{}^{\vee}c}^{can}$, we will write ${}^{\vee}\tilde{G}$ instead of ${}^{\vee}G^{can}$. The preimage of a subgroup ${}^{\vee}H$ of ${}^{\vee}G$ in ${}^{\vee}\tilde{G}$ will be called ${}^{\vee}\tilde{H}$. The discussion after Corollary 7.7 produced a natural surjection

$${}^{\vee}G^{can}_{\vee_c} \to {}^{\vee}G_{\vee_c} \subset {}^{\vee}\tilde{G};$$

consequently finite dimensional representations of ${}^{\vee}\tilde{G}$ may be restricted to ${}^{\vee}G^{can}_{\vee_c}$

8.6 Lemma:

Suppose $T \subset G$ and ${}^dT \subset {}^{\vee}G$ are maximal tori. Write $\Upsilon: \mathfrak{t} \to {}^d\mathfrak{t}^*$ for one of the distinguished isomorphisms. Set

$${}^d\tilde{\Lambda} = \{{}^d\lambda \in {}^d\mathfrak{t} \mid \exp(2\pi i\, {}^d\lambda) \in Z(G)^\theta\}.$$

Then $\Upsilon({}^d\tilde{\Lambda})$ is the lattice of weights of finite-dimensional representations of ${}^{\vee}\tilde{G}$.

This is an elementary consequence of the definition of ${}^{\vee}\tilde{G}$. Now x^2 belongs to $Z(G)^\theta$; so the set ${}^d\mathcal{P}(x^2,P)$ is contained in

8.7 $\quad {}^d\tilde{\mathcal{P}} = \{ {}^d\lambda \in \mathfrak{t} \mid \exp(2\pi i\, {}^d\lambda) \in Z(G)^\theta$, and ${}^d\lambda$ is P-dominant $\}$.

These remarks and the general theory of translation functors establish:

8.8 Corollary:

Suppose ${}^d\pi$ is a ${}^{\vee}G$-translation family of $({}^{\vee}\mathfrak{g}_{\vee_c}, {}^{\vee}K_y^{can})$-modules corresponding to an admissible pair (x,y), parametrized by the set ${}^d\mathcal{P}(x^2,P)$ of (8.5). Then ${}^d\pi$ may be extended in a unique way to a ${}^{\vee}\tilde{G}$-translation family ${}^d\tilde{\pi}$ of $({}^{\vee}\mathfrak{g}_{\vee_c}, {}^{\vee}K_y^{can})$-modules parametrized by the set ${}^d\tilde{\mathcal{P}}$ of (8.7).

Recall that the representations in ${}^d\pi$ all have a fixed type (that is, restriction to $\pi_1({}^{\vee}G_{\vee_c})(\mathbb{R})$). This is no longer true for those in ${}^d\tilde{\pi}$, but the loss will cause no inconvenience.

We write ${}^\vee\tilde{\mathcal{B}}_{TF}(x,y)$ for the set of enlarged translation families constructed in Corollary 8.8, and

$${}^\vee\tilde{\mathcal{B}}_{TF}(y) = \oplus_x {}^\vee\tilde{\mathcal{B}}_{TF}(x,y) .$$

We use other notation (such as ${}^\vee\tilde{\mathcal{B}}(x,y)({}^d\lambda)$ in analogy with section 3. Because of the uniqueness of the extension ${}^d\tilde{\pi}$, ${}^\vee\tilde{\mathcal{B}}_{TF}(x,y)$ is naturally isomorphic to ${}^\vee\mathcal{B}_{TF}(x,y)$. The following corollary to Lemma 8.5 is immediate.

8.9 Corollary:

(1) There is a perfect pairing $\mathcal{B}_{TF}(y)\times{}^\vee\tilde{\mathcal{B}}_{TF}(y)\rightarrow \mathbb{Z}$.

(2) The parameter sets for the translation families in ${}^\vee\tilde{\mathcal{B}}$ are all canonically isomorphic to any one of them ${}^d\tilde{\mathcal{P}}$. Similarly let $\mathcal{P}$ denote any one of the parameter sets for $\mathcal{B}(x,y)$, and fix regular elements $\lambda\in\mathcal{P}$ and ${}^d\lambda\in{}^{\vee d}\tilde{\mathcal{P}}$. Then there is a perfect pairing

$$\mathcal{B}(y)(\lambda)\times{}^\vee\tilde{\mathcal{B}}(y)({}^d\lambda)\rightarrow \mathbb{Z} .$$

We choose normalizations of length. Let (x,y) be an admissible pair with x integral, with corresponding dual blocks $\mathcal{B}\subset\mathcal{U}(\mathfrak{g},K_x)$ and ${}^\vee\mathcal{B}\subset\mathcal{U}({}^\vee\mathfrak{g}_{y2},{}^\vee K_y)$. Let G_{int} be the quasisplit group dual to ${}^\vee G_{y2}$ (cf. §3). Let A_f denote the split part of a fundamental Cartan subgroup of G_{int}. Recall the integer $q(G,x)$ is given by Definition 7.9.

8.10 Definition:

. For $\pi\in\mathcal{B}$, define $\ell(\pi)$ as in 3.26, with

$$c_0=\tfrac{1}{2}\dim(A_f)-[q(G,x)+q({}^\vee G,y)+q(G_{int})].$$

If the infinitesimal character of π is integral, G_{int} is the quasisplit form G_{qs} of G. The definition is more symmetric than it first appears: $q(G_{int})=q({}^{\vee}G^{\vee}{}_{c,qs})$, where ${}^{\vee}G^{\vee}{}_{c,qs}$ is the quasisplit inner form of ${}^{\vee}G^{\vee}{}_{c}$. In particular in the integral case this becomes $q(G_{qs})=q({}^{\vee}G_{qs})$.

8.11 <u>Definition</u>:

(1) Fix a semi-simple element $y\in{}^{\vee}G^{\Gamma}-{}^{\vee}G$, with corresponding super-block $\mathcal{B}=\mathcal{B}(y)$. Fix parameter sets $\mathcal{P}$ and ${}^{d}\tilde{\mathcal{P}}$ as in the previous lemma, and choose regular elements $\lambda\in\mathcal{P}$ and ${}^{d}\lambda\in{}^{d}\tilde{\mathcal{P}}$. Let Θ be a virtual $(\mathfrak{g},K)$-module contained in $\mathcal{B}(y)(\lambda)$.

We say Θ is <u>super</u>-<u>stable</u> if $\langle\Theta,{}^{d}Z\rangle = 0$ for all ${}^{d}Z\in{}^{\vee}\tilde{\mathcal{B}}(y)({}^{d}\lambda)$ vanishing near the identity.

(2) In general (still for regular infinitesimal character) we say Θ is super-stable if the projection of Θ on a super-block $\mathcal{B}$ and infinitesimal character χ is super-stable for all $\mathcal{B}$ and χ.

A virtual $(\mathfrak{g},K)$-module Θ is <u>super</u>-<u>stable</u> if $\langle\Theta,{}^{d}Z\rangle=0$ for all virtual $({}^{\vee}\mathfrak{g},{}^{\vee}K_y^{can})$-modules ${}^{d}Z$ which vanish near the identity.

This notion is <u>not</u> independent of the normalization of length, which comes into the definition of $\langle\,,\rangle$. More precisely, it depends upon how these normalizations vary as the block varies. The normalization was chosen primarily to make Theorem 8.12 hold. (Only the term $q(G,x)$ varies within a superblock; the others are present for aesthetic reasons.) If Θ is super-stable, then the projection of Θ on $\mathcal{U}(\mathfrak{g},K_x)$ is stable for any x. The converse is false as we shall see in a moment.

It may happen that ${}^{d}Z-{}^{d}Z'$ vanishes near the identity, where ${}^{d}Z$ and ${}^{d}Z'$ are virtual $({}^{\vee}\mathfrak{g},{}^{\vee}K_y)$ modules (for the same strong real form y), but ${}^{d}Z$

and ${}^dZ'$ are contained in different blocks. Note that the definition of stability takes no notice of this possibility, but the notion of super-stability does. An example is $\pi^+ - \pi^-$, where π^+ (resp. π^-) denotes the reducible (resp. irreducible) principal series representations of $SL(2,\mathbb{R})$ with infinitesimal character ρ. As a result we have that the discrete series representation X of $PGL(2,\mathbb{R})$ (with infinitesimal character ρ) is stable, as is the trivial representation Y of $PSU(2)$, whereas only the formal sum X-Y is superstable. Note that X+Y in this example is not superstable, even though its projection on $\mathcal{U}(\mathfrak{g},K_x)$ is stable for all x. Thus the condition of super-stability includes a compatibility condition on signs as the real form varies.

Now suppose $\{\varphi\}$ is a conjugacy class of maps of $W_{\mathbb{R}}$ into ${}^\vee G^\Gamma$. Given a strong real form x, let Π_x denote the corresponding L-packet (which is possible empty). Let $\Theta_\Pi = \Sigma_x (-1)^{q(G,x)} \Theta_{\Pi_x}$, where Θ_{Π_x} is the stable virtual $(\mathfrak{g},K_x)$-module corresponding to Π_x.

Let G^{s} denote the set of strongly regular semisimple elements of G satisfying the following condition: there exists a strong real form $x \in G^\Gamma$, and a real form γ corresponding to x, such that $g \in G^\gamma$. By analogy with the situation before Theorem 8.3, if Θ is a virtual $(\mathfrak{g},K)$-module, we try to define a function Φ_Θ on G^{s} by taking $\Phi_\Theta(g)=F_{\Theta_x}(g)$, where x and γ are chosen with $g \in G^\gamma$, and Θ_x is the projection of Θ on $\mathcal{U}(\mathfrak{g},K_x)$. This is not necessarily well defined.

8.12 Theorem:

The following conditions are equivalent:

(1) For all strong real forms x and x' the following condition holds. Choose γ and γ' asociated to x and x', let $G(\mathbb{R})=G^{\gamma}$, and let $G(\mathbb{R})'=G^{\gamma'}$. Then $F_{\Theta}(h)=F_{\Theta'}(h')$ for all strongly regular semisimple elements $h\in G(\mathbb{R})$ and $h'\in G(\mathbb{R})'$ such that there exists $g\in G$, with $ghg^{-1}=h'$.

(2) Φ_{Θ} is well-defined,

(3) $\Theta = \Sigma_{\Pi}\, \alpha_{\Pi}\, \Theta_{\Pi}$ ($\alpha_{\Pi}\in Z$), where the sum is over a set of super L-packets,

(4) Θ is super-stable.

proof:

Conditions (3) and (4) are equivalent as in the proof of Theorem 8.3; the signs come from our normalization of length. Furthermore (2) ⇒ (1) is elementary. Suppose γ and γ' are involutions corresponding to different strong real forms x and x'. Suppose $g\in G^{s}$, and $g\in G^{\gamma}\cap G^{\gamma'}$. Let $\Theta=\Theta_{\Pi}$, and let Θ_{x} denote the projection of Θ onto $\mathcal{U}(\mathfrak{g},K_{x})$. Then by ([S4], Theorem 6.3) $F_{\Theta_x}(g)=(-1)^{q(G,x)-q(G,x')}F_{\Theta_{x'}}(g)$. The equivalence of (1) and (3), and (3) ⇒ (2) follow readily from this; we leave the details of the proof to the reader.

§9

Lifting

We are now in a position to define lifting, and state the main theorem. For the convenience of the reader (and the authors) before doing so we spell out the choices and definitions we have made so far.

9.1 (1) We are given G and an inner class of real forms of G (cf. §1). Let ${}^\vee G$ be a dual group, and choose an L-group $({}^\vee G^\Gamma, {}^\vee\mathcal{D})$ for G (Definition 1.3). Any two such L-groups are isomorphic, by an isomorphism which is canonical up to conjugation by ${}^\vee G$.

(2) Let $(G^\Gamma, \mathcal{D})$ be an L-group for ${}^\vee G$. Any two such choices are isomorphic by an isomorphism canonical up to conjugation by G.

(3) We are given an equivalence class S of endoscopic data (Definition 6.3). Thus if $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H) \in S$, let $s=\pi(\tilde{s})$, and let ${}^\vee H$ be the identity component of the centralizer of s in ${}^\vee G$. Then ${}^\vee H^\Gamma$ has two components, meets ${}^\vee G^\Gamma - {}^\vee G$, and has identity component ${}^\vee H$.

(4) Let H be an endoscopic group determined by S, so H is a quasisplit group with ${}^\vee H^\Gamma$ an E-group for H. Let θ_H be a Cartan involution of H, $K_H = H^{\theta_H}$, and let $\tilde{K}_H$ denote the cover of K_H corresponding to the element $\rho-\rho_H$ (cf. the discussion preceding Lemma 6.10).

We will refer to the data of 9.1(1)-(4) as <u>Lifting Data</u>. We also need to fix a group to which we shall lift.

9.2 Fix a strong real form x of G (Definition 1.8); write $\theta=\text{int}(x)$ and $K_x = G^\theta$.

Choose an extended group $(H^{\Gamma},\mathcal{D}_H)$ containing H as in Definition 6.11. Fix $\delta_H \in \mathcal{D}_H$ with $\mathrm{int}(\delta_H)=\theta_H$; then $(\delta_H)^2=x^2$.

To define lifting, we need some additional structure attached to the particular (virtual) $(\mathfrak{h},\tilde{K}_H)$-module to be lifted.

9.3 In the setting of (9.1) and (9.2), suppose $\Theta_H \in \mathcal{U}(\mathfrak{h},\tilde{K}_H)$ is stable, with G-regular infinitesimal character χ_H. Assume that Θ_H is contained in a block $\mathcal{B}_H$ for H, defined by an admissible pair (δ_H, y) $(y \in {}^{\vee}H^{\Gamma})$. That is we assume Θ_H is representation of the quasisplit form, and is of type ${}^{\vee}z_{\rho}{}^{\vee}z_{\rho_H}$, i.e. it is a genuine representation.

(1) Write ${}^{\vee}c = y^2$, ${}^{\vee}G_{{}^{\vee}c}$ = identity component of the centralizer of ${}^{\vee}c$, ${}^{\vee}\theta=\mathrm{int}(y)$, ${}^{\vee}K_y = ({}^{\vee}G_{{}^{\vee}c})^{{}^{\vee}\theta}$, ${}^{\vee}H_{{}^{\vee}c}$=identity component of the centralizer of ${}^{\vee}c$ in ${}^{\vee}H$, ${}^{\vee}K_H=({}^{\vee}H_{{}^{\vee}c})^{{}^{\vee}\theta}$, ${}^{\vee}K_y^{can}$=preimage of ${}^{\vee}K_y$ in the canonical cover of ${}^{\vee}G_{{}^{\vee}c}$, ${}^{\vee}K_H^{can}$ = preimage of ${}^{\vee}K_H$ in the canonical cover of ${}^{\vee}H_{{}^{\vee}c}$.

(2) Dual to $\mathcal{B}_H$ is a block ${}^{\vee}\mathcal{B}_H$ of $({}^{\vee}\mathfrak{h}_{{}^{\vee}c}, {}^{\vee}K_H^{can})$-modules of type $\tau(x^2 z_{\rho_H}({}^{\vee}c), {}^{\vee}c)$.

(3) Choose a Cartan subgroup ${}^dT$ of ${}^{\vee}H_{{}^{\vee}c}$ and a set ${}^dP_H$ of positive roots for ${}^dT$ in ${}^{\vee}H_{{}^{\vee}c}$. This fixes the parameter set $\mathcal{P}_H \subset {}^d\mathfrak{t}$ for the translation families in $\mathcal{B}_H$. Choose a representative $\lambda \in \mathcal{P}_H$ for the infinitesimal character χ_H of Θ_H. Define ${}^dP \supset {}^dP_H$ to be the set of roots of ${}^d\mathfrak{t}$ in ${}^{\vee}\mathfrak{g}_{{}^{\vee}c}$ positive on λ, and write ${}^dB_{des} \supset {}^dT$ for the corresponding Borel subgroup of ${}^{\vee}G_{{}^{\vee}c}$. This fixes the parameter set $\mathcal{P}$ for translation families for G of infinitesimal character associated to ${}^{\vee}c$; and $\lambda \in \mathcal{P} \subset \mathcal{P}_H \subset {}^d\mathfrak{t}$.

(4) Choose parameter sets ${}^d\mathcal{P}_H$ and ${}^d\mathcal{P}$ for translation families for ${}^{\vee}H_{{}^{\vee}c}$ and ${}^{\vee}G_{{}^{\vee}c}$ respectively, of infinitesimal character associated to $(\delta_H)^2=x^2$.

By the discussion before Theorem 7.4, the Borel subgroup ${}^dB_{des}$ gives rise to a natural injection

$$(\eta_{H,G})^{-1} : {}^d\mathcal{P} \to {}^d\mathcal{P}_H .$$

Pick a ${}^\vee G$-regular element ${}^d\lambda \in {}^d\mathcal{P}$, and let ${}^d\lambda_H$ be the corresponding element of ${}^d\mathcal{P}_H$. Write ${}^d\chi$ for the infinitesimal character of ${}^\vee G_{{}^\vee c}$ defined by ${}^d\lambda$; ${}^d\chi_H$ is defined analogously.

We then define $\mathrm{Des}_{(\mathfrak{s}, {}^dB_{des}, {}^d\chi)}$ as in §7: by Definition 7.3 if $y^2 \in Z({}^\vee G)$, or by the discussion preceding Corollary 7.8 in general. This descent takes (virtual) $({}^\vee\mathfrak{g}, {}^\vee K_y^{can})$-modules of infinitesimal character ${}^d\chi$ to (virtual) $({}^\vee\mathfrak{h}, {}^\vee K_H^{can})$-modules in ${}^\vee\mathcal{B}_H$ defined near the identity (Corollary 7.8).

Of course ${}^dT$ and ${}^dP_H$ are unique up to conjugation by ${}^\vee H_{{}^\vee c}$. The element λ is only determined up to conjugation by the full centralizer of ${}^\vee c$ in ${}^\vee h$, and not just by ${}^\vee H_{{}^\vee c}$. Consequently the pair $(\lambda, {}^dB_{des})$ is unique only up to conjugation by this larger centralizer, and thus Des is not as unique as we might like. Rather than keep careful track of the amount of choice involved, and try to show directly that the definition of Lift is unaffected by it, we will wait until we can calculate Lift explicitly; the formulas will be independent of the choice in (9.3).

9.4 Definition:

In the setting (9.1)-(9.3), define $\mathcal{B}$ to be the block of $(\mathfrak{g}, K_x)$-modules associated to the pair (x,y). The dual block ${}^\vee\mathcal{B}$ is a block of $({}^\vee\mathfrak{g}_{{}^\vee c}, {}^\vee K_y^{can})$-modules of type $\tau(z_{{}^\vee\rho} z_{{}^\vee\rho({}^\vee c)}, {}^\vee c)$ (cf. 4.7). The parameter sets for translation families in these blocks are $\mathcal{P}$ and ${}^d\mathcal{P}$, in which we have elements λ and ${}^d\lambda$ (cf. (9.3)(3)-(4)).

The lift of Θ_H to G (and the strong real form x), written Lift(Θ_H), is the unique virtual $(\mathfrak{g},K_x)$-module contained in $\mathcal{B}(\lambda)$ satisfying the following condition:

9.5 $\quad < \mathrm{Lift}(\Theta_H) \, , {}^dZ > \; = \; < \Theta_H \, , \mathrm{Des}({}^dZ) > \quad$ for all ${}^dZ \in {}^\vee\mathcal{B}({}^d\lambda)$.

Here the pairing on the left is on $\mathcal{B}(\lambda) \times {}^\vee\mathcal{B}({}^d\lambda)$, and that on the right is on $\mathcal{B}_H(\lambda_H) \times {}^\vee\mathcal{B}_H({}^d\lambda_H)$.

Extend the definition of Lift(Θ_H) to all stable virtual $(\mathfrak{h},\tilde{K}_H)$-modules of G-regular infinitesimal character by linearity, via projection on infinitesimal characters and blocks.

If it is necessary to specify the strong real form x to which we are lifting we will write $\mathrm{Lift}_x(\Theta_H)$.

Recall Des(${}^dZ$) is only defined up to virtual characters which vanish on a neighborhood of the identity in ${}^\vee H$. Precisely because Θ_H is stable, and hence orthogonal to such characters, we see that $<\Theta_H, \mathrm{Des}({}^dZ)>$ and hence Lift(Θ_H), is independent of the choice of Des(${}^dZ$). Thus Lift(Θ_H) is well-defined. The only part of the following Lemma which is not immediate is the independence of the choice of λ and ${}^d\lambda$; this will be proved below.

9.6 <u>Lemma</u>:

$\mathrm{Lift}_x(\Theta_H)$ depends only on the data (9.1) and (9.2). In particular it is independent of the choice of $\tilde{s}$ (within the given equivalence class of endoscopic data).

Given lifting data 9.1 and a strong real form x of G, let Π_H be an L-packet of $(\mathfrak{h},\tilde{K}_H)$-modules with G-regular infinitesimal character. We obtain an L-packet (possibly empty) of $(\mathfrak{g},K_x)$-modules as follows. Choose $\varphi_H:W_{\mathbb{R}}\to {}^\vee H^\Gamma$ corresponding to Π_H, and let φ denote the map $W_{\mathbb{R}}\to {}^\vee H^\Gamma\to {}^\vee G^\Gamma$. If φ is admissible we let Π denote the L-packet for G corresponding to φ, otherwise it is empty.

The main theorem is the following.

9.7 Theorem:

Suppose we are in the setting of (9.1) and (9.2), and Π_H is an L-packet of $(\mathfrak{h},\tilde{K}_H)$-modules with G-regular infinitesimal character. If Π_H is relevant to G, let Π denote the L-packet of $(\mathfrak{g},K_x)$-modules associated to Π_H by the preceding discussion. Let Θ_H be the stable virtual character corresponding to Π_H (Theorem 8.2). Then Lift(Θ_H) is defined by Definition 9.4.

(1) Lift(Θ_H)=0 if Π_H is not relevant to G.

(2) Suppose Π_H is relevant to G. Then $\mathrm{Lift}(\Theta_H)=\Sigma_{I\in\Pi}\, c_I\, I$ for some $c_I\in\mathbb{C}^*$.

(3) Let $<\,,\,>: \Pi\times\tilde{S}_\varphi \to \mathbb{C}^*$ be the pairing of Definition 5.2 restricted to Π. Consider $\tilde{s}$ as a representative of an element of $\tilde{S}_\varphi$. Then $c_I=(-1)^{q(G,x)+q(H)}<I,\tilde{s}>$.

We note that $c_I/c_{I'}\in\{\pm 1\}$ for all I and I', and write this in a different form in Theorem 9.11. Since the expression for c_I does not involve the choice of ${}^d\lambda$, we see that the definition of Lift is independent of that choice (cf. Lemma 9.6).

proof:

Choose data $(y,{}^dT^\Gamma,\lambda)$ for the L-packet Π_H as in Proposition 3.4. Define ${}^\vee c=y^2$, and let ${}^dP_H\subset{}^dP$ be the ${}^\vee c$-integral positive systems making λ dominant. This notation is consistent with that of (9.3). Since $x^2\in Z(G)$, the set of positive roots ${}^dP$ defines a Borel subgroup ${}^dB_{des}$ of ${}^\vee G_{{}^\vee c}$ (see the discussion following Definition 3.8).

We first consider (2). Fix ${}^d\lambda$ as in Definition 9.4. It is enough to show that for any standard module ${}^dI\in{}^\vee\mathcal{B}({}^d\lambda)$, $<\mathrm{Lift}(\Theta_H),{}^dI>\neq 0$ implies ${}^dI\in{}^\vee\Pi$. By 9.6, it is enough to show for all standard modules ${}^dI\in{}^\vee\mathcal{B}({}^d\lambda)$:

9.8 $<\Theta_H, \mathrm{Des}({}^dI)>\neq 0 \Rightarrow {}^dI\in{}^\vee\Pi$

Since ${}^\vee\mathcal{B}({}^d\lambda)$ is the block defined by (y,x), we have that ${}^dI$ is given by (or rather dual to) some L-data $S=(x,T^\Gamma,P,y,{}^dT'^\Gamma,{}^dB,\Upsilon)$. We compute $c=<\Theta_H, \mathrm{Des}({}^dI)>$. Write ${}^dB_H={}^dB_{des}\cap{}^\vee H$.

By Theorem 7.4 (applied to ${}^\vee G_{{}^\vee c}$), if $\mathrm{Des}({}^dI)\neq 0$ we may without loss of generality (by conjugating by ${}^\vee K_y$ if necessary) assume $s\in{}^dT'$, and ${}^dB$ is ${}^\vee H_{{}^\vee c}$-conjugate to ${}^dB_{des}$. Then $\mathrm{Des}({}^dI)$ is a non-zero constant times the standard $({}^\vee\mathfrak{h}_{{}^\vee c},{}^\vee K_H^{can})$-module given by some L-data $(\delta_H,T_H^\Gamma,*,y,{}^dT'^\Gamma,{}^dB'\cap{}^\vee H,\Upsilon_H)$. Then $c\neq 0$ implies ${}^dB'\cap{}^\vee H=k\bullet{}^dB_H$, and $k\bullet{}^dT'^\Gamma={}^dT^\Gamma$ for some $k\in{}^\vee K_H$. Thus by replacing S by $\mathrm{int}(k)S$ we may assume ${}^dB'\cap{}^\vee H={}^dB_H$, and ${}^dT'^\Gamma={}^dT$. We still have that ${}^dB'$ is ${}^\vee H_{{}^\vee c}$-conjugate to ${}^dB_{des}$, so suppose ${}^dB'=\mathrm{int}(h){}^dB_{des}$. Then since ${}^dB_{des}\cap{}^\vee H={}^dB_H$, we see $h\in{}^dB_H\subset{}^dB'$, so ${}^dB={}^dB_{des}$. Thus S is conjugate to $(\delta_H,T^\Gamma,B,y,{}^dT^\Gamma,{}^dB_{des},\Upsilon)$. Therefore ${}^dI\in{}^\vee\Pi$, which proves 9.8. Since this holds for any standard module ${}^dI$, this proves (2).

The same argument proves (1), by simply noting $c\neq 0$ implies ${}^dT^\Gamma$ is relevant to G; (1) is the contrapositive of this statement.

Assuming Π_H is relevant, write $\mathrm{Lift}(\Theta_H)=\Sigma\, c_I\, I$ as in (2). Taking ${}^dI={}^\vee I$ for $I\in\Pi$ any standard module, we have the left-hand side of 9.5 is equal to

9.9 $\quad(-1)^{\ell(I)}c_I.$

By Corollary 7.11 the right-hand side of 9.5 is given by:

9.10 $\quad(-1)^{\ell(I_H)}(-1)^{\ell_o({}^\vee I)+\ell_o({}^\vee I_H)}({}^d\Lambda/e^{{}^d\lambda})(\tilde{s}).$

By Lemma 5.5, the final factor is equal to $\langle I,\tilde{s}\rangle$. It is not hard to see that $(-1)^{\ell(I)+\ell_o({}^\vee I)}=(-1)^{q(G,x)}$, and $(-1)^{\ell(I_H)+\ell_o({}^\vee I_H)}=(-1)^{q(H)}$. Thus equating (9.9) and (9.10) we conclude the Theorem.

Definition 9.4 and Theorem 9.6(3) depend on the choices of normalizations of length of §8. Such choices amount to the following: for each conjugacy class of admissible pairs (x,y) we choose real numbers $c_0=c_0(x,y)$ to normalize length on $\mathcal{B}$ (cf. 2.15). (We must make these choices so that all lengths are integral.) Given such choices, define $\mathrm{Lift}^{\#}(\ )$ as above; it is immediate that on each block $\mathcal{B}$, $\mathrm{Lift}^{\#}(\ )=\pm\mathrm{Lift}(\ )$. The definition of Lift( ) is independent of the normalization of length for ${}^\vee G$ and ${}^\vee H$; we have chosen these normalizations for convenience, and to make Corollary 7.11 and Theorem 9.11 have a nice form.

Suppose we are in the setting of Theorem 9.7. Fix a standard module $I \in \Pi$, with Langlands subquotient π. The set of standard characters contained in Π is then $\{ w \times I \mid w \in W_{im} \}$, and these characters span Π. Let $\delta_{I,s}: W_{im} \rightarrow \{\pm 1\}$ be as in Definition 5.2. The following Theorem computes the coefficents in Lifting.

9.11 Theorem:

Suppose we are in the setting of Theorem 9.7, and fix $I \in \Pi$. For $w \in W_{im}$ let $I_w = w \times I$. Define $c \in \mathbb{C}$, $\varepsilon_I(w) \in \{\pm 1\}$ ($w \in W_{im}$) by:

9.12 (i) $\varepsilon_I(1) = 1$,

(ii) $\mathrm{Lift}(\Theta_H) = c \sum \varepsilon_I(w)\, I_w$ (the sum is over $W_{im} \cap W(K_x,T) \backslash W_{im}$).

Then

9.13 (1) $\varepsilon_I(w) = \delta_{I,s}(w)$

(2) $c = (-1)^{q(G,x)+q(H)} \dfrac{^d\Lambda}{e^{d_\lambda}}(\tilde{s})$

In particular,

(3) $\varepsilon_I(xy) = \varepsilon_I(y)\, \varepsilon_{y \times I}(x)$ for all $x,y \in W_{im}$.

(4) For α an imaginary root,

$$\varepsilon_I(s_\alpha) = \begin{cases} \mathrm{sgn}(\alpha^\vee)(s) & \alpha \text{ non-compact} \\ 1 & \alpha \text{ compact.} \end{cases}$$

(5) $\varepsilon_I(xy) = \varepsilon_I(y)$ for all $y \in W_{im}$, $x \in W_{im} \cap W(K_x,T)$.

(6) $\varepsilon_I(xy) = \varepsilon_I(x)$ for all $x \in W_{im}$, $y \in W_{im}(H,T)$.

proof:

The first assertion follows immediately from theorem 9.7 and the definition of δ, and (2) is Lemma 5.5. Similarly (3) and (4) are immediate consequences of Lemma 5.8, and (5) follows immediately from (4). We leave the proof of (6) as an exercise using Lemma 5.5. This completes the proof of the Theorem.

This definition of lifting differs from the usual one in part because we have taken $\tilde{s}\in{}^\vee G^{can}$. This only plays a serious role in super-lifting (discussed below) as we see by the following Lemma, which follows immediately from Lemma 5.6. Recall (cf. §4 and Lemma 5.6) the definition of the characters $\tau(z,1)$ of $\pi_1({}^\vee G^{can})(\mathbb{R})$ associated to $z\in Z(G)^\theta$.

9.14 <u>Corollary</u>:

Suppose we are in the setting of Definition 9.4, with $S=(\tilde{s},{}^\vee H^\Gamma,{}^\vee\mathcal{D}_H)$. Let $S'=(\tilde{z}\tilde{s},{}^\vee H^\Gamma,{}^\vee\mathcal{D}_H)$ for some $\tilde{z}\in\pi_1({}^\vee G^{can})(\mathbb{R})$, and let Lift'( ) be defined accordingly. Let $\tau=\tau(z_{{}^\vee\rho}x^{-2},1)$.

<u>Then</u> Lift'( )$=\tau(\tilde{z})$ Lift( ).

As a special case, suppose $\tilde{s}\in\pi_1({}^\vee G^{can})(\mathbb{R})$, and $({}^\vee H^\Gamma,{}^\vee\mathcal{D}_H)=({}^\vee G^\Gamma,{}^\vee\mathcal{D})$; then H is the quasiplit inner form of G.

9.15 <u>Corollary</u>:

In the setting of the preceding Corollary, suppose $S=(\tilde{z},{}^\vee G^\Gamma,{}^\vee\mathcal{D})$, with $\tilde{z}\in\pi_1({}^\vee G^{can})(\mathbb{R})$.

<u>Then</u> $\mathrm{Lift}(\Theta_{\pi_H})=(-1)^{q(G,x)+q(H)}\tau(\tilde{z})\,\Theta_\pi$.

In particular $\mathrm{Lift}(\Theta_{\pi_H})$ is stable.

<u>proof</u>:

Recall (cf. §8) $\Theta_\Pi=\Sigma_{I\in\Pi}\, I$; which we write $\Theta_H=\Sigma_w\, I_w$ as in Theorem 9.11. Now ${}^\vee H={}^\vee G$, so by Theorem 9.11 (6) $\varepsilon_I(x)=1$ for all $x\in W_{im}$; the result follows immediately.

Endoscopy is used to study the structure of L-packets, so we should fix $\varphi : W_{\mathbb{R}} \to {}^\vee G^\Gamma$, and consider endoscopic groups through which this map factors. Thus suppose we are given a conjugacy class of homomorphisms $\{\varphi_0\} : W_{\mathbb{R}} \to {}^\vee G^\Gamma$. Suppose x is a strong real form of G, and $\{\varphi_0\}$ is relevant for G_x. Suppose $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$ is a set of endoscopic data such that φ factors through ${}^\vee H^\Gamma$ for some $\varphi\in\{\varphi_0\}$. Let Π_H be the corresponding L-packet, with stable virtual character Θ_H. This data determines blocks for G, ${}^\vee G$, H and ${}^\vee H$, hence all the choices of 9. 3, and $\mathrm{Lift}_x(\ )$ is defined.

9.16 <u>Corollary</u>:

Suppose we are given $\varphi: W_{\mathbb{R}} \to {}^\vee G^\Gamma$, and endoscopic data $(\tilde{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$ such that φ factors through ${}^\vee H^\Gamma$.

<u>Then</u> $\mathrm{Lift}_x(\Theta_H) = (-1)^{q(G,x)+q(H)}\Sigma_{I\in\Pi} <I,\tilde{s}>I$

(and is independent of the choice of ${}^\vee\mathcal{D}_H$).

From Corollary 9.16 and Fourier inversion on the group $\tilde{\mathcal{S}}_\varphi$ we immediately obtain an inversion formula, which we defer until we have discussed super-lifting (cf. Theorem 9.24).

Theorem 9.11 enables us to compare our results with those of [S1]. Suppose we are in the setting of Corollary 9.16. Fix $I\in\Pi$, and define c and $\varepsilon_I(w)$ by 9.11.

Recall (Theorem 6.13) there is a surjective map $\Psi:\mathcal{S}\to\mathcal{S}_{[S1]}$, where $\mathcal{S}_{[S1]}$ is the set of equivalence classes of endoscopic data of [S1]. For $S\in\mathcal{S}$ let (s,ω) be a representative of $\Psi(S)$. Given the choices of ([S1], §4) we obtain an L-packet Π'_H of $(\mathfrak{h},K_H)$-modules.

Let Lift'( ) denote lifting of stable virtual characters from H to G as defined in ([S1],§4). As in [A2] we extend this to non-tempered distributions. Thus we obtain Lift'(Θ'_H), which is a virtual $(\mathfrak{g},K_x)$-module. With $I\in\Pi$ fixed as above, we have ([S1], Theorem 4.1.1):

Lift'$(\Theta'_H)=\Sigma_w\ \beta(w)(w\times I)$ for some numbers $\beta(w)\in\{\pm1\}$, where the sum runs over $w\in(W_{im}\cap W(K_x,T))\backslash W_{im}$.

9.17 Theorem:

There exists $\xi\in\{\pm1\}$ and $c\in\mathbb{C}^*$ such that Lift(Θ_H) = ξc Lift'(Θ'_H). The constant c is given by Theorem 9.11(2); up to sign it depends only on the block $\mathcal{B}$ containing Π_H.

proof:

This follows from ([S1], 4.4.3, 4.4.10, and 4.5.1). The essential point is that given I, in the notation of 5.12, we may choose η such that $\beta(w)/\beta(1)=\kappa_{\eta,s}(w)=\kappa(w)$. The Theorem follows immediately from Corollary 5.14.

It is the authors' opinion that the question of whether ξ is equal to +1 or −1 is probably undecidable.

The L-packets Π_H and Π'_H are similar but not necessarily the same. First of all Π_H and Π have the same infinitesimal character (considered as an element of $\mathfrak{t}^*$) whereas the infinitesimal character of Π'_H is shifted by the choice of embedding ${}^LH \hookrightarrow {}^LG$. Finally Π_H is an L-packet for a covering group $\tilde{H}(\mathbb{R})$ of $H(\mathbb{R})$. Let $\tilde{\Pi}'_H$ denote the pullback of Π'_H to $\tilde{H}(\mathbb{R})$ via projection. We leave the proof of the following lemma to the conscience of the reader.

9.18 Lemma:

There is a translation functor Ψ on $\tilde{H}(\mathbb{R})$ so that $\Pi_H = \Psi(\tilde{\Pi}'_H)$.

We proceed to discuss super-lifting. Note that in Definition 9.4 we may vary the strong real form x of G. Recall the notion of virtual $(\mathfrak{g},K)$-module (cf. §8): a virtual $(\mathfrak{g},K)$-module is a formal sum $\Sigma_x \Theta_x$ where Θ_x is a virtual $(\mathfrak{g},K_x)$-module, and the sum runs over equivalence classes of strong real forms. (If G has infinite center the formal sum may be infinite).

9.19 Definition:

Suppose we are given lifting data 9.1 and a stable virtual $(\mathfrak{h},\tilde{K}_H)$-module Θ_H, with G-regular infinitesimal character. Then we define the virtual $(\mathfrak{g},K)$-module $\mathrm{Lift}^*(\Theta_H)$ as follows:

9.20 $\mathrm{Lift}^*(\Theta_H)=\Sigma_x \mathrm{Lift}_x(\Theta_H)$

where $\mathrm{Lift}_x(\Theta_H)$ is the virtual $(\mathfrak{g},K_x)$-module of Definition 9.4, and the sum runs over equivalence classes of strong real forms of G.

Recall the definition of a super L-packet Π, and the super-stable virtual $(\mathfrak{g},K)$-module Θ_Π associated to Π (cf. §8). The following two results are immediate consequences of Theorem 9.7.

9.21 Theorem:

Suppose Θ_H is the stable virtual $(\mathfrak{h},\tilde{K}_H)$-module Θ_H associated to an L-packet Π_H of G-regular infinitesimal character. Then $Lift^*(\Theta_H)$ is contained the super L-packet Π defined in the discussion preceding Theorem 9.7, and

$$Lift^*(\Theta_H)=(-1)^{q(H)}\sum_{I\in\Pi}(-1)^{q(G,x(I))}\langle I,\tilde{s}\rangle I .$$

Here x(I) is the strong real form of which I is a representation.

9.22 Theorem:

Suppose $S=(1,{}^\vee G^\Gamma,{}^\vee\mathcal{D})$. Then $Lift^*(\Theta_H)=(-1)^{q(H)}\Theta_\Pi$. In particular $Lift^*(\Theta_H)$ is super-stable.

Inversion now takes the following form. Let $\varphi:W_{\mathbb{R}}\to{}^\vee G^\Gamma$ be an admissible homomorphism (with regular infinitesimal character), with corresponding super L-packet Π_φ. Let ${}^dT$ be the centralizer of $\varphi(\mathbb{C}^*)$, and pick ${}^\vee\delta\in{}^\vee\mathcal{D}$ such that ${}^\vee\delta|_{{}^dT}=int(\varphi(j))|_{{}^dT}$. Then S_φ is the group of fixed points of $int({}^\vee\delta)$ on ${}^dT$. By Lemma 6.6, to define endoscopic data it is enough to choose $\tilde{s}\in{}^\vee G^{can}$ such that the image of $\tilde{s}$ is elliptic and fixed by $int({}^\vee\delta)$. We see immediately that we may pick a set of representatives of $\tilde{S}_\varphi/\tilde{S}_\varphi{}^0$ satisfying these conditions. We change notation slightly and let $\tilde{\mathbb{S}}_\varphi$ denote this set of representatives. For $\tilde{s}\in\tilde{\mathbb{S}}_\varphi$ we choose ${}^\vee\mathcal{D}_{H_{\tilde{s}}}$ making a set $(\tilde{s},{}^\vee H_{\tilde{s}}^\Gamma,{}^\vee\mathcal{D}_{H_{\tilde{s}}})$ of endoscopic data, and let $H_{\tilde{s}}$ denote a corresponding endoscopic group. Now φ factors through ${}^\vee H_{\tilde{s}}{}^\Gamma$, and we obtain a stable virtual $(\mathfrak{h},\tilde{K}_H)$-module $\Theta_{H_{\tilde{s}}}$. Let $Lift^*(\tilde{s},\Theta_{H_{\tilde{s}}})$ denote the lift of this character to $(\mathfrak{g},K)$. By Corollary 9.16 it is independent of the choice of ${}^\vee\mathcal{D}_{H_{\tilde{s}}}$.

To avoid technicalities we assume the center of G is finite, so Π and $\tilde{\mathbb{S}}_\varphi$ are finite. The following Theorem follows from Theorems 5.1, 9.7 and Fourier inversion on the group $\tilde{\mathbb{S}}_\varphi$.

9.23 Theorem:

For $I \in \Pi_\varphi$ a standard $(\mathfrak{g},K_x)$-module,

$$I = \frac{(-1)^{q(G,x)}}{|\tilde{\mathbb{S}}_\varphi|} \sum_{\tilde{s}\in\tilde{\mathbb{S}}_\varphi} (-1)^{q(H_{\tilde{s}})} \overline{\langle I,\tilde{s}\rangle} \mathrm{Lift}(\tilde{s},\Theta_{H_{\tilde{s}}})$$

Here x is the strong real form of which I is a representation.

Suppose we fix a strong real form x of G. Letting Lift( ) denote lifting for this strong real form, and projecting both sides of 9.2 onto $(\mathfrak{g},K_x)$-modules, we obtain:

$$9.24 \quad \frac{(-1)^{q(G,x)}}{|\tilde{\mathbb{S}}_\varphi|} \sum_{\tilde{s}\in\tilde{\mathbb{S}}_\varphi} (-1)^{q(H_{\tilde{s}})} \overline{\langle I,\tilde{s}\rangle} \mathrm{Lift}(\tilde{s},\Theta_{H_{\tilde{s}}})$$

$$= \begin{cases} I & \text{if } I \text{ is a } (\mathfrak{g},K_x)\text{-module} \\ 0 & \text{if } I \text{ is a } (\mathfrak{g},K_{x'})\text{-module for } x' \text{ not conjugate to } x. \end{cases}$$

This explains the "ghosts" of [S1].

§10

Singular Infinitesimal Character

In this section we will extend the material of sections 5 through 9 to the case of singular infinitesimal character. We begin with some generalities, followed by a discussion of L-packets. Theorem 10.19 describes the parametrization of L-packets, generalizing Theorem 5.1, and Theorem 10.24 generalizes Theorem 8.2 on stable characters. Finally lifting is carried over from section in in Theorem 10.42.

To begin, we take G and $^\vee G$ as usual, and a semisimple element $^\vee c$ of $^\vee G$ corresponding to a translation family of infinitesimal characters for G as in section 2. (The pair $(c, {}^\vee c)$ of section 2 will here be taken to be $(z, {}^\vee c)$, with z some element of Z(G).) We recall and extend some of the notation of section 2. Fix maximal tori

10.1 (a) $\quad T \subset G, \ {}^dT \subset {}^\vee G_{^\vee c}$

and a distinguished isomorphism

10.1 (b) $\quad \Upsilon: {}^\vee T \to {}^dT$.

To each root α of T in G corresponds a coroot $\alpha^\vee$, which (via Υ) may be regarded as a character of $^\vee T$. Set

10.1 (c) $\quad R_{^\vee c} = R_{^\vee c}(\Upsilon) = \{\alpha \in R(G,T) \mid \Upsilon(\alpha^\vee)({}^\vee c) = 1\}$

(cf. (2.4)). The corresponding set of coroots is written $^\vee R_{^\vee c}$. It is identified by Υ with

10.1(d) $\quad {}^dR_{^\vee c} = \{{}^d\alpha \in R({}^\vee G, {}^dT) \mid {}^d\alpha({}^\vee c) = 1\}$.

As in section 5, we will sometimes find it convenient to write $^\vee\alpha = \Upsilon(\alpha^\vee)$ for the root in $^\vee G$ corresponding to α.

Choose positive root systems

10.1 (e) $P \subset R_{{}^\vee c}$, ${}^dP \subset {}^dR_{{}^\vee c}$

corresponding by ζ. Definition 2.9 now provides a parameter set

10.1 (f) $\mathcal{P} = \mathcal{P}({}^\vee c, {}^dP) \subset {}^d\mathfrak{t}$

for a translation family. Now write

10.1 (g) $B \subset P$, ${}^dB \subset {}^dP$

for the corresponding sets of simple roots.

10.2 Definition:

In the setting of (10.1), suppose π is a translation family of irreducible $\mathfrak{g}$-modules (or Harish-Chandra modules) based on $\mathcal{P}$. The Borho-Jantzen-Duflo τ-invariant of π is a subset $\tau(\pi) \subset B$, defined to be the τ-invariant of the primitive ideal $\mathrm{Ann}(\pi(\lambda))$ for any regular $\lambda \in \mathcal{P}$ (cf. [D2] or [V1]).

We have not indicated the dependence on the various choices of (10.1) in the notation. Since Lemma 2.11 provides a canonical bijection between the corresponding sets attached to any two choices, this leads to no confusion.

We have not explained the definition of the τ-invariant of an ideal. The following lemma does not quite characterize $\tau(\pi)$ in general, but for our purposes it can be taken more or less as a definition.

10.3 Lemma (cf. [D2] or [V1], Corollary 7.3.23):

In the setting of Definition 10.2, suppose $\mu \in \mathcal{P}$. Then $\pi(\mu)=0$ if and only if there is a root $\alpha \in \tau(\pi)$ such that the corresponding root $\zeta(\alpha^\vee) \in {}^dB$ satisfies $\zeta(\alpha^\vee)(\mu)=0$.

We need the analogous structure on the dual side as well. In (10.1) we have made enough choices to fix a parameter set

$${}^d\mathcal{P} = {}^d\mathcal{P}(z,P) \subset \mathfrak{t}$$

for translation families of ${}^\vee\mathfrak{g}^\vee{}_c$-modules.

10.4 Definition:

Suppose ${}^\vee\pi$ is a translation family of irreducible ${}^\vee\mathfrak{g}^\vee{}_c$-modules (or Harish-Chandra modules) based on ${}^d\mathcal{P}$. The τ-invariant $\tau({}^\vee\pi)$ is a subset of ${}^dB$, defined to be the τ-invariant of the primitive ideal $\mathrm{Ann}({}^\vee\pi({}^d\lambda))$ for any regular ${}^d\lambda\in{}^d\mathcal{P}$.

The obvious analogue of Lemma 10.2 applies to translation families for ${}^\vee\mathfrak{g}^\vee{}_c$; we leave its formulation to the reader.

We turn next to the calculation of τ-invariants. Fix a set of L-data

$$S = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \Upsilon)$$

(with $x^2 = z \in Z(G)$, $y^2 = {}^\vee c$) corresponding to irreducible translation families $J(S)$ and $J({}^\vee S)$ for $\mathfrak{g}$ and ${}^\vee\mathfrak{g}^\vee{}_c$ respectively. As the notation indicates, we assume that T, P, ${}^dT$, ${}^dP$, and Υ are as in (10.1); this can always be arranged by conjugation of S by G and ${}^\vee G^\vee{}_c$. Write as usual $\theta=\mathrm{int}(x)$ and ${}^\vee\theta=\mathrm{int}(y)$.

10.5 Lemma ([V1], Theorem 8.5.18):

In the setting just described, a root $\alpha\in B \subset P$ belongs to $\tau(J(S))$ if and only if one of the following conditions holds:

(a) $\theta\alpha = \alpha$, and α is compact; or

(b) $\theta\alpha = -\alpha$, and α satisfies the parity condition; or

(c) $\theta\alpha \neq \pm\alpha$, and $\theta\alpha \notin P$.

An analogous result holds for ${}^{\vee}\tau(J({}^{\vee}S))$. Since (using the condition of (10.1)) ${}^{\vee}(\theta\alpha) = -{}^{\vee}\theta({}^{\vee}\alpha)$, (5.10) implies

10.6 Lemma:

In the setting of (10.4), the τ-invariants of $J(S)$ and $J({}^{\vee}S)$ correspond to complementary sets of simple roots:

$\alpha \in \tau(J(S)) \Leftrightarrow {}^{\vee}\alpha \notin \tau(J({}^{\vee}S))$.

We now fix a parameter

10.7 (a) $\lambda_0 \in \mathcal{P} \subset {}^{d}\mathfrak{t} \subset {}^{\vee}\mathfrak{g}_{{}^{\vee}c}$,

generally singular; we will be studying representations of $\mathfrak{g}$ having infinitesimal character attached to λ_0. Set

10.7 (b) $B_0 = \{ \alpha \in B \mid \Upsilon(\alpha^{\vee})(\lambda_0) = 0 \}$,

the set of singular simple roots for λ_0, and let ${}^{d}B_0$ be the corresponding subset of ${}^{d}B$.

10.8 Theorem:

In the setting of (10.7), fix regular parameters $\lambda \in \mathcal{P}$ and ${}^{d}\lambda \in {}^{d}\mathcal{P}$. Suppose $\mathcal{B}$ is a block of $(\mathfrak{g}, K_x)$-modules of infinitesimal character associated to ${}^{\vee}c$, and ${}^{d}\mathcal{B}$ is a dual block of $({}^{\vee}\mathfrak{g}_{{}^{\vee}c}, {}^{\vee}K_y^{can})$-modules (Definition 4.9). Define

$\mathcal{B}_{0,TF}$ = span of translation families of irreducible modules π having some root $\alpha \in B_0$ in $\tau(\pi)$

${}^{\vee}\mathcal{B}_{1,TF}$ = span of translation families of irreducible modules ${}^{\vee}\pi$ with ${}^{\vee}\tau({}^{\vee}\pi) \supset {}^{d}B_0$.

(a) The translation functor $\Psi_\lambda^{\lambda_0}$ is a surjective map from $\mathcal{B}(\lambda)$ to $\mathcal{B}(\lambda_0)$ with kernel $\mathcal{B}_0(\lambda) = \mathcal{B}_{0,TF}(\lambda)$. Consequently there is a natural isomorphism

$$\mathcal{B}(\lambda_0) \approx \mathcal{B}(\lambda)/\mathcal{B}_0(\lambda).$$

(b) In the pairing between $\mathcal{B}_{TF}$ and ${}^\vee\mathcal{B}_{TF}$ of Definition 3.30 and 4.9,

$$(\mathcal{B}_{0,TF})^\perp = {}^\vee\mathcal{B}_{1,TF}.$$

Consequently there is a natural perfect pairing

$$< , >_0 : \mathcal{B}(\lambda_0) \times {}^\vee\mathcal{B}_1({}^d\lambda) \rightarrow \mathbb{Z},$$

between modules for $\mathfrak{g}$ of singular infinitesimal character, and modules for ${}^\vee\mathfrak{g}^\vee{}_c$ of regular infinitesimal character and sufficently large τ-invariant.

(c) Suppose S and S' are two sets of L-data in $\mathcal{B}$, and

$$\tau(J(S)) \cap B_0 = \tau(J(S')) \cap B_0 = \emptyset.$$

Then

$$< J(S,\lambda_0), J({}^\vee S', {}^d\lambda) >_0 = \begin{cases} (-1)^{\ell(J(S,\lambda))}, & S \approx S' \\ 0 & S \not\approx S'. \end{cases}$$

This theorem is a formal consequence of Lemmas 10.2 and 10.6 (and the definition of $< , >$). There is no immediately obvious analogue of Corollary 3.31, since ${}^\vee\mathcal{B}_1({}^d\lambda)$ may contain no standard modules. This gap will be filled by Corollary 10.26 below.

We extend these ideas to their "super" form as in Section 8. Recall from Corollary 8.8 the identification ${}^\vee\mathcal{B}_{TF} \approx {}^\vee\tilde{\mathcal{B}}_{TF}$ of ${}^\vee G$-coherent families of $({}^\vee\mathfrak{g}^\vee{}_c, {}^\vee K_y^{can})$-modules with ${}^\vee\tilde{G}$-coherent families. The parameter set for thse larger families is

10.9 (a) ${}^d\tilde{\mathcal{P}} = \{{}^d\mu \in \mathfrak{t} \mid \exp(2\pi i\, {}^d\mu) \in Z(G)^\theta$, and ${}^d\mu$ is P-dominant$\}$

(cf. (8.7)). Let $\mathcal{B}(y)$ be a super-block (Definition 8.4) and

$$^\vee\tilde{\mathcal{B}}_{1,TF}(y) = \oplus_x\, ^\vee\tilde{\mathcal{B}}_{1,TF}(x,y)\ ;$$

these are translation families having all roots in ${}^dB_0$ in the τ-invariants of all irreducible constituents. We use other notation (such as $^\vee\tilde{\mathcal{B}}_1^{irr}(y)$ for the corresponding collection of irreducible modules) analogously. The analogue of Corollary 8.9 is a perfect pairing

10.9 (c) $\quad \langle\ ,\ \rangle_0 : \mathcal{B}(y)(\lambda_0) \times {}^\vee\tilde{\mathcal{B}}_1(y)({}^d\lambda) \to \mathbb{Z}$.

We turn now to the parametrization of L-packets. We temporarily set aside the notation introduced so far; it will be reconstructed in this new context. Fix a quasi-admissible homomorphism

10.10 (a) $\quad \varphi : W_{\mathbb{R}} \to {}^\vee G^\Gamma$.

Define

10.10 (b) $\quad {}^\vee G_0 = \text{centralizer of } \varphi(\mathbb{C}^*),\ {}^\vee G_0^\Gamma = \langle {}^\vee G_0, \varphi(W_{\mathbb{R}})\rangle,$

$\quad\quad {}^\vee\theta_0 = \text{int}(\varphi(j)).$

Thus ${}^\vee\theta_0$ is an involution of ${}^\vee G_0$. The centralizer S_φ of $\varphi(W_{\mathbb{R}})$ in ${}^\vee G$ is

10.10 (c) $\quad S_\varphi = \{\, g \in {}^\vee G_0 \mid {}^\vee\theta_0 g = g \,\} = ({}^\vee G_0)^{{}^\vee\theta_0};$

this is the complexification of a maximal compact subgroup of the real form of ${}^\vee G_0$ with Cartan involution ${}^\vee\theta_0$. Define λ_0 from φ as in Proposition 3.4. _A priori_ λ_0 belongs to a particular Cartan subalgebra of ${}^\vee\mathfrak{g}_0$, but clearly it is central in ${}^\vee\mathfrak{g}_0$, so

10.10 (d) $\quad \lambda_0 = (\text{center of } {}^\vee\mathfrak{g}_0) \subset {}^d\mathfrak{t}_0,$ for any Cartan subalgebra ${}^d\mathfrak{t}_0 \subset {}^\vee\mathfrak{g}_0$.

Define

10.10 (e) $\quad {}^\vee c = \exp(2\pi i \lambda_0),$

$\quad\quad {}^\vee G_1 = \text{centralizer of } \lambda_0 \text{ in } {}^\vee G,$

and let ${}^\vee G_{{}^\vee c} \supset {}^\vee G_1 \supset {}^\vee G_0$ be the identity component of the centralizer of ${}^\vee c$ as usual. Finally, define

10.11 (f) $y \in {}^\vee G_0^\Gamma - {}^\vee G_0$

as in Proposition 3.4; then $\mathrm{int}(y) = {}^\vee\theta_0$ on ${}^\vee G_0$.

If ${}^dT$ is any ${}^\vee\theta_0$-stable Cartan subgroup of ${}^\vee G_0^\Gamma$, write ${}^dT^\Gamma = \langle {}^dT, y \rangle$.

Then

10.11 (a) $\varphi = \varphi(y, {}^dT^\Gamma, \lambda_0)$

(cf. (3.3)). Choose a set ${}^dP$ of positive roots for ${}^dT$ in ${}^\vee G_{{}^\vee c}$ making λ_0 dominant. (Notice that the choice involved is just an arbitrary set ${}^dP_1$ of positive roots for ${}^dT$ in ${}^\vee G_1$.) This gives a parameter set

10.11 (b) $\mathcal{P} = \mathcal{P}({}^\vee c, {}^dP) \subset {}^d\mathfrak{t}$

containing λ_0. We want to get the representations in the L-packet attached to φ by translation from an L-packet at a regular infinitesimal character. It is natural to try an L-packet attached to some $\varphi(y, {}^dT^\Gamma, \mu)$.

10.12 Proposition:

In the setting of (10.10) - (10.11), fix a G-regular weight $\mu \in \mathcal{P}$, and an irreducible translation family J with $J(\mu)$ in the L-packet of $\varphi(y, {}^dT^\Gamma, \mu)$.

(a) If $J(\lambda_0)$ is non-zero, then it belongs to the L-packet of $\varphi(y, {}^dT^\Gamma, \lambda_0)$; and every element of the L-packet arises in this way.

(b) Suppose that there is a simple root ${}^d\alpha$ of ${}^dT$ in ${}^\vee G_1$ such that either

(1) ${}^d\alpha$ is complex, and ${}^\vee\theta_0({}^d\alpha)$ is positive; or

(2) ${}^d\alpha$ is non-compact imaginary.

Then $J(\lambda_0) = 0$.

proof:

Write I for the standard translation family with $I(\mu) \supset J(\mu)$. Then $I(\lambda_0) \supset J(\lambda_0)$ by the exactness of translation functors. By Theorem 3.6,

$I(\lambda_0)$ is a direct summand of a standard module I_0 attached to $\varphi(y,{}^dT^\Gamma,\lambda_0)$; so $I_0\supset J(\lambda_0)$. Consequently $J(\lambda_0)$ belongs to the L-packet of $\varphi(y,{}^dT^\Gamma,\lambda_0)$. For the converse, let J be an irreducible translation family with $J(\lambda_0)$ attached to $\varphi(y,{}^dT^\Gamma,\lambda_0)$. Then $J(\mu)$ belongs to some L-packet $\varphi(y',({}^dT^\Gamma)',\mu')$, so (by the first part) $J(\lambda_0)$ is attached to $\varphi(\lambda',({}^dT^\Gamma)',\lambda_0')$. By the Langlands classification, $\varphi(y',({}^dT^\Gamma)',\lambda'_0)$ is conjugate to $\varphi(y',({}^dT^\Gamma)',\lambda_0)$. After replacing $\varphi(y',({}^dT^\Gamma)',\mu')$ by a conjugate, we may therefore assume that $y=y'$and $\lambda_0=\lambda_0'$ (regarded as elements of ${}^\vee\mathfrak{g}$ rather then ${}^d\mathfrak{t}$ and ${}^d\mathfrak{t}'$). Now $({}^dT^\Gamma)'$ is clearly a ${}^\vee\theta_0$-stable Cartan subgroup of ${}^\vee G_0^\Gamma$, and (a) is established. Part (b) follows from Lemmas 10.3, 10.5, and 10.6.

Part (b) of the Proposition suggests a way to pin down the choices of ${}^dT$ and ${}^dP$ in (10.11). The next Lemma is stated in terms of G to make the notation as familiar as possible; it will be applied to ${}^\vee G_0$. We first establish some notation.

10.13 Definition:

Suppose G is a connected complex reductive group, and θ is an involution of G; write $K=G^\theta$ for the fixed points, and $\mathfrak{g}=\mathcal{K}\oplus\mathcal{P}$ for the decomposition of the Lie algebra into the +1 and -1 eigenspaces of K. Choose a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{P}$ consisting of semisimple elements, and write $A=\exp(\mathcal{A})$ for the corresponding torus. Let L be the centralizer of A in G, and let $M=L\cap K$. Then L is a connected θ-stable reductive group, and $L=MA$; here the factors commute, but the product may not be direct. Choose a set of positive restricted roots of A in G, corresponding to a parabolic subgroup $Q_{IW}=LN=MAN$, an Iwasawa parabloic subgroup. A maximal torus $T_{IW}\subset L$ is called an Iwasawa maximal

<u>torus</u>, and a Borel subgroup B_{IW} with $T_{IW} \subset B_{IW} \subset Q_{IW}$ is called an <u>Iwasawa Borel subgroup</u>. Finally, the positive root system P_{IW} corresponding to B_{IW} is called an <u>Iwasawa positive root system</u>.

10.14 <u>Lemma</u>:

In the setting of Definition 10.13, each of the structures A, L, M, N, Q_{IW}, T_{IW}, B_{IW}, and P_{IW} is unique up to conjugation by K.

Suppose (T,P) is a pair consisting of a θ-stable maximal torus in G and a positive root system for T in G. Then (T,P) is K-conjugate to (T_{IW}, P_{IW}) if and only if there is no simple root α in P such that either

(1) α is complex, and $\theta\alpha$ is positive; or

(2) α is noncompact imaginary.

This is standard and easy; we omit the proof.

We return now to the setting (10.10), and consider the problem of specifying the positive root system ${}^dP$ chosen in (10.11).

10.15 <u>Lemma</u>:

In the setting of (10.10), suppose ${}^dP$ is a set of positive roots for ${}^dT$ in ${}^\vee G^\vee{}_c$ making λ_0 dominant. Write ${}^dP_1 \supset {}^dP_0$ for the roots in ${}^dP$ and ${}^\vee G_1$, ${}^\vee G_0$ respectively.

(a) Suppose that there is a root ${}^d\beta$ of ${}^dT$ in ${}^dP_1$ such that $\langle {}^d\beta, {}^\vee\theta_0\lambda_0 \rangle > 0$. Then there is a complex simple root ${}^d\alpha$ of ${}^dT$ in ${}^\vee G_1$ such that ${}^\vee\theta_0({}^d\alpha)$ is positive.

(b) Suppose there is <u>no</u> root satisfying the condition in (a). Then

$$ {}^dP_1 - {}^dP_0 = \{ {}^d\beta \text{ a root of } {}^dT \text{ in } {}^\vee G_1 \mid \langle {}^d\beta, {}^\vee\theta_0\lambda_0 \rangle < 0 \}. $$

In this case ${}^dP_0$ is spanned by simple roots of ${}^dT$ in ${}^dP$.

proof:

For (a), write ${}^d\beta = \Sigma\, n_{{}^d\alpha}\, {}^d\alpha$, a sum of simple roots in ${}^dP_1$. Obviously $\langle {}^d\alpha, {}^\vee\theta_0\lambda_0 \rangle > 0$ for some ${}^d\alpha$. Since ${}^d\alpha$ is a root in ${}^dP_1$, it is orthogonal to λ_0; so ${}^d\alpha$ cannot be real or imaginary. This proves (a). The first claim of (b) is clear, and the second follows immediately.

We can now explain how to choose ${}^dP$ in (10.11). In the setting of (10.10), choose an Iwasawa maximal torus

10.16 (a) ${}^dT \subset {}^\vee G_0$; put ${}^dT^\Gamma = \langle {}^dT, y \rangle$.

Choose an Iwasawa positive root system ${}^dP_0$ for ${}^dT$ in ${}^\vee G_0$, and extend it to a positive system

10.16 (b) ${}^dP_1 \supset {}^dP_0$ for ${}^dT$ in ${}^\vee G_1$ making $-{}^\vee\theta_0\lambda_0$ dominant. Finally, extend ${}^dP_1$ to

10.16 (c) ${}^dP \supset {}^dP_1$ for ${}^dT$ in ${}^\vee G_1$ making λ_0 dominant.

This gives a parameter set

10.16 (d) $\mathcal{P} = \mathcal{P}({}^\vee c, {}^dP) \subset {}^d\mathfrak{t}$

containing λ. The data $(y, {}^dT^\Gamma, {}^dP, \lambda_0)$ is unique up to conjugation by S_φ (cf. (10.10)(c)).

10.17 Corollary:

In the setting of (10.10) and (10.16), fix a G-regular weight $\mu \in \mathcal{P}$. List the irreducible translation families $\{J_i\}_{i\in I}$ with $J_i(\mu)$ in the L-packet of $\varphi_{reg} = \varphi(y, {}^dT^\Gamma, \mu)$. Then the L-packet of φ consists precisely of the non-zero representations in $\{J_i(\lambda_0)\}$.

This follows from Proposition 10.12 and Lemmas 10.14 and 10.15.

To state the final parametrization of the L-packet, we need some coverings. Let ${}^\vee\tilde{G}$ be the canonical cover of ${}^\vee G$, and use the tilde for the preimages in ${}^\vee\tilde{G}$ of subgroups of ${}^\vee G$. We have

10.18 (a) $({}^dT\cap S_\varphi)^\sim \subset \tilde{S}_\varphi \subset {}^\vee\tilde{G}$.

Define

10.18 (b) $\tilde{\mathbb{S}}_\varphi = \tilde{S}_\varphi/(\tilde{S}_\varphi)^0$.

The group ${}^dT\cap S_\varphi$ is the centralizer of y in ${}^dT$. It is therefore equal to $S_{\varphi_{reg}}$ (cf. Corollary 10.17), and we get a natural map

10.18 (c) $\tilde{\mathbb{S}}_{\varphi_{reg}} \to \tilde{\mathbb{S}}_\varphi$.

10.19 <u>Theorem</u>:

Suppose we are in the setting of (10.10) and (10.16)-(10.18). Then there is a natural bijection between the super L-packet Π attached to φ and $(\tilde{\mathbb{S}}_\varphi)^\wedge$, the group of characters of $\tilde{\mathbb{S}}_\varphi$.

More precisely, the map (10.18)(c) is surjective. A character of $\tilde{\mathbb{S}}_{\varphi_{reg}}$ is trivial on the kernel of this map (and so lifts to $\tilde{\mathbb{S}}_\varphi$) if and only if the corresponding irreducible translation family J_i has $J_i(\lambda_0)\neq 0$.

We will first give a proof of the parametrization alone, and then a somewhat different argument leading to the second assertion.

<u>first proof</u>:

Fix μ as in Corollary 10.17, and choose

10.20 (a) $T^\Gamma \subset G^\Gamma$, $\zeta : {}^\vee T \to {}^dT$, P

as in the discussion after Definition 3.12 for $\varphi_{reg}=\varphi(y,{}^dT^\Gamma,\mu)$. By Lemma 3.13, L-data for the translation families J_i may be taken to be

10.20 (b) $S(x) = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \zeta)$ $(x^2\in Z(G))$

as x runs over a set of representatives of the T-conjugacy classes in $T^\Gamma - T$, subject to the restriction indicated. Define $G_0 \supset T$ to be the (Levi) subgroup of G corresponding to ${}^\vee G_0$ under $\mathfrak{I}$. Since ${}^\vee G_0$ is normalized by y, the corresponding set of roots is ${}^\vee\theta$-stable; so the set of roots of T in G_0 is θ-stable, and T^Γ normalizes G_0. Set

$$G_0^\Gamma = \langle G_0, T^\Gamma \rangle .$$

Fix $\delta \in \mathcal{D} \cap T^\Gamma$ making the set of positive imaginary roots in P distinguished; this δ is unique up to conjugation by T. Choose an extended group structure $\mathcal{D}_0$ on G_0^Γ with $\delta \in \mathcal{D}_0$, so that the set of positive imaginary roots in P_0 is distinguished with respect to δ. Then G_0^Γ is an E-group structure for ${}^\vee G_0$ associated to $z_{{}^\vee\rho} z_{{}^\vee\rho_0}$; the factors here are the obvious elements of $Z(G_0)$. Now L-data for $({}^\vee G_0^\Gamma, G_0^\Gamma)$ involving y therefore correspond to translation families of $({}^\vee\mathfrak{g}_0, S_\varphi^{can})$-modules of type $z_{{}^\vee\rho} z_{{}^\vee\rho_0}$. (Here S_φ^{can} is contained in the canonical cover of ${}^\vee G_0$.) Each S(x) gives such a family

$${}^\vee S(x)_0 = (y, {}^d T^\Gamma, {}^d P_0, x, T^\Gamma, P_0, \mathfrak{I}^{-1}) .$$

We are interested in those S(x) for which no simple root in G_0 belongs to the τ-invariant. They correspond to those ${}^\vee S(x)_0$ for which <u>every</u> simple root in ${}^\vee G_0$ belongs to the τ-invariant (Lemma 10.6), and thus to finite-dimensional $({}^\vee\mathfrak{g}_0, S_\varphi^{can})$-modules.

Suppose finally that ${}^\vee S(x)_0$ does correspond to a translation family ${}^\vee J$ of finite-dimensional representations. Fix a regular element ${}^d\mu \in \mathcal{P}(x^2, {}^d P_0) \subset \mathfrak{t}$. Then ${}^\vee J({}^d\mu)$ is a finite-dimensional $({}^\vee\mathfrak{g}_0, S_\varphi^{can})$-module of type $z_{{}^\vee\rho} z_{{}^\vee\rho_0}$ and infinitesimal character ${}^d\mu$. On the other hand, there is a finite-dimensional holomorphic representation ${}^\vee F({}^d\mu)$ of ${}^\vee G_0^{can}$ having infinitesimal character ${}^d\mu$; it has type $\exp(2\pi i\, {}^d\mu) z_{{}^\vee\rho_0} = x^2 z_{{}^\vee\rho_0}$. By restriction we can regard ${}^\vee F({}^d\mu)$ as a $({}^\vee\mathfrak{g}_0, S_\varphi^{can})$-module. Define

10.21 $\check{E} = (\check{J}(^d\mu) \otimes [\check{F}(^d\mu)^*])^{\check{g}_0}$;

this is an irreducible $(\check{g}_0, S_\varphi^{can})$-module of type $z_{\check{\rho}}x^{-2}$ and trivial infinitesimal character. By the first of these properties, it factors to $\tilde{S}_\varphi$. By the second, $\check{E}$ is trivial on $\check{g}_0$, and so on $(\tilde{S}_\varphi)^0$. We may therefore regard $\check{E}$ as a character of $\tilde{\mathbb{S}}_\varphi$.

Conversely, suppose $\check{E}$ is a character of $\tilde{\mathbb{S}}_\varphi$ of type $z \epsilon Z(G)^\theta$. Regard $\check{E}$ as an $\tilde{S}_\varphi$-module trivial on $(\tilde{S}_\varphi)^0$, then as a $(\check{g}_0, \tilde{S}_\varphi)$-module trivial on $\check{g}_0$. Then we can define a translation family $\check{J}$ by $\check{J}(^d\mu) = \check{F}(^d\mu) \otimes \check{E}$, for $^d\mu \epsilon \mathcal{P}(z, {}^dP)$. The family $\check{J}$ must be attached to L-data $\check{S}(x)_0$, and so to L-data $S(x)$. This establishes the bijection of the theorem.

<u>second proof</u>:

Begin as in (10.20). By Lemmas 10.5 and 10.6, we are interested in those x such that all the simple imaginary roots in P_0 are noncompact. Recall the proof of Theorem 5.1; since δ makes <u>all</u> the simple imaginary roots (including those in P_0) noncompact, we are interested in elements t such that $\alpha(t)=1$ for all simple imaginary roots α in P_0: the quotient

$$\mathcal{F}_0 = \{t \epsilon T \mid t\theta(t) \epsilon Z(G),\ \alpha(t) = 1 \text{ for all simple imaginary roots } \alpha \text{ in } P_0\} / \{ s\theta(s^{-1}) \mid s \epsilon T \}$$

parametrizes the L-packet. This is obviously a closed subgroup of the group $\mathcal{F}$ of Theorem 5.1, which we identified with characters of $\tilde{S}_{\varphi_{reg}}$. Consequently there is a subgroup

$$\bar{C} \subset \tilde{S}_{\varphi_{reg}} = (^dT \cap S_\varphi)^\sim / [(^dT \cap S_\varphi]^\sim]^0$$

such that $\mathcal{F}_0$ consists of characters trivial on $\bar{C}$. Write C for the pre-image of $\bar{C}$ in $(^dT \cap S_\varphi)^\sim$; then $\mathcal{F}_0$ corresponds to characters of $(^dT \cap S_\varphi)^\sim$ trivial on C. By inspection of the proof of Theorem 5.1, we can identify C:

$$C = < ((^dT \cap S_\varphi)^\sim)^0, \{ m_{d\alpha} \mid {}^d\alpha \text{ a simple real root in } {}^dP_0\} >.$$

Here $m_{d\alpha}$ is the element of order 2 attached to a root; that is if $({}^d\alpha)^\vee : \mathbb{C}^* \to {}^d\tilde{T}$ is the coroot, then $m_{d\alpha} = ({}^d\alpha)^\vee(-1)$. Theorem 10.19 now follows by applying the following structural result to ${}^\vee G_0$.

10.22 Proposition:

In the setting of Definition 10.13, suppose T is an Iwasawa maximal torus in G. Then

(a) T meets every connected component of K.

(b) The kernel of the map $T \cap K \to K/K^0$ is generated by $(T \cap K)^0$ and the various $m_\alpha = \alpha^\vee(-1)$, for α a simple real root of T in G (with respect to an Iwasawa Borel subgroup).

The proof is elementary (using for example the fact that K is connected when G is simply connected, and the uniqueness of Iwasawa Borel subgroups) and we omit it. This completes the proof of Theorem 10.19.

Next, we need to describe the notion of stability for singular infinitesimal characters. In the setting of Theorem 10.8, there are two natural candidates for a definition of stable: the image of stable characters in $\mathcal{B}(\lambda)$ under $\Psi_{\lambda}^{\lambda_0}$, and the characters orthogonal to characters in ${}^{\vee}\mathcal{B}_1({}^d\lambda)$ that vanish near the identity. The main problem will be to show that these are equivalent.

10.23 Definition:

In the setting of Theorem 10.8, suppose Θ is a virtual $(\mathfrak{g},K_x)$-module in $\mathcal{B}(\lambda_0)$. We say that Θ is stable if $< \Theta, {}^dZ >_0 = 0$ for all ${}^dZ \in {}^{\vee}\mathcal{B}_1({}^d\lambda)$ such that ${}^dZ$ vanishes near the identity. An arbitrary virtual $(\mathfrak{g},K_x)$-module is called stable if its projection on a block $\mathcal{B}$ and an infinitesimal character χ is stable for all $\mathcal{B}$ and χ.

10.24 Theorem:

(1) In the setting of Theorem 10.8, let Π be an L-packet of $(\mathfrak{g},K_x)$-modules in $\mathcal{B}(\lambda_0)$. List the standard limit modules in Π having unique Langlands subquotients as $I_1, I_2, \ldots, I_r$, and define $\Theta_\Pi = \Sigma_i\, I_i$. Then Θ_Π is stable.

(2) Any stable virtual $(\mathfrak{g},K_x)$-module is a finite sum of various Θ_Π.

To prove this we will need a better understanding of the characters in ${}^{\vee}\mathcal{B}_1({}^d\lambda)$, so we pause to develop that.

10.25 Lemma:

In the setting of Theorem 10.8, suppose ${}^\vee I$ is a standard module whose Langlands subquotient ${}^\vee J$ belongs to ${}^\vee\mathcal{B}_1({}^d\lambda)$. Then we can find other standard modules ${}^\vee I_1, {}^\vee I_2, \ldots, {}^\vee I_r$ in ${}^\vee\mathcal{B}({}^d\lambda)$, and integers n_j, so that

(1) ${}^\vee I + \Sigma_j n_j\, {}^\vee I_j \in {}^\vee\mathcal{B}_1({}^d\lambda)$

(2) $\ell({}^\vee I_j) < \ell({}^\vee I)$ for all j.

We may in addition require

(3) the Langlands quotient ${}^\vee J_j$ of ${}^\vee I_j$ is not in ${}^\vee\mathcal{B}_1({}^d\lambda)$;

and then the virtual character in (1) is unique.

proof:

One knows that ${}^\vee J$ itself may be written in the form (1); this proves the first claim. For the second, we proceed by induction on $\ell({}^\vee I)$. For every term ${}^\vee I_j$ with ${}^\vee J_j$ in ${}^\vee\mathcal{B}_1({}^d\lambda)$, find (by induction) an expression

$${}^\vee I_j + \Sigma n_{jk}\, {}^\vee I_k \in {}^\vee\mathcal{B}_1({}^d\lambda)$$

with $\ell({}^\vee I_k) > \ell({}^\vee I)$ and ${}^\vee J_k$ not in ${}^\vee\mathcal{B}_1({}^d\lambda)$ when $n_{jk} \neq 0$. Then the expression

$${}^\vee I + \Sigma n_j\, {}^\vee I_j - \sum_{{}^\vee J_j \in \mathcal{B}_1} n_j({}^\vee I_j + \Sigma n_{jk}\, {}^\vee I_k)$$

belongs to ${}^\vee\mathcal{B}_1$ and satisfies (2) and (3). The uniqueness is easy.

The virtual characters satisfying (1) - (3) of the lemma are called pseudo-standard (virtual) characters of type ${}^\vee B_0$ (where ${}^\vee B_0$ is the set of simple roots defining the τ-invariant condition for belonging to ${}^\vee\mathcal{B}_1$). (They can also be described using D-modules on a partial flag variety, but we will have no need for this.) If ${}^dS$ is a set of L-data for an irreducible module in ${}^\vee\mathcal{B}_1$, we write $\Psi({}^dS, {}^d\lambda)$ for the corresponding pseudo-standard character.

10.26 Corollary:

In the setting of Theorem 10.8, the pseudo-standard characters form a $\mathbb{Z}$-basis of ${}^\vee\mathcal{B}_1({}^d\lambda)$, which is related to the basis of irreducible characters by an upper triangular matrix. Each pseudo-standard character has a unique irreducible constituent of maximal length, called its Langlands subquotient. In the setting of (10.9),

$$< I(S,\lambda_0), \Psi({}^\vee S', {}^d\lambda) >_0 = \begin{cases} (-1)^{\ell(J(S,\lambda))} & S \approx S' \\ 0 & S \not\approx S'. \end{cases}$$

proof:

The only part which is not quite immediate is the formula for $< , >_0$. For that, recall that $I(S,\lambda_0) = \Psi_\lambda^{\lambda_0}(I(S,\lambda))$. Therefore by definition we have

$$< I(S,\lambda_0) , \Psi({}^\vee S', {}^d\lambda) >_0 = < I(S,\lambda) , I({}^\vee S', {}^d\lambda) + \Sigma n_i I({}^dS_i, {}^d\lambda) >$$

where the various $J({}^dS_i, {}^d\lambda)$ do not belong to ${}^\vee\mathcal{B}_1$. Consequently no term ${}^dS_i$ can be dual to S; they contribute 0 to the pairing by Corollary 3.31, and we get

$$< I(S,\lambda_0) , \Psi({}^\vee S', {}^d\lambda) >_0 = < I(S,\lambda) , I({}^\vee S', {}^d\lambda) >.$$

The result follows from Corollary 3.31.

The following generalization of Corollary 7.6 is the heart of the proof of Theorem 10.24.

10.27 Proposition:

(a) In the setting of (10.9), suppose ${}^dS=(y,{}^dT^\Gamma,{}^dP,x,T^\Gamma,P,{}^d\Upsilon)$ is a set of L-data for a pseudo-standard translation family $\Psi=\Psi({}^dS)$ in ${}^\vee\mathcal{B}_1$. Suppose x' is another element of $T^\Gamma-T$ with $x^2\in Z(G)$, and that ${}^dS'=(y,{}^dT^\Gamma,{}^dP,x',T^\Gamma,P,{}^d\Upsilon)$ also corresponds to a pseudo-standard translation family Ψ' in ${}^\vee\mathcal{B}_1$. Extend Ψ and Ψ' to ${}^d\tilde{\mathcal{P}}$ as in (10.9). Then the character of $\Psi({}^d\lambda)-\Psi'({}^d\lambda)$ vanishes near the identity.

(b) Any virtual $({}^\vee\mathfrak{g}^\vee{}_c,K_y^{can})$-module in ${}^\vee\mathcal{B}_1({}^d\lambda)$ vanishing near the identity must be a sum of a finite number of terms as in (a).

The proof of this proposition rests on a refined description of the pseudo-standard characters. To get that we need the coherent continuation representation ([V1], Definition 7.2.28) of the integral Weyl group $W({}^d\lambda)$ of ${}^\vee\mathfrak{g}^\vee{}_c$ (Definition 3.17) on ${}^\vee\tilde{\mathcal{B}}({}^d\lambda)$. This is very closely related to the cross action of Definition 3.17, but the two are not quite the same. We write the coherent continuation action with a dot. The most important fact about the coherent continuation representation is

10.28 Proposition:

A virtual character Θ in ${}^\vee\tilde{\mathcal{B}}({}^d\lambda)$ belongs to ${}^\vee\tilde{\mathcal{B}}_1({}^d\lambda)$ if and only if $s\bullet\Theta=-\Theta$ for every (simple) reflection s corresponding to a root in ${}^\vee B_0$.

10.29 Lemma ([V1], Chapter 8):

In the setting just described, fix a simple reflection $s\in W({}^d\lambda)$. Then the set of equivalence classes of L-data for elements of ${}^\vee\tilde{\mathcal{B}}({}^d\lambda)$ may be uniquely partitioned into sets of one, two, or three classes, each of which is one of the following five types. (We will always write $^{d}S=(y,{}^{d}T^{\Gamma},{}^{d}P,x,T^{\Gamma},P,{}^{d}\Upsilon)$, and $^{d}\alpha$ for the simple root of ^{d}T in ^{d}P corresponding to s.)

(a) $\{\, {}^{d}S,\ s\times{}^{d}S \,\}$; the root $^{d}\alpha$ is complex, $\theta({}^{d}\alpha)$ is negative, $\ell(s\times{}^{d}S)=\ell({}^{d}S)-1$, and

$$s\bullet I({}^{d}S,{}^{d}\lambda) = I(s\times{}^{d}S,{}^{d}\lambda).$$

(b) $\{\, {}^{d}S \,\}$; the root $^{d}\alpha$ is compact imaginary, and

$$s\bullet I({}^{d}S,{}^{d}\lambda) = -I({}^{d}S,{}^{d}\lambda).$$

(c) $\{\, {}^{d}S \,\}$; the root $^{d}\alpha$ is real and fails to satisfy the parity condition, and

$$s\bullet I({}^{d}S,{}^{d}\lambda) = I({}^{d}S,{}^{d}\lambda).$$

(d) $\{\, {}^{d}S,\ {}^{d}S^{\alpha}_{-},\ {}^{d}S^{\alpha}_{+} \,\}$; the root $^{d}\alpha$ is real and satisfies the parity condition, $\ell({}^{d}S^{\alpha}_{\pm}) = \ell({}^{d}S)-1$, $s\times{}^{d}S={}^{d}S$, $s\times{}^{d}S^{\alpha}_{+}={}^{d}S^{\alpha}_{-}$, $s\times{}^{d}S^{\alpha}_{-}={}^{d}S^{\alpha}_{+}$, and

$$s\bullet I({}^{d}S,{}^{d}\lambda) = I({}^{d}S,{}^{d}\lambda)$$
$$s\bullet I({}^{d}S^{\alpha}_{+},{}^{d}\lambda) = I({}^{d}S,{}^{d}\lambda) - I({}^{d}S^{\alpha}_{-},{}^{d}\lambda)$$
$$s\bullet I({}^{d}S^{\alpha}_{-},{}^{d}\lambda) = I({}^{d}S,{}^{d}\lambda) - I({}^{d}S^{\alpha}_{+},{}^{d}\lambda).$$

The simple root for $^{d}S^{\alpha}_{\pm}$ corresponding to s is noncompact imaginary.

(e) $\{\, {}^{d}S,\ {}^{d}S',\ {}^{d}S^{\alpha} \,\}$; the root $^{d}\alpha$ is real and satisfies the parity condition, $\ell({}^{d}S')=\ell({}^{d}S)$, $\ell({}^{d}S^{\alpha})=\ell({}^{d}S)-1$, $s\times{}^{d}S={}^{d}S'$, $s\times{}^{d}S^{\alpha}={}^{d}S^{\alpha}$, and

$$s\bullet I({}^{d}S,{}^{d}\lambda) = I({}^{d}S',{}^{d}\lambda),\ s\bullet I({}^{d}S',{}^{d}\lambda) = I({}^{d}S,{}^{d}\lambda),$$
$$s\bullet I({}^{d}S^{\alpha},{}^{d}\lambda) = I({}^{d}S,{}^{d}\lambda) + I({}^{d}S',{}^{d}\lambda) - I({}^{d}S^{\alpha},{}^{d}\lambda).$$

The simple root for $^{d}S^{\alpha}$ corresponding to s is noncompact imaginary.

10.30 Corollary:

Any virtual character Θ in $^{\vee}\tilde{\mathbb{B}}({}^{d}\lambda)$ such that $s\bullet\Theta=-\Theta$ is a linear combination of contributions from the various classes in the partition of

Lemma 10.28. The possible contributions (using the same letters to separate cases are:

(a) $I({}^dS,{}^d\lambda) - I(s\times{}^dS,{}^d\lambda)$;

(b) $I({}^dS,{}^d\lambda)$;

(c) no contribution;

(d) $I({}^dS,{}^d\lambda) - I({}^dS^\alpha_+,{}^d\lambda) - I({}^dS^\alpha_-,{}^d\lambda)$;

(e) $I({}^dS,{}^d\lambda) - I({}^dS^\alpha,{}^d\lambda)$ and $I({}^dS',{}^d\lambda) - I({}^dS^\alpha,{}^d\lambda)$.

This Corollary amounts to explicit formulas for the pseudo-standard characters in the case when λ_0 is singular with respect to just one root. It leads to a fairly explicit formula for the pseudo-standard representations in general. To state this, we need some notation.

10.31 Definition:

In the setting of Lemma 10.29, fix a simple reflection $s\in W({}^d\lambda)$. Suppose I and I' are standard representation in ${}^\vee\tilde{\mathcal{B}}({}^d\lambda)$. Write

$$I \overset{s}{\leftarrow} I'$$

if $\ell(I') = \ell(I) - 1$, and I and I' belong to a common set of one of the five types in Lemma 10.29.

Define $\leftarrow$ to be the smallest transitive relation on standard modules in ${}^\vee\tilde{\mathcal{B}}({}^d\lambda)$ containing all the relations $\overset{s}{\leftarrow}$ for s corresponding to a simple root in ${}^\vee B_0$. Explicitly, $I \leftarrow I'$ means that there is a simple reflection s (corresponding to a root in ${}^\vee B_0$) and a standard module I'' such that $I'' \overset{s}{\leftarrow} I'$, and either

(a) $\ell(I') = \ell(I) - 1$ and $I'' = I$; or

(b) $\ell(I') < \ell(I) - 1$, and $I \leftarrow I''$.

We may write the relation $\leftarrow$ betwen sets of L-data as well, and for representations and L-data in $\mathcal{B}$ instead of ${}^\vee\mathcal{B}$. (Note that $I \overset{s}{\leftarrow} I'$ in $\mathcal{B}$ is equivalent to ${}^dI' \overset{s}{\leftarrow} {}^dI$ in ${}^\vee\mathcal{B}$.) Corollary 10.30 says that the virtual characters satisfying $s \cdot \Theta = -\Theta$ are spanned by certain characters $I - \Sigma I'$, where the sum runs over I' satisfying $I \overset{s}{\leftarrow} I'$. We deduce immediately:

10.32 Corollary:

In the setting of Theorem 10.8, suppose that ${}^dS$ is a set of L-data for a pseudo-standard translation family $\Psi({}^dS)$ in ${}^\vee\mathcal{B}_{1,TF}$:

$$\Psi({}^dS) = I({}^dS) + \Sigma\, (-1)^{\ell({}^dS') - \ell({}^dS)}\, I({}^dS')$$

with the sum running over ${}^dS$ satisfying ${}^dS \leftarrow {}^dS'$.

The reader may wonder why we did not use the formula of Corollary 10.32 as the definition of $\Psi({}^dS)$. The difficulty is that this formula does not obviously define an element of ${}^\vee\mathcal{B}_{1,TF}$. The problem is that one might imagine relations $I'' \overset{s}{\leftarrow} I'$, $I \leftarrow I'$, but $I \not\leftarrow I''$. What is obvious is that this formula is the only one that might lead to a character in ${}^\vee\mathcal{B}_{1,TF}$.

10.33 Corollary:

In the setting of Theorem 10.24, let S be a set of L-data with $J(S,\lambda_0)$ in the L-packet of Π, and let S' be any other set of L-data for $\mathcal{B}$. Then $I(S,\lambda_0)$ is a direct summand of $I(S',\lambda_0)$ if and only if $S'=S$ or $S' \leftarrow S$.

proof:

By Corollary 10.26, the multiplicity of $I(S,\lambda_0)$ as a summand of $I(S',\lambda_0)$ is

$$(-1)^{\ell(S)} < I(S',\lambda_0),\ \Psi({}^dS,{}^d\lambda) >_0$$

$$= (-1)^{\ell(S')} < I(S',\lambda),\ \Psi({}^dS,{}^d\lambda) >$$

$$= (-1)^{\ell(S)} < I(S',\lambda), I({}^dS,{}^d\lambda) + \sum_{{}^dS \leftarrow {}^dS''} (-1)^{\ell({}^dS'')-\ell({}^dS)} I({}^dS'',{}^d\lambda) >$$
$$= (-1)^{\ell(S)} < I(S',\lambda), I({}^dS,{}^d\lambda) + \sum_{S'' \leftarrow S} (-1)^{\ell(S'')-\ell(S)} I({}^dS'',{}^d\lambda) > .$$

Now apply Corollary 3.31.

<u>proof of Proposition 10.27</u>:

For (a), write $I=I({}^dS,{}^d\lambda)$, $I'=I({}^dS', {}^d\lambda)$. Suppose that either $I=I_1$ or $I \leftarrow I_1$. List the standard characters $I_1, \ldots, I_r$ such that $I_i = I_1$ near the identity, and $I \leftarrow I_i$. Similarly write $I'_1, \ldots, I'_{r'}$ for the standard characters equal to I'_1 near the identity such that $I' \leftarrow I'_j$. We must prove that $r=r'$. We proceed by induction on $\ell(I_1)$. If $I=I_1$, then $r=r'=1$; so we may suppose $\ell(I_1) < \ell(I)$. Choose a simple reflection s corresponding to a simple root in ${}^\vee B_0$, and a standard character I_0 so that $I_0 \overset{s}{\leftarrow} I_1$, and either $I_0=I$ or $I \leftarrow I_0$. Let ${}^d\alpha$ be the root for (L-data for) I_1 corresponding to s, as in Lemma 10.29. We may take L-data for I_1 and I'_1 with the same Cartan subgroup and positive root system in ${}^\vee G^\vee{}_c$; then ${}^d\alpha$ is the same root for each. There are two cases.

<u>Case 1 ${}^d\alpha$ is complex</u>:

In this case $I_0 = s \times I_1$. Corollary 10.30 guarantees that (since $s \bullet \Psi = -\Psi$) the characters $s \times I_i$ must all appear in Ψ, and that no others equal to I_0 near the identity can appear. That is, exactly r terms equal to I_0 near the identity apear in Ψ. Similarly, Ψ' has exactly r' terms equal to $I'_0 = s \times I'_1$ near the identity. By induction $r=r'$.

<u>Case 2 ${}^d\alpha$ is noncompact imaginary</u>:

Once again Corollary 10.30 implies that Ψ contains exactly r standard characters equal to I_0 near the identity. (The only difference is that in this case we cannot say exactly what they must be, since the condition $I_0 \overset{s}{\leftarrow} I_1$ does not determine I_0 from I_1.) By induction, $r=r'$.

Part (b) of Proposition 10.27 is an easy consequence of Lemma 7.5; we omit the argument.

The proof actually shows that the numbers r defined in it are equal to 1. Now two standard $({}^\vee\mathfrak{g}^\vee{}_c, {}^\vee K_y^{can})$-modules of regular infinitesimal character agree near the identity if and only if the dual $(\mathfrak{g},K)$-modules belong to the same L-packet. We therefore obtain:

10.34 Corollary:

In the setting of Theorem 10.24, suppose S is a set of L-data with $J(S,\lambda_0)$ non-zero. Suppose S_1 and S_2 are any two other distinct sets of L-data for $\mathcal{B}$, and for i=1,2 either $S_i=S$ or $S_i \leftarrow S$. Then $I(S_1,\lambda)$ and $I(S_2,\lambda)$ must belong to distinct L-packets.

Theorem 10.24 follows from Proposition 10.27 and Corollary 10.26 in exactly the same way as Theorem 8.2 follows from Lemma 7.5 and Corollary 3.31.

We turn now to the generalization of Theorem 8.3, characterizing stable characters in terms of a strong invariance property of their distribution characters.

10.35 Lemma:

Suppose $\mathcal{B}$ is a block of $(\mathfrak{g},K_x)$-modules, and $\mathcal{P}$ is the parameter set for translation families in the block. Fix $\lambda_0 \in \mathcal{P}$ arbitrary, and $\lambda \in \mathcal{P}$ regular.

(a) The space of stable virtual characters in $\mathcal{B}(\lambda)$ is invariant under the coherent continuation representation of the integral Weyl group $W(\lambda)$.

(b) The translation functor $\Psi_\lambda^{\lambda_0}$ is a surjective map from stable characters in $\mathcal{B}(\lambda)$ onto stable characters in $\mathcal{B}(\lambda_0)$.

(c) Write $\Psi_{\lambda_0}^{\lambda}$ for the translation functor from $\mathcal{B}(\lambda_0)$ to $\mathcal{B}(\lambda)$. Then a virtual character Θ_0 in $\mathcal{B}(\lambda_0)$ is stable if and only if $\Psi_{\lambda_0}^{\lambda}(\Theta_0)$ is stable in $\mathcal{B}(\lambda)$.

(d) The space of stable virtual characters in $\mathcal{B}$ is invariant under tensor products with holomorphic finite-dimensional representations and projection on an infinitesimal character.

<u>proof</u>:

Suppose s is a simple reflection in $W(\lambda)$, Θ is a virtual character in $\mathcal{B}(\lambda)$, and ${}^dZ$ is a virtual character in the dual block ${}^\vee\mathcal{B}({}^d\lambda)$. A fundamental property of the pairing of Definition 3.30 is

$$< s\bullet\Theta, {}^dZ > = - <\Theta, s\bullet{}^dZ > .$$

This can be computed from Corollary 3.31 and Corollary 10.31. To prove (a), it is therefore sufficient to prove that the space of virtual characters in ${}^\vee\mathcal{B}({}^d\lambda)$ vanishing near the identity is stable under the coherent continuation action. But this is obvious from the explicit form of the action on distribution characters.

For (b), the fact that $\Psi_\lambda^{\lambda_0}$ carries stable charactes to stable characters is a formal consequence of Definition 10.23 and Theorem 10.8. Surjectivity folows from the explicit description of stable characters in Theorem 10.24.

For (c), suppose Θ_0 is stable; write $\Theta_0=\Psi_\lambda^{\lambda_0}(\Theta)$, with Θ a stable character in $\mathcal{B}(\lambda)$. Then

$$\Psi_{\lambda_0}^{\lambda}(\Theta_0)= \sum_{w\in W(\lambda)^{\lambda_0}} w\bullet\Theta$$

(by the theory of the coherent continuation representation) which is stable by (a). Conversely, suppose $\Psi_{\lambda_0}^{\lambda}(\Theta_0)$ is stable. Then so is $\Psi_{\lambda}^{\lambda_0}\Psi_{\lambda_0}^{\lambda}(\Theta_0)$, by (b). But

$$\Psi_{\lambda}^{\lambda_0}\Psi_{\lambda_0}^{\lambda}(\Theta_0) = |W(\lambda)^{\lambda_0}|\,\Theta_0$$

(by the theory of coherent continuation again). By inspection of the definition, it follows that Θ_0 is stable.

For (d), the statement about projection on infinitesimal character is part of the definition. Suppose Θ_0 is stable in $\mathcal{B}(\lambda_0)$. Choose Θ_1 stable in $\mathcal{B}(\lambda)$ with $\Psi_{\lambda}^{\lambda_0}(\Theta_1)=\Theta_0$. Like any virtual character with regular infinitesimal character, Θ_1 gives rise to a unique coherent family Θ of virtual $(\mathfrak{g},K)$-modules based on $\lambda + X_*({}^dT)$ ([V1], Definition 7.2.5) such that $\Theta(\lambda)=\Theta_1$. We have (by definition of the coherent continuation representation) $\Theta(w\lambda)=w^{-1}\Theta_1$, which is stable by (a). If μ lies in the same Weyl chamber as $w\lambda$, then $\Theta(\mu)=\Psi_{w\lambda}^{\mu}(\Theta(w\lambda))$ ([V1], Proposition 7.2.22). This is stable by (b); the equality also shows that $\Theta(\lambda_0)=\Theta_0$. Hence all values of $\Theta(\mu)$ are stable. By the definition of coherent family, $\Theta_0\otimes F$ is a sum of such values, for any holomorphic finite-dimensional representation F. This completes the proof.

10.36 <u>Corollary</u>:

Suppose x is a strong real form of G and Θ is a virtual $(\mathfrak{g},K_x)$-module. The following conditions are equivalent:

(a) Fix a real form $G(\mathbb{R})$ associated to x, and let F_Θ be the locally L^1 function on $G(\mathbb{R})$ representing Θ. Then $F_\Theta(h)=F_\Theta(h')$ whenever h and h' are strongly regular semisimple elements of $G(\mathbb{R})$ conjugate in G.

(b) Suppose $G(\mathbb{R})$ and $G(\mathbb{R})'$ are two real forms associated to x, and F_Θ and F'_Θ are the corresonding locally L^1 functions. Let g be a strongly regular semisimple element of $G(\mathbb{R}) \cap G(\mathbb{R})'$. Then $F_\Theta(g)=F'_\Theta(g)$.

(c) Θ is stable.

As in Theorem 8.3, we may use (b) to associate to Θ a function Φ_Θ on the set $G^\#$ of strongly regular semisimple elements of G belonging to some real form associated to x.

<u>proof</u>:

The equivalence of (a) and (b) is proved as in Theorem 8.3. The space of virtual charaters satisfying (a) or (b) is clearly closed under projection on an infinitesimal character or tensoring with a finite-dimensional representation of G. Since (b) and (c) are equivalent for regular infinitesimal character, they are equivalent for singular infinitesimal character by Lemma 10.35 (b) and (c).

In the same way we get the analogue of Theorem 8.12; its formulation and proof are left to the reader.

We will need another description of the stable character Θ_π of Theorem 10.24.

10.37 Proposition:

Suppose ${}^dT^\Gamma$ is a Cartan subgroup of ${}^\vee G^\Gamma$, and $y \in {}^dT^\Gamma - {}^dT$. Write ${}^\vee c = y^2$, and let ${}^\vee G_{{}^\vee c}$ be the identity component of the centralizer of ${}^\vee c$ as usual. Fix a set ${}^dP$ of positive roots for ${}^dT$ in ${}^\vee G_{{}^\vee c}$, and let $\mathcal{P} = \mathcal{P}({}^\vee c, {}^dP)$ be the corresponding parameter set for translation families in G (Definition 2.9). For $\mu \in \mathcal{P}$, write

$\varphi(\mu) = \varphi(y, {}^dT^\Gamma, \mu)$: $W_{\mathbb{R}} \to {}^\vee G^\Gamma$,

and define $\Theta(\mu)$ to be the stable character associated to $\varphi(\mu)$ by Theorem 10.24.

(a) The family $\{\Theta(\mu) \mid \mu \in \mathcal{P}\}$ is a translation family.

(b) If $\lambda \in \mathcal{P}$ is regular and $\lambda_0 \in \mathcal{P}$ is arbitrary, then $\Psi_\lambda^{\lambda_0}(\Theta(\lambda)) = \Theta(\lambda_0)$.

(c) $\Theta(\lambda_0) = \Sigma_{S'}\ I(S', \lambda_0)$

where the sum extends over equivalence classes of L-data $S' = (x', (T')^\Gamma, P', y, {}^dT^\Gamma, {}^dP, \Upsilon')$, with y, ${}^dT^\Gamma$, and ${}^dP$ fixed as above, and x' conjugate to x.

sketch of proof:

For λ_0 regular, the assertion in (c) is just the definition. The right side of the formula in (c) obviously obeys the rule in (c), so (b) is equivalent to (c). Statement (a) is equivalent to (b) by definition; so it is enough to prove (b) (or (c)). By Lemma 10.35(b), $\Psi_\lambda^{\lambda_0}(\Theta(\lambda))$ is stable. By Theorem 3.6(a), it is a sum of standard limit modules attached to $\varphi(\lambda_0)$. By Theorem 10.24, it is therefore a multiple of $\Theta(\lambda_0)$. To see that this multiple is one, it is necessary to study the formula in (c) more carefully- in particular, to show that different terms cannot give the same non-zero representations. We omit the argument.

We turn now to the definition of lifting for singular infinitesimal character. As in section 9, we begin with endoscopic data $(\breve{s}, {}^\vee H^\Gamma, {}^\vee\mathcal{D}_H)$ for an endoscopic group H (cf. 9.1)), a strong real form x of G, and an extended group $H^\Gamma \supset H$ (cf. (9.2)). The analogue of (9.3) requires a little more care.

10.38 In the setting of (9.1) and (9.2), suppose $\Theta_H \in \mathcal{U}(\mathfrak{h}, \tilde{K}_H)$ is stable, with (possibly singular) infinitesimal character χ_H. Assume as in (9.3) that Θ_H is contained in the block $\mathcal{B}_H$ corresponding to (δ_H, y).

(1) Define ${}^\vee c = y^2$, ${}^\vee G_{{}^\vee c}$, ${}^\vee\theta$, ${}^\vee K_y$, ${}^\vee H_{{}^\vee c}$, ${}^\vee K_H$, ${}^\vee K_y^{can}$, and ${}^\vee K_H^{can}$ as in (9.3(1)).

(2) Define the dual block ${}^\vee\mathcal{B}_H$ of $({}^\vee\mathfrak{h}_{{}^\vee c}, {}^\vee K_H^{can})$-modules as in (9.3(2)).

(3) Choose a Cartan subgroup ${}^dT$ of ${}^\vee H_{{}^\vee c}$ and a set ${}^dP_H$ of positive roots for ${}^dT$ in ${}^\vee H_{{}^\vee c}$, thus fixing the parameter set $\mathcal{P}_H \subset {}^d\mathfrak{t}$ for translation families in $\mathcal{B}_H$. Choose a representative $\lambda_0 \in \mathcal{P}_H$ for the infinitesimal character χ_H. Choose a set ${}^dP \supset {}^dP_H$ of positive roots of ${}^d\mathfrak{t}$ in ${}^\vee\mathfrak{g}_{{}^\vee c}$ making λ_0 dominant, and write ${}^dB_{des} \supset {}^dT$ for the corresponding Borel subgroup of ${}^\vee G_{{}^\vee c}$. This fixes the parameter set $\mathcal{P}$ for translation families for G of infinitesimal character associated to ${}^\vee c$, and $\lambda \in \mathcal{P} \subset \mathcal{P}_H \subset {}^d\mathfrak{t}$.

(4) Choose parameter sets ${}^d\mathcal{P}_H$ and ${}^d\mathcal{P}$ for translation families for ${}^\vee H_{{}^\vee c}$ and ${}^\vee G_{{}^\vee c}$ respectively, of infinitesimal character associated to ${\delta^2}_H = x^2$. As in (9.3)(4), we get a natural injection $(\iota_{H,G})^{-1}: {}^d\mathcal{P} \to {}^d\mathcal{P}_H$; choose ${}^d\lambda \in {}^d\mathcal{P}$

${}^{\vee}G$-regular, and define ${}^d\lambda_H \in {}^d\mathcal{P}_H$, ${}^d\chi$, ${}^d\chi_H$, and $\mathrm{Des}=\mathrm{Des}_{(\mathfrak{s}, {}^dB_{des}, {}^d\chi)}$ as in (9.3)(4).

(5) Let ${}^dB \subset {}^dP$ be the set of simple roots, and ${}^dB_0$ the subset of ${}^dB$ vanishing on λ_0. Define ${}^{\vee}\tilde{\mathcal{B}}_1({}^d\lambda)$ as in (10.9) (as the span of the irreducible modules in ${}^{\vee}\tilde{\mathcal{B}}({}^d\lambda)$ having all roots in B_0 in their τ-invariants). Similarly, define ${}^dB_H$, ${}^dB_{H,0}$, and $({}^{\vee}\mathcal{B}_H)_1^{\sim}({}^d\lambda_H)$. (The roots in ${}^dB_{H,0}$ are positive integral combinations of roots in ${}^dB_0$, but they need not belong to ${}^dB_0$).

10.39 Lemma:

In the setting of (10.38), Des( ) takes ${}^{\vee}\mathcal{B}_1({}^d\lambda)$ into $({}^{\vee}\mathcal{B}_H)_1^{\sim}({}^d\lambda_H)$.

proof:

Write W_0 for the stabilizer of λ_0 in $W({}^{\vee}\mathfrak{g}, {}^d\mathfrak{t})$; this is the subgroup generated by the reflections corresponding to roots in B_0. The intersection of W_0 with $W({}^{\vee}\mathfrak{h}_{{}^{\vee}c}, {}^d\mathfrak{t})$ is the corresponding group $W_{H,0}$ for H. In terms of the coherent continuation action of $W({}^{\vee}\mathfrak{g}_{{}^{\vee}c}, {}^d\mathfrak{t})$ on ${}^{\vee}\mathcal{B}({}^d\lambda)$, the virtual characters in ${}^{\vee}\mathcal{B}_1({}^d\lambda)$ are characterized by

$$w \cdot \Theta = \det(w)\Theta \quad (w \in W_0)$$

(Proposition 10.28). On the other hand, from the explicit description of Des( ) and coherent continuation on the level of distribution characters, one can read off (cf. [AV1], Theorem 4.2)

$$\mathrm{Des}(x \cdot \Theta) = x \cdot \mathrm{Des}(\Theta)$$

for $\Theta \in {}^{\vee}\mathcal{B}({}^d\lambda)$ and $x \in W({}^{\vee}\mathfrak{h}_{{}^{\vee}c}, {}^d\mathfrak{t})$. The lemma follows.

10.40 Definition:

In the setting of (9.1), (9.2), and (10.36), define $\mathcal{B}$ to be the block of $(\mathfrak{g}, K_x)$-modules associated to the pair (x,y). Write ${}^{\vee}\mathcal{B}$ for the dual block of

$({}^\vee\mathfrak{g}_{{}^\vee c}, {}^\vee K_y^{can})$-modules. We have parameter sets $\mathcal{P}$ and ${}^d\mathcal{P}$ for these blocks (cf. (10.36)(3)-(4)), and elements ${}^d\lambda\in{}^d\mathcal{P}$, $\lambda_0\in\mathcal{P}$. The lift of Θ_H to G (and the strong real form x), written Lift(Θ_H), is defined to be the unique virtual $(\mathfrak{g},K_x)$-module in $\mathcal{B}(\lambda_0)$ satisfying

10.41 $$< \mathrm{Lift}(\Theta_H), {}^dZ >_0 = <\Theta_H, \mathrm{Des}({}^dZ) >_0$$

for all ${}^dZ\in{}^\vee\mathcal{B}_1({}^d\lambda)$. The pairings are those of Theorem 10.8 for G and H respectively.

Extend the definition of lift to all stable $\Theta_H\in\mathcal{U}(\mathfrak{h},\tilde{K}_H)$ by linearity. As in Definition 9.4, we may write Lift_x if it is necessary to specify the strong real form x.

This definition involves more choices than the one in section 9, notably of the positive roots system ${}^dP$ in (10.38)(3). As in Lemma 9.6, these choices do not affect the notion of Lift; this follows from the next theorem.

10.42 Theorem:

Suppose we are in the setting of (9.1)-(9.2), and Π_H is an L-packet of standard limit $(\mathfrak{h},\tilde{K}_H)$-modules associated to $\varphi_H:W_{\mathbb{R}}\to{}^\vee H^\Gamma$. Let $\varphi:W_{\mathbb{R}}\to{}^\vee G^\Gamma$ be the composition of φ with the inclusion ${}^\vee H^\Gamma\hookrightarrow{}^\vee G^\Gamma$. If φ is admissible for G and x, let Π denote the corresponding L-packet of standard limit $(\mathfrak{g},K)$-modules (each having an irreducible Langlands quotient); otherwise Π is empty. Let Θ_H be the stable virtual character corresponding to Π_H (Theorem 10.24). Then Lift(Θ_H) is defined by Definition 10.40.

(1) $$\mathrm{Lift}(\Theta_H) = \sum_{I\in\Pi} c_I I \text{ , some } c_I\in\mathbb{C}^* .$$

(2) Let $< \, , \, >: \Pi \times \tilde{\mathbb{S}}_\varphi \to \mathbb{C}^*$ be the pairing of Theorem 10.19 restricted to Π. Regard $\tilde{s}$ as a representative of an element of $\tilde{\mathbb{S}}_\varphi$. Then

$$c_I = (-1)^{q(G,x) + q(H)} < I, \tilde{s} >.$$

proof:

The example of section 9 would suggest that we begin by calculating the descent of pseudo-standard characters. This is possible but painful; we prefer to invoke Theorem 9.7, and say as little about descent as possible.

Choose $(y, {}^dT_H^\Gamma, {}^dP_H, \lambda_0)$ for φ_H as in (10.10) and (10.16) (applied to H and ${}^\vee H^\Gamma$ instead of G). Extend ${}^dP_H$ to a set of positive roots ${}^dP'$ for ${}^dT_H$ in ${}^\vee\mathfrak{g}_{{}^\vee c}$ in such a way that the pair $({}^dP', {}^dP_H)$ is conjugate to ${}^\vee H_{{}^\vee c}$ to the pair chosen in (10.37)(3). For every weight μ in the parameter set $\mathcal{P}_H = \mathcal{P}({}^\vee c, {}^dP_H)$ we get a Weil group homomorphism

$$\varphi_H(\mu) = \varphi_H(y, {}^dT_H^\Gamma, \mu) : W_{\mathbb{R}} \to {}^\vee H^\Gamma;$$

we write $\varphi(\mu)$ for this map regarded as a map into ${}^\vee G^\Gamma$. The corresponding stable characters $\Theta_H(\mu)$ form a translation family for H, and $\Theta_H(\lambda_0) = \Theta_H$ (Proposition 10.36).

Write $\mathcal{B}$ for the block defined by x and y. Suppose

$$S = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \Upsilon)$$

is a set of L-data with $J(S,\lambda_0)$ a non-zero irreducible module. Then $I(S,\lambda_0)$ is a standard limit $(\mathfrak{g},K_x)$-module in $\mathcal{B}(\lambda_0)$ having a unique Langlands subquotient, and all such standard limit modules arise in this way. To prove the theorem, we must calculate the multiplicity of $I(S,\lambda_0)$ in Lift (Θ_H). Fix a regular weight $\lambda \in \mathcal{P}({}^\vee c, {}^dP)$, and let $\lambda_H \in \mathcal{P}({}^\vee c, {}^dP')$ be the corresponding element. By Corollary 10.26, the multiplicity we want is

$$(-1)^{\ell(S)} < \mathrm{Lift}(\Theta_H(\lambda_0)), \Psi({}^dS, {}^d\lambda) >_0$$
$$= (-1)^{\ell(S)} < \Theta_H(\lambda_0), \mathrm{Des}(\Psi({}^dS, {}^d\lambda)) >_0 \qquad \text{(Def. 10.41)}$$
$$= (-1)^{\ell(S)} < (\Psi_H)_{\lambda_H}^{\lambda_0}(\Theta_H(\lambda)), \mathrm{Des}(\Psi({}^dS, {}^d\lambda)) >_0 \quad \text{(Prop. 10.36)}$$
$$= (-1)^{\ell(S)} < \Theta_H(\lambda_H), \mathrm{Des}(\Psi({}^dS, {}^d\lambda)) > \qquad \text{(Thm. 10.8)}$$
$$= (-1)^{\ell(S)} < \mathrm{Lift}(\Theta_H(\lambda_H)), \Psi({}^dS, {}^d\lambda) > \qquad \text{(Def. 9.4)}$$
$$= (-1)^{\ell(S)} < \mathrm{Lift}(\Theta_H(\lambda_H)), I({}^dS, {}^d\lambda) + \Sigma(-1)^{\ell(S)-\ell(S')} I({}^dS', {}^d\lambda) >$$
$$\text{(Cor. 10.32).}$$

(The final sum is over $\{ S' \mid S' \leftarrow S \}$.) We can compute this by Theorem 9.7. We obtain:

$$\Sigma(-1)^{q(G,x) + q(H)} < I(S',\lambda), \tilde{s} > ;$$

with the sum running over all L-data S' such that

(a) $S = S'$ or $S' \leftarrow S$; and

(b) $I(S', \lambda)$ belongs to the L-packet of $\varphi(\lambda_H)$.

(Condition (b) insures that the pairing $< I(S',\lambda),\tilde{s} >$ makes sense.) By Corollary 10.34, there is at most one equivalence class S' of L-data satisfying (a) and (b); so the sum computing the multiplicity reduces to at most a single term.

We want to show that S' satisfying (a) and (b) exists if and only if $I(S,\lambda_0)$ belongs to the L-packet of $\varphi(\lambda_0)$. Suppose first that S' exists. By Corollary 10.33, $I(S,\lambda_0)$ is a a direct summand of $I(S',\lambda_0)$. By Theorem 3.6(a), $I(S',\lambda_0)$ is a sum of standard limit representations in the L-packet of $\varphi(\lambda_0)$. Conversely, suppose $I(S,\lambda_0)$ is in the L-packet of $\varphi(\lambda_0)$. By Proposition 10.37(c) there is an S' with $I(S',\lambda_H)$ in the L-packet of $\varphi(\lambda_H)$, such that $I(S,\lambda_0)$ is a direct summand of $I(S,\lambda_0)$. By Corollary 10.33, S satisfies (a) and (b) above.

Finally we must compare the constant we have calculated with the one in the Theorem. This is accomplished by

10.43 Lemma:

Suppose we are in the setting of (10.10) and (10.16)-(10.18). Fix a set of L-data $S = (x, T^\Gamma, P, y, {}^dT^\Gamma, {}^dP, \Upsilon)$ with $J(S,\lambda_0)$ an irreducible module in the L-packet of φ. Let $({}^dT')^\Gamma$ be another Cartan subgroup of ${}^\vee G_0^\Gamma$, and $S' = (x', (T')^\Gamma, P', y, ({}^dT')^\Gamma, {}^dP', \Upsilon)$ another set of L-data with $\lambda_0 \in \mathcal{P}({}^\vee c, {}^dP')$. Assume that $S=S'$ or $S' \leftarrow S$, so that $I(S,\lambda_0)$ is a direct summand of $I(S',\lambda_0)$. Suppose $\tilde{s} \in ({}^dT' \cap S_\varphi)^\sim$. Fix $\lambda' \in \mathcal{P}({}^\vee c, {}^dP')$ regular. Then

$$< I(S,\lambda_0), \tilde{s} > = < I(S', \lambda), \tilde{s} >.$$

The pairing on the left is defined in Theorem 10.19, and that on the right in Theorem 5.1.

proof:

In case $S'=S$, this lemma is precisely the definition used in the second proof of Theorem 10.19. In general we proceed by downward induction on $\ell(S')$. If $S' \neq S$, we get a simple reflection t fixing λ_0, and some S'' with $S' \overset{t}{\leftarrow} S''$ and $S''=S$ or $S'' \leftarrow S$. Now the relationship between S' and S'' is very simple and explicit (cf. Lemma 10.29). One checks first of all that $\tilde{s}$ still belongs to the Cartan subgroup ${}^dT''$ for S''. By the inductive hypothesis, therefore

$$< I(S,\lambda_0), \tilde{s} > = < I(S'',\lambda), \tilde{s} >.$$

Now one can use the explicit relationship between S' and S'' to prove that

$$< I(S',\lambda), \tilde{s} > = < I(S'',\lambda), \tilde{s} >;$$

it is convenient to use Lemma 5.4 to calculate the pairings. (Essentially one is reduced to the case of $SL(2,\mathbb{R})$.) We omit the details. This completes the proof of the Lemma.

Now Lemma 10.43 shows that the calculation in the proof of Theorem 10.42 agrees with statement (b), completing the proof of the Theorem.

The definition of super-lifting (Definition 9.19) extends word for word to arbitrary infinitesimal character, along with Theorems 9.21-9.23 and the inversion formula (9.24). Corollaries 9.14-9.16 also extend immediately. Comparison with Shelstad's definition of lifting is not quite so trivial; but it is straightforward, and we leave it to the reader.

§11

Examples

We discuss a few examples. A great deal can be learned from a consideration of real forms of SL(2) and PGL(2).

First we mention a few general facts which are of assistance in computing examples. (For the moment we ignore coverings.) Fix infinitesimal character ρ for G and ${}^\vee\rho$ for ${}^\vee G$. Suppose $x \in G^\Gamma$ defines a strong real form of G, and $y \in {}^\vee G^\Gamma$ defines a block $\mathcal{B}$. We work in terms of representations of $G(\mathbb{R})_x$ in place of $(\mathfrak{g},K_x)$ modules. Suppose $\mathcal{B}$ is the block containing the discrete series of $G(\mathbb{R})_x$. Then ${}^\vee G(\mathbb{R})_y$ contains a split Cartan subgroup, and dual to the discrete series of $G(\mathbb{R})_x$ are certain minimal principal series of ${}^\vee G(\mathbb{R})_y$. Suppose π is a one-dimensional representation of a split group $G(\mathbb{R})_x$. Then ${}^\vee G(\mathbb{R})_y$ is quasisplit and ${}^\vee\pi$ is a discrete series representation, such that every simple root in the corresponding positive system is non-compact. Suppose the block $\mathcal{B}$ is one-dimensional, i.e. $\mathcal{B}=\langle\pi\rangle$. Thus the standard module containing π is irreducible, and the same holds for ${}^\vee\pi$. For example this holds if $G(\mathbb{R})_x$ is compact; then ${}^\vee G(\mathbb{R})_y$ is split, and ${}^\vee\pi$ is an irreducible minimal principal series. (This is also the case if the infinitesimal character of π has non-integral inner product with all roots; then ${}^\vee G(\mathbb{R})_y$ is a torus and ${}^\vee\pi$ is a character of ${}^\vee G(\mathbb{R})_y$.)

Now let G be SL(2); we identify the algebraic group with its points over $\mathbb{C}$. This has only one inner class of real forms. An L-group for G is obtained as follows. The group ${}^\vee G$ may be taken to be PGL(2). Let ${}^\vee\delta$ act by conjugation by diag(1,-1). Choose ${}^\vee B$ to be upper triangular matrices, and let ${}^\vee\mathcal{D}$ be the conjuacy class of $({}^\vee\delta,{}^\vee B)$. Then taking ${}^\vee G^\Gamma={}^\vee G \cup {}^\vee G\,{}^\vee\delta$,

$({}^{\vee}G^{\Gamma},{}^{\vee}\mathcal{D})$ is an L-group for G. Note ${}^{\vee}\delta^2=z_\rho=1$. Similary $(G^{\Gamma},\mathcal{D})$ is an L-group for ${}^{\vee}G$, with δ acting by conjugation by diag(1,-1), B is the set of upper triangular matrices, and δ^2=diag(-1,-1).

Up to equivalence G has three strong real forms: $x=\delta$, or $x=\pm\mathrm{diag}(i,-i)\delta$ (note that δ is conjugate to $-\delta$). If $x=\delta$ then $G(\mathbb{R})_x$ is quasisplit, i.e. isomorphic to $SL(2,\mathbb{R})$. If $x=\pm\mathrm{diag}(i,-i)\delta$ then G_x is isomorphic to SU(2). It is instructive to write these three cases as SU(2,0), SU(1,1) and SU(0,2).

Similarly ${}^{\vee}G$ has two equivalence classes of strong real forms. If $y=\delta$ then ${}^{\vee}G(\mathbb{R})_y$ is isomorphic to $PGL(2,\mathbb{R})$, and for $y=\mathrm{diag}(i,-i){}^{\vee}\delta$, ${}^{\vee}G(\mathbb{R})_y$ is isomorphic to $SO(3,\mathbb{R})$.

Let T and ${}^dT$ be the subgroups of G and ${}^{\vee}G$ respectively consisting of diagonal matrices. Fix infinitesimal character ρ for G and ${}^{\vee}\rho$ for ${}^{\vee}G$. Consider the blocks $\mathcal{B}$ and ${}^{\vee}\mathcal{B}$ defined by the pair $(\delta,{}^{\vee}\delta)$, so $G(\mathbb{R})_\delta\approx SL(2,\mathbb{R})$ and ${}^{\vee}G(\mathbb{R})_{{}^{\vee}\delta}\approx PGL(2,\mathbb{R})$. Let T' (respectively ${}^dT'$) denote SO(2) embedded in G (resp. ${}^{\vee}G$) in the usual way, with B' and ${}^dB'$ Borels containing T' and ${}^dT'$. Thus T and ${}^dT$ are compact, whereas T' and ${}^dT'$ are split (with respect to $\theta=\mathrm{int}(\delta)$ and ${}^{\vee}\theta=\mathrm{int}({}^{\vee}\delta)$ respectively). Let B and ${}^dB$ respectively denote the subgroups of G and ${}^{\vee}G$ respectively, consisting of upper triangular matrices. Let B^{opp} denote the lower triangular matrices.

Let S be a set of L-data. Recall (cf. the discussion following Definition 3.8) we may omit Υ from the data. Furthermore we write Borel subgroups in place of the positive root systems in the data.

11.1 L-data S	π	${}^\vee\pi$
$(\delta,T,B,{}^\vee\delta,{}^dT',{}^dB')$	π_+=holomorphic discrete series	${}^\vee\pi_0$=trivial
$(\delta,T,B^{opp},{}^\vee\delta,{}^dT',{}^dB')$	π_-=antihol. discrete series	${}^\vee\pi_1$=sgn
$(\delta,T',B',{}^\vee\delta,{}^dT,{}^dB)$	π_0=trivial	${}^\vee\pi_d$=discrete series

In fact we have that $\mathcal{B}=<\pi_+,\pi_-,\pi_0>$ and ${}^\vee\mathcal{B}=<{}^\vee\pi_0,{}^\vee\pi_1,{}^\vee\pi_d>$. Note that $(\delta,T,B,{}^\vee\delta,{}^dT',{}^dB')$ is conjugate via G to $(-\delta,T,B^{opp},{}^\vee\delta,{}^dT',{}^dB')$. The first set of data corresponds to the holomorphic discrete series of $G(\mathbb{R})_\delta\approx SL(2,\mathbb{R})$, whereas the latter produces the antiholomorphic discrete series of $G(\mathbb{R})_{-\delta}\approx SL(2,\mathbb{R})$. This is an example in which it is necessary to keep track not only of the Cartan involution defining the real form of G, but the element of G^Γ producing it: in this example there is no contradiction because we distinguish between $G(\mathbb{R})_\delta$ and $G(\mathbb{R})_{-\delta}$.

There are three pairs of dual blocks for G and ${}^\vee G$ with integral infinitesimal character. We list these below, with choices of infinitesimal character indicated by λ and ${}^\vee\lambda$. Note that $PGL(2,\mathbb{R})$ has two irreducible principal series with infinitesimal character $2{}^\vee\rho$, which we write $ps_\pm$. We let ps_- denote the irreducible principle series representation of $SL(2,\mathbb{R})$ with infinitesimal character ρ. Let
$x^1=diag(i,-i)\delta$, $y^1=diag(i,-i){}^\vee\delta$.

11.2 (1) $(x,y)=(\delta,{}^\vee\delta)$ $G(\mathbb{R})_x\approx SL(2,\mathbb{R})$, ${}^\vee G(\mathbb{R})_y\approx PGL(2,\mathbb{R})$,

$\lambda=\rho$, ${}^\vee\lambda={}^\vee\rho$, $\mathcal{B}_0=<\pi_+,\pi_-,\pi_0>$, ${}^\vee\mathcal{B}_0=<{}^\vee\pi_0,{}^\vee\pi_1,{}^\vee\pi_d>$.

(2) $(x,y)=(x^1,{}^\vee\delta)$, $G(\mathbb{R})_x\approx SU(2,0)$, ${}^\vee G(\mathbb{R})_y\approx PGL(2,\mathbb{R})$,

$\lambda=\rho$, ${}^\vee\lambda=2{}^\vee\rho$, $\mathcal{B}_{(2,0)}=<trivial>$, ${}^\vee\mathcal{B}_+=<ps_+>$.

(3) $(x,y)=(-x^1,{}^\vee\delta)$, $G(\mathbb{R})_x\approx SU(0,2)$, ${}^\vee G(\mathbb{R})_y\approx PGL(2,\mathbb{R})$,

$\lambda=\rho$, ${}^\vee\lambda=2{}^\vee\rho$, $\mathcal{B}_{(0,2)}=<trivial>$, ${}^\vee\mathcal{B}_-=<ps_->$.

(4) $(x,y)=(\delta,y^1)$, $G(\mathbb{R})_x\approx SL(2,\mathbb{R})$, ${}^\vee G(\mathbb{R})_y\approx PU(2)$,

$\lambda=\rho$, ${}^\vee\lambda={}^\vee\rho$, $\mathcal{B}_-=<ps_->$, ${}^\vee\mathcal{B}_{(2,0)}=<trivial>$.

In the following tables we will refer to these blocks by the indicated labels.

We next consider endoscopic data for G, and spell out the constructions of 9.1-9.3 in this case. We have ${}^\vee G^{can}\approx SL(2)$. By Lemma 6.7 we fix ${}^\vee\delta$ as above. Then as a set of representatives for the equivalence classes of weak endoscopic data we may take $\tilde{s}$ equal to $\pm I$, $\tilde{s}=\tilde{s}_h=\mathrm{diag}(i,-i)$, or $\tilde{s}=\tilde{s}_e=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. If $\tilde{s}=\pm I$ or s_e, then there is a unique choice of endoscopic data extending each choice of weak endoscopic data (cf. Corollary 6.6). If $\tilde{s}=s_h$, then there are two such choices, given by ${}^\vee\delta_H=\pm{}^\vee\delta$.

Let $\tilde{\mathbb{R}}^*$ denote the non-trivial covering group of $\mathbb{R}^*$ given by $\tilde{\mathbb{R}}^*=\mathbb{R}^*\cup i\mathbb{R}^*$, with covering map $z\to z^2/|z|$. Define a genuine character $\tilde{\alpha}$ of $\tilde{\mathbb{R}}^*$: $\tilde{\alpha}(z)=z/|z|^{\frac{1}{2}}$. Let $\alpha^\pm$ denote the characters of $\mathbb{R}^*$ defined by $\alpha^\pm(t)=t$ $(t\in\mathbb{R}^+)$, $\alpha^+(-1)=1$, $\alpha^-(-1)=-1$.

We will use the following convention about covering groups. Suppose $\tilde{G}$ is a covering group of G, with real points $\tilde{G}(\mathbb{R})$ covering $G(\mathbb{R})$. The representations of $\tilde{G}(\mathbb{R})$ of trivial type (i.e. trivial on the kernel of the covering map) are canoncially identified with representations of $G(\mathbb{R})$. Similarly if the covering splits the representations of $\tilde{G}(\mathbb{R})$ of a fixed type are in canonical bijection with the representations of $G(\mathbb{R})$. In either case by abuse of notation we identify the representations of $\tilde{G}(\mathbb{R})$ with those of $G(\mathbb{R})$, and write $\tilde{G}=G$ and $\tilde{G}(\mathbb{R})=G(\mathbb{R})$. This will be applied mainly to the group ${}^\vee H$.

11.3 (1) $\tilde{s}=\pm I$

Then ${}^\vee H^\Gamma={}^\vee G^\Gamma$, $H=G$, and $(H^\Gamma,\mathcal{D}_H)$ may be taken to be $(G^\Gamma,\mathcal{D})$. There is only one choice for ${}^\vee\mathcal{D}_H$. Then $H\approx SL(2)$, $H(\mathbb{R})\approx SL(2,\mathbb{R})$, and ${}^\vee H\approx PGL(2)$. The covering $\tilde{H}\to H$ is split.

(2) $\tilde{s}=\tilde{s}_e$.

Then ${}^\vee H=\mathbb{C}^*$. In this case ${}^\vee H^\Gamma$ is isomorphic to an L-group for $H(\mathbb{R})\approx S^1$ (not just an E-group) so $\tilde{H}\to H$ splits. There is only one choice for ${}^\vee\mathcal{D}_H$ in this case also.

(3) $\tilde{s}=\tilde{s}_h$.

Here ${}^\vee H=\mathbb{C}^*$, and ${}^\vee H^\Gamma$ is isomorphic to an L-group for $H(\mathbb{R})\approx\mathbb{R}^*$; and again $\tilde{H}\to H$ splits. Now there are two choices for ${}^\vee\mathcal{D}_H$; changing the choice corresponds to tensoring with the sign representations of $\mathbb{R}^*$.

Thus by the discussion preceding 11.3 we will identify representations of $\tilde{H}(\mathbb{R})$ in each case with representations of $H(\mathbb{R})$.

Furthermore introducing x and y as in 9.2 we obtain a covering group ${}^\vee H_{y2}^{can}$ of ${}^\vee H_{y2}$ and blocks as follows. We describe the covering group of ${}^\vee H$, blocks $\mathcal{B}$ and ${}^\vee\mathcal{B}$ for G and ${}^\vee G_{y2}$ defined by (x,y), and blocks $\mathcal{B}_H$ and ${}^\vee\mathcal{B}_H$ for $\tilde{H}$ and ${}^\vee\tilde{H}_{y2}$ defined by (δ_H,y).

11.4 (1) $\tilde{s}=\pm I$

We have ${}^{\vee}H=PGL(2)$ and ${}^{\vee}H^{can}=SL(2)$. There is only one choice of endoscopic data extending this weak endoscopic data.

i. $(x,y)=(\delta,{}^{\vee}\delta)$

Then ${}^{\vee}H(\mathbb{R})^{can}$ is isomorphic to $PGL(2,\mathbb{R})$, the inverse image of $PGL(2,\mathbb{R})$ in $SL(2,\mathbb{C})$. The representations in ${}^{\vee}\mathcal{B}_H$ factor to ${}^{\vee}H(\mathbb{R})\approx PGL(2,\mathbb{R})$. Then $(\mathcal{B},{}^{\vee}\mathcal{B})=(\mathcal{B}_H,{}^{\vee}\mathcal{B}_H)=(\mathcal{B}_0,{}^{\vee}\mathcal{B}_0)$.

ii. $(x,y)=(\pm x^1,{}^{\vee}\delta)$

Now $(\mathcal{B},{}^{\vee}\mathcal{B})=(\mathcal{B}_{(2,0)},{}^{\vee}\mathcal{B}_{\pm})$ $(x=x^1)$ or $(\mathcal{B}_{(0,2)},{}^{\vee}\mathcal{B}_{\pm})$ $(x=-x^1)$. Again we have ${}^{\vee}H(\mathbb{R})^{can}\approx PGL(2,\mathbb{R})^{can}$, and now $(\mathcal{B}_H,{}^{\vee}\mathcal{B}_H)=(\mathcal{B}_0,{}^{\vee}\tilde{\mathcal{B}}_0)$. Here ${}^{\vee}\tilde{\mathcal{B}}_0$ is ${}^{\vee}\mathcal{B}_0$ pulled back from $PGL(2,\mathbb{R})$ to $PGL(2,\mathbb{R})^{can}$, followed by applying the translation principle from infinitesimal character ${}^{\vee}\rho$ to infinitiesmal character $2{}^{\vee}\rho$. This involves tensoring with the (genuine) two-dimensional representation of $PGL(2,\mathbb{R})^{can}$.

iii. $(x,y)=(\delta,y^1)$

Now ${}^{\vee}H(\mathbb{R})^{can}\approx SU(2)$, and $(\mathcal{B},{}^{\vee}\mathcal{B})=(\mathcal{B}_H,{}^{\vee}\mathcal{B}_H)=(\mathcal{B}_-,{}^{\vee}\mathcal{B}_{(2,0)})$ with ${}^{\vee}\mathcal{B}_{(2,0)}$. Here ${}^{\vee}\mathcal{B}_{(2,0)}$ is the block consisiting of the (non-genuine) trivial representation of $SU(2)$.

(2) $\tilde{s}=\tilde{s}_e$

We have ${}^{\vee}H(\mathbb{R})\approx\mathbb{R}^*$, ${}^{\vee}\tilde{H}$ is the genuine two-fold cover $\mathbb{C}^*\to\mathbb{C}^*$ given by $z\to z^2$, and ${}^{\vee}\tilde{H}(\mathbb{R})\approx\tilde{\mathbb{R}}^*$. Again the choice of endoscopic data is unique.

i. $(x,y)=(\delta,{}^{\vee}\delta)$

We have $(\mathcal{B},{}^{\vee}\mathcal{B})=(\mathcal{B}_0,{}^{\vee}\mathcal{B}_0)$. Now ${}^{\vee}\tilde{H}(\mathbb{R})\approx\tilde{\mathbb{R}}^*$, ${}^{\vee}\mathcal{B}_H$ consists of genuine representations, and $(\mathcal{B}_H,{}^{\vee}\mathcal{B}_H)=(\langle e^{i\theta}\rangle,\langle\tilde{\alpha}\rangle)$.

ii. $(x,y)=(x^1,{}^{\vee}\delta)$

In this case $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_{(2,0)},{}^\vee\mathcal{B}_\pm)$. In this case ${}^\vee\mathcal{B}_H$ factors to $\mathbb{R}^*$, and $(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<e^{i\theta}>,<\alpha^+>)$.

iii. $(x,y)=(-x^1,{}^\vee\delta)$

This case is similar to (ii): $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_{(0,2)},{}^\vee\mathcal{B}_-)$, ${}^\vee\mathcal{B}_H$ factors to $\mathbb{R}^*$, and $(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<e^{i\theta}>,<\alpha^->)$.

iv. $(x,y)=(\delta,y^1)$

We have $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_-,{}^\vee\mathcal{B}_{(2,0)})$, ${}^\vee\mathcal{B}_H$ consists of genuine representations, and$(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<e^{i\theta}>,<\tilde{\alpha}>)$.

(3) $\tilde{s}=\tilde{s}_h$

In this case ${}^\vee H(\mathbb{R})\approx S^1$. As in (2) we have ${}^\vee\tilde{H}$ is the connected two-fold cover of $\mathbb{C}^*$, and ${}^\vee\tilde{H}(\mathbb{R})\approx S^1\rightarrow{}^\vee H(\mathbb{R})\approx S^1$ is the two-fold cover of ${}^\vee H(\mathbb{R})$. IN this case there are two choices of endoscopic data extending the given weak endoscopic data, given by ${}^\vee\delta_H=\pm{}^\vee\delta$.

i. $(x,y)=(\delta,{}^\vee\delta)$

We have $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_0,{}^\vee\mathcal{B}_0)$. The representations in ${}^\vee\mathcal{B}_H$ are genuine, and $(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<\alpha^+>,<e^{i\theta/2}>)$ $({}^\vee\delta_H={}^\vee\delta)$ or $(<\alpha^->,<e^{i\theta/2}>)$ $({}^\vee\delta_H=-{}^\vee\delta)$.

ii. $(x,y)=(x^1,{}^\vee\delta)$

Now $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_{(2,0)},{}^\vee\mathcal{B}_\pm)$ the representations in ${}^\vee\mathcal{B}_H$ factor to ${}^\vee H(\mathbb{R})$, and $(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<\alpha^\pm>,<e^{i\theta}>)$ $({}^\vee\delta_H=\pm{}^\vee\delta)$.

iii. $(x,y)=(-x^1,{}^\vee\delta)$

Now $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_{(0,2)},{}^\vee\mathcal{B}_-)$ and $(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<\alpha^\pm>,<e^{i\theta}>)$ $({}^\vee\delta_H=\pm{}^\vee\delta)$.

iv. $(x,y)=(\delta,y^1)$

We have $(\mathcal{B},{}^\vee\mathcal{B})=(\mathcal{B}_-,{}^\vee\mathcal{B}_{(2,0)})$, the representations in ${}^\vee\mathcal{B}_H$ are genuine, and $(\mathcal{B}_H,{}^\vee\mathcal{B}_H)=(<\alpha^\pm>,<e^{i\theta/2}>)$ $({}^\vee\delta_H=\pm{}^\vee\delta)$. Note that this is a different strong real form of ${}^\vee H$ than in (i), with the same real points.

The stable virtual characters of $SL(2,\mathbb{R})$ with infinitesimal character ρ are $\pi_+ + \pi_-$, π_0, and ps_-. The trivial representation $\mathbb{C}_{(2,0)}$ of $SU(2,0)$ or $\mathbb{C}_{(0,2)}$ of $SU(0,2)$ is stable.

Now G has three super L-packets $\Pi_1 = <\pi_+, \pi_-, \mathbb{C}_{(2,0)}, \mathbb{C}_{(0,2)}>$ and $\Pi_2 = < ps_- >$. The superstable sums are $\pi_+ + \pi_- - \mathbb{C}_{(2,0)} - \mathbb{C}_{(0,2)}$ and ps_-.

In the following table we list all the liftings (with integral infinitesimal character) for real forms of SL(2) (up to the translation principle). The infinitesimal character for G is always ρ, for ${}^\vee G$ it is either ${}^\vee\rho$ or $2{}^\vee\rho$. We identify $\tilde{H}(\mathbb{R})$ with $H(\mathbb{R})$ in each case.

11.5

$G(\mathbb{R}) \approx$	$SU(2,0)$	$SL(2,\mathbb{R})$	$SU(0,2)$
$\tilde{s}=I$ $H(\mathbb{R})\approx SL(2,\mathbb{R})$, ${}^{\vee}\tilde{H}(\mathbb{R})\approx PGL(2,\mathbb{R})^{can}$:			
$Lift(\pi_+ + \pi_-) =$	$-\mathbb{C}_{(2,0)}$	$\pi_+ + \pi_-$	$-\mathbb{C}_{(0,2)}$
$Lift(\pi_0) =$	$\mathbb{C}_{(2,0)}$	π_0	$\mathbb{C}_{(0,2)}$
$\tilde{s}=-I$ $H(\mathbb{R})\approx SL(2,\mathbb{R})$, ${}^{\vee}\tilde{H}(\mathbb{R})\approx PGL(2,\mathbb{R})^{can}$:			
${}^{\vee}\tilde{H}(\mathbb{R}) \approx$	$SL(2,\mathbb{R})^{\sim}$	$SL(2,\mathbb{R})$	$SL(2,\mathbb{R})^{\sim}$
$Lift(\pi_+ + \pi_-) =$	$\longleftarrow$	$\mathbb{C}_{(2,0)}$ $\longleftarrow$	$\pi_+ + \pi_-$ $\longleftarrow$ $\mathbb{C}_{(0,2)}$
$Lift(\pi_0) =$	$-\mathbb{C}_{(2,0)}$	π_0	$-\mathbb{C}_{(0,2)}$
$\tilde{s}=\tilde{s}_e$ $H(\mathbb{R})\approx S^1$, ${}^{\vee}\tilde{H}(\mathbb{R})\approx\tilde{\mathbb{R}}^*$:			
$Lift(e^{i\theta}) =$	$i\mathbb{C}_{(2,0)}$	$\pi_+ + \pi_-$	$-i\mathbb{C}_{(0,2)}$
$Lift(e^{i\theta}) =$	$-i\mathbb{C}_{(2,0)}$	$\pi_+ + \pi_-$	$i\mathbb{C}_{(0,2)}$
$\tilde{s}=diag(i,-i)$, $\tilde{H}(\mathbb{R})\approx\mathbb{R}^*$, ${}^{\vee}\tilde{H}(\mathbb{R})\approx\tilde{S}^1$:			
$Lift(\alpha^{\pm}) =$	0	$-(\pi_+ + \pi_- + \pi_0)$	0
$Lift(\alpha^{\pm}) =$	0	$-(ps_-)$	0

Interesting examples of most of the constructions of this paper can be found in the preceding tables.

For example consider the super L-packet $\Pi_1 = <\pi_+, \pi_-, \mathbb{C}_{(2,0)}, \mathbb{C}_{(0,2)}>$. The super-stable element of this packet is $\pi_+ + \pi_- - \mathbb{C}_{(2,0)} - \mathbb{C}_{(0,2)}$, which is the lift from $\tilde{s}=I$ of $\pi_+ + \pi_-$. Four linearly independent sums in this packet are found in Table 11.5, which correspond to the four elements of $\tilde{\mathbb{S}}_\varphi \approx Z/4Z$. Note that the coefficents in these liftings are ±1 or ±i.

Note that the effect of varying the endoscopic data for given weak endoscopic data may be seen in 11.5, with $\tilde{s}=\mathrm{diag}(i,-i)$. Thus that wheras what we are lifting from $(\alpha^{\pm})$ depends on this choice, the image of the lift is not affected by such a change (cf. Corollary 9.16).

Suppose next that G is S^1. We identify G with $\mathbb{C}^*$ so that $G^\Gamma \approx \mathbb{C}^* \times \Gamma = <\mathbb{C}^*, \delta>$, with $\delta^2=1$. We have ${}^\vee G^\Gamma \approx <\mathbb{C}^*, {}^\vee\delta>$, with ${}^\vee\delta(z)=z^{-1}$, and ${}^\vee\delta^2=1$. An element $z\delta$ ($z \in \mathbb{C}^*$) defines a strong real form, denoted G_z, of G, and these are all inequivalent. On the other hand up to equivalence the only strong real form of ${}^\vee G$ is ${}^\vee\delta$, and ${}^\vee G(\mathbb{R})_\delta \approx \mathbb{R}^*$. In this case blocks coincide with translation families of L-packets. Thus a pair of dual blocks (or L-packets) is given by a pair $(z\delta, {}^\vee\delta)$. The corresponding translation family of L-packets is the coherent family of characters $\{e^{in\theta} \mid n \in Z\}$ of the strong real from $z\delta$ of G. Write this L-packet Π_z to indicate the strong real form. Fix w so that $e^{-i\pi w}=z$. For $\mu \in w+Z$, let ${}^\vee\pi_\mu$ be the character ${}^\vee\pi_\mu(t)=t^\mu$ $(t \in \mathbb{R}^+)$, ${}^\vee\pi_\mu(-1)=(-1)^{\mu-w}$. Then the dual translation family is $\{\pi_\mu \mid \mu \in w+Z\}$.

We specialize to fixed infinitesimal characters by choosing infinitesimal character 0 for G. Thus $\Pi_z=<\pi_z>$, where π_z denotes the trivial representation of the strong real form $z\delta$. The unique super L-packet Π for G is $\cup_{z \in \mathbb{C}^*} \Pi_z$. The (formal) sum $\Sigma_{z \in \mathbb{C}^*} \pi_z$ is the unique super-stable element of Π (up to a constant multiple).

Note that ${}^\vee G^{can}={}^\vee G^{sc}=\mathbb{C}$, with covering map the exponential map. Fix ${}^\vee\delta$, with ${}^\vee K=\{\pm 1\}$. Then endoscopic data for G is given as in Lemma 6.7 by $\tilde{s} \in \mathbb{C}$ such that $\exp(\tilde{s})=\pm 1$. Let $\tilde{s}_k=\pi ik$ $(k \in Z)$; these all define inequivalent endoscopic data, though in each case $H_k \approx S^1$. Note that $S_\varphi=\{\pm 1\}$, $\tilde{\mathbb{S}}_\varphi=\tilde{S}_\varphi=\{\tilde{s}_k\}$. These countably many distinct liftings from endoscopic groups are necessary to distinguish the (uncountably many) distinct representations π_z.

Given $\tilde{s}_k$, the lift from H_k to G_z of the trivial representation is $z^k\pi_z$. The super lift is therefore the formal sum $\mathrm{Lift}^*(\mathbb{C})=\Sigma_{z\in\mathbb{C}^*}z^k\pi_z$. Formally an element of Π is a sum $\Sigma_{z\in\mathbb{C}^*}\alpha(z)\pi_z$ for α a function on $\mathbb{C}^*$. Thus formally inversion (Theorem 9.24) amounts to ordinary Fourier inversion for the function $\alpha(z)$.

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